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Noninformative priors for the common location parameter in half-t distributions

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Abstract

In this paper, we want to develop objective priors for the common location parameter in two half-t distributions with unequal scale parameters. The half-t distribution is a non-regular class of distribution. One can not develop the reference prior by using the algorithm of Berger of Bernardo (1989). Specially, we derive the reference priors and prove the propriety of joint posterior distribution under the developed priors. Through the simulation study, we show that the proposed reference prior matches the target coverage probabilities in a frequentist sense.

Keywords: Half-t distribution, location parameter, nonregular case, reference prior.

1. Introduction

The half-normal distribution has been used as a model for truncated data from application areas as diverse as fibre buckling (Haberle, 1991), blowfly dispersion (Dobzhansky and Wright, 1943), sports science physiology (Pewsey, 2002, 2004) and stochastic frontier modeling (Aigner *et al.*, 1977; Meeusen and van den Broeck, 1977). However, for heavy-tailed data, the half-normal distribution will not be an adequate model and then a half-t distribution might be considered as a more flexible alternative. For the half-t distribution, little work appears to have been published, but see Trancredi (2002). Recently, Wiper *et al.* (2008) gave Bayesian inference for half-t distribution, and they presented the latent variable so as to define Gibbs sampling algorithm similar to the one of Geweke (1993) designed to perform Bayesian inference.

Consider X and Y are independently distributed random variables according to the half-t distribution $\mathcal{HT}_{d_1}(\xi, \tau_1)$ with the location parameter ξ and the scale parameter τ_1, and the

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half-t distribution $\mathcal{HT}_{d_2}(\xi, \tau_2)$ with the location parameter ξ and the scale parameter τ_2. Then the probability density functions of half-t distributions of X and Y are given by

$$f(x|\xi,\tau_1,d_1) = \frac{2\Gamma(\frac{d_1+1}{2})\tau_1^{-\frac{1}{2}}}{\Gamma(\frac{d_1}{2})\sqrt{d_1\pi}}\left[1+\frac{1}{d_1}\left(\frac{x-\xi}{\tau_1^{1/2}}\right)^2\right]^{-\frac{d_1+1}{2}}, \quad (1.1)$$

where $x > \xi, -\infty < \xi < \infty, \tau_1 > 0, d_1 > 0$ and

$$f(y|\xi,\tau_2,d_2) = \frac{2\Gamma(\frac{d_2+1}{2})\tau_2^{-\frac{1}{2}}}{\Gamma(\frac{d_2}{2})\sqrt{d_2\pi}}\left[1+\frac{1}{d_2}\left(\frac{y-\xi}{\tau_2^{1/2}}\right)^2\right]^{-\frac{d_2+1}{2}}, \quad (1.2)$$

where $y > \xi, -\infty < \xi < \infty, \tau_2 > 0, d_2 > 0$, respectively. The parameter ξ is the common location parameter. This common parameter plays an important role in this model, because this parameter is a threshold value and also represents a location of the distribution.

In this article, we want to develop the reference priors as noninformative priors for the common location parameter. Subjective priors are ideal when sufficient information from past experience, expert opinion or previously collected data exist. However, often even without adequate prior information, one can use Bayesian techniques efficiently with some noninformative or default priors.

In recent years, the notion of a noninformative prior has attracted much attention. There are different notions of noninformative prior. One is the reference prior approach of Bernardo (1979) which extended by Berger and Beranrdo (1989,1992) to a general algorithm to derive a reference prior by splitting the parameters into several groups according to their order of inferential importance. The other approach is a probability matching prior initiated by Welch and Peers (1963) which matches the posterior and frequentist probabilities of confidence intervals. Lately, Stein (1985), Tibshirani (1989), DiCiccio and Stern (1994), Datta and Ghosh (1995), Datta (1996), Mukerjee and Ghosh (1997), Kim *et al.* (2009a, 2009b) developed probability matching priors and studied their properties in many statistical models.

The works about the noninformative priors mentioned above are developed under the regular families of distributions. However, nonregular families, such as the uniform or shifted exponential, are also important in many practical problems. The method developed for regular families of distribution can not be applied to nonregular distributions. In developing reference priors, an asymptotic expansion of the posterior density is required. The expansion of the posterior in nonregular case is different from regular case.

Ghosal and Samanta (1995) studied such nonregular families extensively. Ghosal and Smanta (1997) developed the reference priors for the case of one parameter families of discontinuous densities in the sense of Bernardo (1979). Ghosal (1997) proposed the general method to develop the reference priors for the multiparameter nonregular cases when except a threshold parameter, the distribution under consideration is regular with respect to the other parameters. Based on his results, Kang *et al.* (2008, 2010) developed the reference priors for exponential and half-normal distributions, and showed that the proposed reference prior matches the target coverage probabilities very well.

In spite of importance of half-t distribution, there was no study related with an objective Bayesian inference. We want to set up an objective Bayesian inference which can be utilized in the situation that there was a little prior information about location parameter.

The outline of the remaining sections is as follows. In Section 2, we develop reference priors for the common location parameter. In Section 3, we provide that the propriety of the posterior distribution for the general prior including the reference priors. In Section 4, simulated frequentist coverage probabilities under the derived priors are given.

2. The reference priors

In this section, we will develop the reference priors for different groups of orderings of importance by following Ghosal (1997).

Let $X_i, i = 1, \cdots, n_1$ denote observations from the half-t distribution $\mathcal{HT}_{d_1}(\xi, \tau_1)$, and $Y_i, i = 1, \cdots, n_2$ denote observations from the half-t distribution $\mathcal{HT}_{d_2}(\xi, \tau_2)$. Then likelihood function is given by

$$f(\mathbf{x}, \mathbf{y}|\xi, \tau_1, \tau_2) = 2^{n_1+n_2} \left[\frac{\Gamma(\frac{d_1+1}{2})}{\Gamma(\frac{d_1}{2})\sqrt{d_1 \pi}} \right]^{n_1} \tau_1^{-\frac{n_1}{2}} \prod_{i=1}^{n_1} \left[1 + \frac{1}{d_1} \left(\frac{x_i - \xi}{\tau_1^{1/2}} \right)^2 \right]^{-\frac{d_1+1}{2}}$$
$$\times \left[\frac{\Gamma(\frac{d_2+1}{2})}{\Gamma(\frac{d_2}{2})\sqrt{d_2 \pi}} \right]^{n_2} \tau_2^{-\frac{n_2}{2}} \prod_{j=1}^{n_2} \left[1 + \frac{1}{d_2} \left(\frac{y_j - \xi}{\tau_2^{1/2}} \right)^2 \right]^{-\frac{d_2+1}{2}}, \quad (2.1)$$

where $-\infty < \xi < \infty$, $\tau_1 > 0, \tau_2 > 0, d_1 > 0$ and $d_2 > 0$.

We will derive the reference prior when ξ is parameter of interest. The reference prior is developed by considering a sequence of compact subsets of the parameter space, and taking the limit of a sequence of priors as these compact subsets fill out of the parameter space. The compact subsets were taken to be Cartesian products of sets of the form

$$\tau_1 \in [a_1, b_1], \tau_2 \in [a_2, b_2].$$

In the limit a_1, a_2 will tend to 0 and b_1, b_2 will tend to ∞. Here, and below, a subscripted Q denotes a function that is constant and does not depend on any parameter but any Q may depend on the ranges of the parameters.

Let the matrix $F(\xi, \tau_1, \tau_2)$ be defined by

$$F(\xi, \tau_1, \tau_2) = Diag \left\{ \frac{n_1}{2} \frac{d_1}{d_1+3} \tau_1^{-2}, \frac{n_2}{2} \frac{d_2}{d_2+3} \tau_2^{-2} \right\},$$

where $F(\xi, \tau_1, \tau_2) = \{4 J_{jk}(\xi, \tau_1, \tau_2)\}, j, k = 1, 2,$

$$J_{jk}(\xi, \tau_1, \tau_2) = \int \int g_{\tau_j}(\mathbf{x}, \mathbf{y}; \xi, \tau_1, \tau_2) g_{\tau_k}(\mathbf{x}, \mathbf{y}; \xi, \tau_1, \tau_2) d\mathbf{x} d\mathbf{y},$$

$g_{\tau_j} = \partial g / \partial \tau_j$ and $g = f^{\frac{1}{2}}$, where f is given in (2.1). Then the reference prior for (τ_1, τ_2) given ξ is

$$\begin{aligned} \pi(\tau_1, \tau_2|\xi) &= [\det F(\xi, \tau_1, \tau_2)]^{\frac{1}{2}} \\ &= \left(\frac{n_1 n_2}{4} \right)^{\frac{1}{2}} \left(\frac{d_1}{d_1+3} \frac{d_2}{d_2+3} \right)^{\frac{1}{2}} \tau_1^{-1} \tau_2^{-1}. \end{aligned} \quad (2.2)$$

The normalizing constant $K_l(\xi)$ of the reference prior $\pi(\tau_1, \tau_2|\xi)$ is given by

$$\begin{aligned} K_l(\xi)^{-1} &= \int_{a_2}^{b_2}\int_{a_1}^{b_1} [\det F(\xi,\tau_1,\tau_2)]^{\frac{1}{2}} d\tau_1 d\tau_2 \\ &= \int_{a_2}^{b_2}\int_{a_1}^{b_1} \left(\frac{n_1 n_2}{4}\right)^{\frac{1}{2}} \left(\frac{d_1}{d_1+3}\frac{d_2}{d_2+3}\right)^{\frac{1}{2}} \tau_1^{-1}\tau_2^{-1} d\tau_1 d\tau_2 \\ &= \left(\frac{n_1 n_2}{4}\right)^{\frac{1}{2}} \left(\frac{d_1}{d_1+3}\frac{d_2}{d_2+3}\right)^{\frac{1}{2}} [\log(b_1/a_1)\log(b_2/a_2)], \end{aligned} \tag{2.3}$$

and so we can obtain joint conditional probability density function of (τ_1, τ_2) given ξ over the compact set. This is given by

$$p_l(\tau_1,\tau_2|\xi) = K_l(\xi)\pi(\tau_1,\tau_2|\xi) = [\log(b_1/a_1)\log(b_2/a_2)]^{-1}\tau_1^{-1}\tau_2^{-1}. \tag{2.4}$$

Thus the marginal reference prior for ξ is given by

$$\pi_l(\xi) = \exp\left\{\int_{a_2}^{b_2}\int_{a_1}^{b_1} p_l(\tau_1,\tau_2|\xi)\log c(\xi,\tau_1,\tau_2) d\tau_1 d\tau_2\right\} = Q(a_1,b_1,a_2,b_2), \tag{2.5}$$

where

$$c(\xi,\tau_1,\tau_2) = E_{\xi,\tau_1,\tau_2}[\partial \log f/\partial\xi] = \frac{2n_1\Gamma(\frac{d_1+1}{2})}{\Gamma(\frac{d_1}{2})\sqrt{d_1\pi}}\tau_1^{-\frac{1}{2}} + \frac{2n_2\Gamma(\frac{d_2+1}{2})}{\Gamma(\frac{d_2}{2})\sqrt{d_2\pi}}\tau_2^{-\frac{1}{2}}.$$

Therefore the reference prior for (ξ, τ_1, τ_2), when ξ is parameter of interest, is given by

$$\begin{aligned} \pi_1(\xi,\tau_1,\tau_2) &= \lim_{l\to\infty}\left[\frac{K_l(\xi)\pi_l(\xi)}{K_l(\xi_0)\pi_l(\xi_0)}\right]\pi(\tau_1,\tau_2|\xi) \\ &\propto \tau_1^{-1}\tau_2^{-1}, \end{aligned} \tag{2.6}$$

where ξ_0 is a fixed point. Also when both ξ and (τ_1, τ_2) are parameters of interest, the reference prior for (ξ, τ_1, τ_2) is given by

$$\begin{aligned} \pi_2(\xi,\tau_1,\tau_2) &= c(\xi,\tau_1,\tau_2)[\det F(\xi,\tau_1,\tau_2)]^{\frac{1}{2}} \\ &\propto \tau_1^{-1}\tau_2^{-1}\left(\frac{n_1\Gamma(\frac{d_1+1}{2})}{\sqrt{d_1}\Gamma(\frac{d_1}{2})}\tau_1^{-\frac{1}{2}} + \frac{n_2\Gamma(\frac{d_2+1}{2})}{\sqrt{d_2}\Gamma(\frac{d_2}{2})}\tau_2^{-\frac{1}{2}}\right). \end{aligned} \tag{2.7}$$

When ξ is parameter of interest, the reference prior for (ξ, τ_1, τ_2) based on an appropriate penalty term of Ghosh and Mukerjee (1992) (Ghosal, 1997) is given by

$$\pi_3(\xi,\tau_1,\tau_2) = c(\xi,\tau_1,\tau_2) \propto \frac{n_1\Gamma(\frac{d_1+1}{2})}{\sqrt{d_1}\Gamma(\frac{d_1}{2})}\tau_1^{-\frac{1}{2}} + \frac{n_2\Gamma(\frac{d_2+1}{2})}{\sqrt{d_2}\Gamma(\frac{d_2}{2})}\tau_2^{-\frac{1}{2}}. \tag{2.8}$$

Note that the developed reference priors (2.6), (2.7) and (2.8) are different.

3. Implementation of the Bayesian procedure

The reference priors developed in the previous section are improper. So, we must check whether these priors give proper posterior distribution or not. We prove the propriety of posteriors for a general class of priors which include the reference priors (2.6), (2.7) and (2.8). We consider the class of priors

$$\pi_g(\xi,\tau_1,\tau_2) \propto \tau_1^{-a}\tau_2^{-b}\left(\frac{n_1\Gamma(\frac{d_1+1}{2})}{\sqrt{d_1}\Gamma(\frac{d_1}{2})}\tau_1^{-\frac{1}{2}}+\frac{n_2\Gamma(\frac{d_2+1}{2})}{\sqrt{d_2}\Gamma(\frac{d_2}{2})}\tau_2^{-\frac{1}{2}}\right)^c. \tag{3.1}$$

where $a \geq 0, b \geq 0$ and $c \geq 0$. The following theorem proves the propriety of posterior under the prior (3.1).

Theorem 3.1 The posterior distribution of (ξ,τ_1,τ_2) under the general prior (3.1) is proper if $n_1+2a+c-2>0$, $(n_1-1)d_1-2a-c+1>0$, $n_2+2b+c-2>0$, $d_2n_2-2b-c+2>0$ and $n_1+n_2+2a+2b+2c-5>0$, or $n_2+2b+c-2>0$, $(n_2-1)d_2-2b-c+1>0$, $n_1+2a+c-2>0$, $d_1n_1-2a-c+2>0$ and $n_1+n_2+2a+2b+2c-5>0$.

Proof: Under the general prior (3.1), the joint posterior for ξ,τ_1,τ_2 given $\mathbf{x}$ and $\mathbf{y}$ is

$$\pi(\xi,\tau_1,\tau_2|\mathbf{x},\mathbf{y}) \propto \tau_1^{-\frac{n_1}{2}-a}\tau_2^{-\frac{n_2}{2}-b}\left(\frac{n_1\Gamma(\frac{d_1+1}{2})}{\sqrt{d_1}\Gamma(\frac{d_1}{2})}\tau_1^{-\frac{1}{2}}+\frac{n_2\Gamma(\frac{d_2+1}{2})}{\sqrt{d_2}\Gamma(\frac{d_2}{2})}\tau_2^{-\frac{1}{2}}\right)^c$$
$$\times\prod_{i=1}^{n_1}\left[1+\frac{1}{d_1}\left(\frac{x_i-\xi}{\tau_1^{1/2}}\right)^2\right]^{-\frac{d_1+1}{2}}\prod_{j=1}^{n_2}\left[1+\frac{1}{d_2}\left(\frac{y_j-\xi}{\tau_2^{1/2}}\right)^2\right]^{-\frac{d_2+1}{2}}. \tag{3.2}$$

Now

$$\begin{aligned}\pi(\xi,\tau_1,\tau_2|\mathbf{x},\mathbf{y}) &\leq k_1\tau_1^{-\frac{n_1+c}{2}-a}\tau_2^{-\frac{n_2+c}{2}-b}\\ &\times\prod_{i=1}^{n_1}\left[1+\frac{1}{d_1}\left(\frac{x_i-\xi}{\tau_1^{1/2}}\right)^2\right]^{-\frac{d_1+1}{2}}\prod_{j=1}^{n_2}\left[1+\frac{1}{d_2}\left(\frac{y_j-\xi}{\tau_2^{1/2}}\right)^2\right]^{-\frac{d_2+1}{2}}\\ &= k_1\tau_1^{-\frac{n_1+c}{2}-a}\tau_2^{-\frac{n_2+c}{2}-b}\\ &\times\prod_{i=1}^{n_1}\left[1+\frac{1}{d_1}\frac{(w_i+z_M-\xi)^2}{\tau_1}\right]^{-\frac{d_1+1}{2}}\prod_{j=1}^{n_2}\left[1+\frac{1}{d_2}\frac{(v_j+z_M-\xi)^2}{\tau_2}\right]^{-\frac{d_2+1}{2}}\\ &\equiv \pi'(\xi,\tau_1,\tau_2|\mathbf{x},\mathbf{y})\end{aligned} \tag{3.3}$$

where $z_M=\min\{x_1,\cdots,x_{n_1},y_1,\cdots,y_{n_2}\}$, $w_i=x_i-z_M$, $v_j=y_j-z_M$, and k_1 is a constant.

We take $w_1 = 0$ without loss of generality. Then

$$\begin{aligned}
\pi'(\xi,\tau_1,\tau_2|\mathbf{x},\mathbf{y}) &\propto k_1 \tau_1^{-\frac{n_1+c}{2}-a}\tau_2^{-\frac{n_2+c}{2}-b}\left[1+\frac{1}{d_1}\frac{(z_M-\xi)^2}{\tau_1}\right]^{-\frac{d_1+1}{2}} \\
&\times \prod_{i=2}^{n_1}\left[1+\frac{1}{d_1}\frac{(w_i+z_M-\xi)^2}{\tau_1}\right]^{-\frac{d_1+1}{2}}\prod_{j=1}^{n_2}\left[1+\frac{1}{d_2}\frac{(v_j+z_M-\xi)^2}{\tau_2}\right]^{-\frac{d_2+1}{2}} \\
&\le k_1 \tau_1^{-\frac{n_1+c}{2}-a}\tau_2^{-\frac{n_2+c}{2}-b} \\
&\times \prod_{i=2}^{n_1}\left[1+\frac{1}{d_1}\frac{(w_i+z_M-\xi)^2}{\tau_1}\right]^{-\frac{d_1+1}{2}}\prod_{j=1}^{n_2}\left[1+\frac{1}{d_2}\frac{(v_j+z_M-\xi)^2}{\tau_2}\right]^{-\frac{d_2+1}{2}} \qquad (3.4)\\
&\le k_2 \tau_1^{-\frac{n_1+c}{2}-a}\tau_2^{-\frac{n_2+c}{2}-b}\left[1+\frac{\prod_{i=2}^{n_1}(w_i+z_M-\xi)^2}{d_1^{n_1-1}\tau_1^{n_1-1}}\right]^{-\frac{d_1+1}{2}} \\
&\times \left[1+\frac{\prod_{j=1}^{n_2}(v_j+z_M-\xi)^2}{d_2^{n_2}\tau_2^{n_2}}\right]^{-\frac{d_2+1}{2}},
\end{aligned}$$

where k_2 is a constant. Then integrating with respect to τ_1 and τ_2 in (3.4), we obtain

$$\pi''(\xi|\mathbf{x},\mathbf{y}) \propto k_3\left[\prod_{i=2}^{n_1}(w_i+z_M-\xi)^2\right]^{-\frac{n_1+2a+c-2}{2(n_1-1)}}\left[\prod_{j=1}^{n_2}(v_j+z_M-\xi)^2\right]^{-\frac{n_2+2b+c-2}{2n_2}}, \qquad (3.5)$$

where k_3 is a constant, if $n_1+2a+c-2>0$, $(n_1-1)d_1-2a-c+1>0$, $n_2+2b+c-2>0$ and $d_2n_2-2b-c+2>0$. Then

$$\begin{aligned}
\pi''(\xi|\mathbf{x},\mathbf{y}) &\propto k_4\left[\prod_{i=2}^{n_1}\left(1+\frac{z_M-\xi}{w_i}\right)^2\right]^{-\frac{n_1+2a+c-2}{2(n_1-1)}}\left[\prod_{j=1}^{n_2}\left(1+\frac{z_M-\xi}{v_j}\right)^2\right]^{-\frac{n_2+2b+c-2}{2n_2}} \\
&\le k_5\left(1+\frac{z_M-\xi}{w_M}\right)^{-(n_1+n_2+2a+2b+2c-4)}, \qquad (3.6)
\end{aligned}$$

where k_4 and k_5 are a constants and $w_M = \max\{w_2,\cdots,w_{n_1},v_1,\cdots,v_{n_2}\}$. Thus the function (3.6) is proper if $n_1+n_2+2a+2b+2c-5>0$. This completes the proof. □

Theorem 3.2 Under the reference prior π_1, the marginal posterior density of ξ is given by

$$\begin{aligned}
\pi(\xi|\mathbf{x},\mathbf{y}) &\propto \int_0^\infty\int_0^\infty \tau_1^{-\frac{n_1}{2}-1}\tau_2^{-\frac{n_2}{2}-1}\prod_{i=1}^{n_1}\left[1+\frac{1}{d_1}\left(\frac{x_i-\xi}{\tau_1^{1/2}}\right)^2\right]^{-\frac{d_1+1}{2}} \\
&\times \prod_{j=1}^{n_2}\left[1+\frac{1}{d_2}\left(\frac{y_j-\xi}{\tau_2^{1/2}}\right)^2\right]^{-\frac{d_2+1}{2}} d\tau_1 d\tau_2. \qquad (3.7)
\end{aligned}$$

Under the reference prior π_2, the marginal posterior density of ξ is given by

$$\pi(\xi|\mathbf{x},\mathbf{y}) \propto \int_0^\infty \int_0^\infty \tau_1^{-\frac{n_1}{2}-1} \tau_2^{-\frac{n_2}{2}-1} \left(\frac{n_1\Gamma(\frac{d_1+1}{2})}{\sqrt{d_1}\Gamma(\frac{d_1}{2})}\tau_1^{-\frac{1}{2}} + \frac{n_2\Gamma(\frac{d_2+1}{2})}{\sqrt{d_2}\Gamma(\frac{d_2}{2})}\tau_2^{-\frac{1}{2}} \right) \times \prod_{i=1}^{n_1} \left[1 + \frac{1}{d_1}\left(\frac{x_i-\xi}{\tau_1^{1/2}}\right)^2 \right]^{-\frac{d_1+1}{2}} \prod_{j=1}^{n_2} \left[1 + \frac{1}{d_2}\left(\frac{y_j-\xi}{\tau_2^{1/2}}\right)^2 \right]^{-\frac{d_2+1}{2}} d\tau_1 d\tau_2. \quad (3.8)$$

Under the reference prior π_3, the marginal posterior density of ξ is given by

$$\pi(\xi|\mathbf{x},\mathbf{y}) \propto \int_0^\infty \int_0^\infty \tau_1^{-\frac{n_1}{2}} \tau_2^{-\frac{n_2}{2}} \left(\frac{n_1\Gamma(\frac{d_1+1}{2})}{\sqrt{d_1}\Gamma(\frac{d_1}{2})}\tau_1^{-\frac{1}{2}} + \frac{n_2\Gamma(\frac{d_2+1}{2})}{\sqrt{d_2}\Gamma(\frac{d_2}{2})}\tau_2^{-\frac{1}{2}} \right) \times \prod_{i=1}^{n_1} \left[1 + \frac{1}{d_1}\left(\frac{x_i-\xi}{\tau_1^{1/2}}\right)^2 \right]^{-\frac{d_1+1}{2}} \prod_{j=1}^{n_2} \left[1 + \frac{1}{d_2}\left(\frac{y_j-\xi}{\tau_2^{1/2}}\right)^2 \right]^{-\frac{d_2+1}{2}} d\tau_1 d\tau_2. \quad (3.9)$$

Note that actually, normalizing constant for the marginal density of ξ requires two dimensional integration. Therefore we can have the marginal posterior density of ξ and so we compute the marginal moment of ξ. In Section 4, we investigate the frequentist coverage probabilities for the reference priors π_1, π_2 and π_3, respectively.

4. Numerical study

We compute the frequentist coverage probability by investigating the credible interval of the marginal posteriors density of ξ under the reference prior π given in Section 3 for several configurations (ξ, τ_1, τ_2) and (n_1, n_2). That is to say, the frequentist coverage of a posterior $(1-\alpha)$-quantile should be close to $1-\alpha$. This is done numerically. Table 4.1 gives numerical values of the frequentist coverage probabilities of 0.05 (0.95) posterior quantiles for the proposed priors. The computation of these numerical values is based on the following algorithm for any fixed true (ξ, τ_1, τ_2) and any prespecified value α. Here α is 0.05 (0.95). Let $\xi^\pi(\alpha|\mathbf{x},\mathbf{y})$ be the posterior α-quantile of ξ given $\mathbf{x}$ and $\mathbf{y}$. That is to say, $F(\xi^\pi(\alpha|\mathbf{x},\mathbf{y})|\mathbf{x},\mathbf{y}) = \alpha$, where $F(\cdot|\mathbf{x},\mathbf{y})$ is the marginal posterior distribution of ξ. Then the frequentist coverage probability of this one sided credible interval of ξ is

$$P_{(\xi,\tau_1,\tau_2)}(\alpha;\xi) = P_{(\xi,\tau_1,\tau_2)}(0 < \xi < \xi^\pi(\alpha|\mathbf{x},\mathbf{y})). \quad (4.1)$$

The estimated $P_{(\xi,\tau_1,\tau_2)}(\alpha;\xi)$ when $\alpha = 0.05(0.95)$ is shown in Table 4.1. In particular, for fixed (ξ, τ_1, τ_2), we take 10,000 independent random samples of $\mathbf{X} = (X_1, \cdots, X_{n_1})$ and $\mathbf{Y} = (Y_1, \cdots, Y_{n_2})$ from the model (2.1).

In Table 4.1, we see that the reference prior π_1 matches the target coverage probability much more accurately than the reference priors π_2 and π_3. The reference prior π_1 meets the target coverage probabilities well even in small samples. Note that the reference prior π_1 is the prior when ξ is parameter of interest, and the results of tables are not much sensitive to change of the values of (τ_1, τ_2). Thus we recommend to use the reference prior π_1 in the sense of asymptotic frequentist coverage property.

Table 4.1 Frequentist coverage probabilities of 0.05 (0.95) posterior quantiles for ξ

(d_1, d_2)	ξ	τ_1, τ_2	(n_1, n_2)	π_1	π_2	π_3
(3,3)	1.0	1,1	5,5	0.072 (0.957)	0.102 (0.963)	0.016 (0.931)
			5,10	0.063 (0.955)	0.081 (0.959)	0.022 (0.937)
			10,10	0.059 (0.950)	0.070 (0.953)	0.027 (0.941)
			10,15	0.056 (0.953)	0.064 (0.955)	0.034 (0.942)
		1,10	5,5	0.063 (0.954)	0.102 (0.961)	0.017 (0.931)
			5,10	0.063 (0.956)	0.094 (0.963)	0.018 (0.934)
			10,10	0.054 (0.953)	0.072 (0.957)	0.027 (0.941)
			10,15	0.056 (0.952)	0.072 (0.956)	0.032 (0.944)
		1,100	5,5	0.058 (0.953)	0.102 (0.963)	0.017 (0.927)
			5,10	0.060 (0.956)	0.096 (0.962)	0.018 (0.934)
			10,10	0.056 (0.951)	0.073 (0.955)	0.028 (0.940)
			10,15	0.055 (0.957)	0.073 (0.960)	0.029 (0.940)
	10.0	1,1	5,5	0.074 (0.959)	0.100 (0.965)	0.016 (0.935)
			5,10	0.063 (0.957)	0.081 (0.960)	0.026 (0.940)
			10,10	0.061 (0.958)	0.075 (0.960)	0.029 (0.942)
			10,15	0.059 (0.952)	0.069 (0.955)	0.034 (0.942)
		1,10	5,5	0.066 (0.954)	0.104 (0.962)	0.018 (0.930)
			5,10	0.058 (0.957)	0.089 (0.962)	0.016 (0.932)
			10,10	0.055 (0.954)	0.071 (0.958)	0.032 (0.945)
			10,15	0.058 (0.954)	0.072 (0.959)	0.034 (0.940)
		1,100	5,5	0.061 (0.953)	0.106 (0.961)	0.017 (0.925)
			5,10	0.057 (0.951)	0.094 (0.958)	0.016 (0.932)
			10,10	0.052 (0.955)	0.074 (0.960)	0.030 (0.941)
			10,15	0.050 (0.950)	0.066 (0.955)	0.033 (0.942)
	100.0	1,1	5,5	0.070 (0.956)	0.103 (0.962)	0.016 (0.927)
			5,10	0.069 (0.953)	0.085 (0.957)	0.025 (0.937)
			10,10	0.058 (0.954)	0.071 (0.956)	0.033 (0.944)
			10,15	0.057 (0.951)	0.067 (0.954)	0.032 (0.944)
		1,10	5,5	0.060 (0.954)	0.096 (0.961)	0.014 (0.931)
			5,10	0.065 (0.956)	0.094 (0.961)	0.018 (0.934)
			10,10	0.057 (0.952)	0.073 (0.956)	0.028 (0.941)
			10,15	0.053 (0.951)	0.070 (0.954)	0.029 (0.946)
		1,100	5,5	0.060 (0.954)	0.104 (0.961)	0.017 (0.931)
			5,10	0.057 (0.951)	0.094 (0.958)	0.018 (0.931)
			10,10	0.050 (0.952)	0.067 (0.957)	0.031 (0.943)
			10,15	0.054 (0.948)	0.072 (0.953)	0.028 (0.941)
(3,10)	1.0	1,1	5,5	0.070 (0.957)	0.097 (0.962)	0.016 (0.929)
			5,10	0.065 (0.956)	0.083 (0.959)	0.026 (0.936)
			10,10	0.059 (0.954)	0.071 (0.956)	0.031 (0.942)
			10,15	0.054 (0.952)	0.062 (0.955)	0.032 (0.944)
		1,10	5,5	0.059 (0.953)	0.098 (0.960)	0.013 (0.928)
			5,10	0.060 (0.954)	0.086 (0.961)	0.017 (0.930)
			10,10	0.058 (0.955)	0.075 (0.959)	0.029 (0.944)
			10,15	0.057 (0.952)	0.074 (0.955)	0.032 (0.941)
		1,100	5,5	0.055 (0.957)	0.098 (0.965)	0.018 (0.928)
			5,10	0.059 (0.953)	0.094 (0.962)	0.017 (0.935)
			10,10	0.057 (0.951)	0.076 (0.957)	0.032 (0.941)
			10,15	0.052 (0.949)	0.068 (0.953)	0.030 (0.942)
	10.0	1,1	5,5	0.071 (0.957)	0.097 (0.962)	0.016 (0.930)
			5,10	0.062 (0.951)	0.077 (0.956)	0.026 (0.940)
			10,10	0.059 (0.952)	0.069 (0.955)	0.028 (0.940)
			10,15	0.059 (0.953)	0.069 (0.955)	0.037 (0.942)
		1,10	5,5	0.063 (0.954)	0.102 (0.961)	0.015 (0.927)
			5,10	0.059 (0.952)	0.087 (0.959)	0.019 (0.933)
			10,10	0.056 (0.949)	0.072 (0.954)	0.033 (0.940)
			10,15	0.053 (0.951)	0.067 (0.955)	0.030 (0.941)
		1,100	5,5	0.058 (0.951)	0.100 (0.960)	0.015 (0.930)
			5,10	0.058 (0.956)	0.094 (0.963)	0.017 (0.927)
			10,10	0.053 (0.955)	0.069 (0.959)	0.030 (0.943)
			10,15	0.052 (0.951)	0.071 (0.955)	0.028 (0.936)
	100.0	1,1	5,5	0.071 (0.958)	0.099 (0.964)	0.017 (0.929)
			5,10	0.064 (0.955)	0.079 (0.958)	0.026 (0.931)
			10,10	0.059 (0.953)	0.071 (0.955)	0.030 (0.945)
			10,15	0.053 (0.954)	0.060 (0.956)	0.031 (0.944)
		1,10	5,5	0.058 (0.954)	0.098 (0.962)	0.013 (0.926)
			5,10	0.061 (0.955)	0.093 (0.961)	0.019 (0.936)
			10,10	0.056 (0.954)	0.072 (0.958)	0.033 (0.943)
			10,15	0.062 (0.955)	0.077 (0.958)	0.029 (0.943)
		1,100	5,5	0.057 (0.954)	0.098 (0.962)	0.015 (0.931)
			5,10	0.059 (0.952)	0.093 (0.960)	0.018 (0.929)
			10,10	0.054 (0.953)	0.072 (0.958)	0.030 (0.944)
			10,15	0.052 (0.948)	0.065 (0.953)	0.034 (0.939)

5. Concluding remarks

In two independent half-t distributions with common location parameter, we have found reference priors for the purpose of Byaesian inference. We derived the reference priors when ξ is parameter of interest, and both ξ and (τ_1, τ_2) are parameters of interest. We showed that the reference prior π_1 performs better than the reference priors π_2 and π_3 in matching the target coverage probabilities. Thus we recommend the use of the reference prior π_1 for the Bayesian inference in two independent half-t distributions with common location parameter.

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