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A continuous time asymmetric power GARCH process driven by a Lévy process[†]

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Abstract

A continuous time asymmetric power GARCH(1,1) model is suggested, based on a single background driving Lévy process. The stochastic differential equation for the given process is derived and the strict stationarity and kth order moment conditions are examined.

Keywords: COGARCH(1,1) process, continuous time asymmetric power GARCH(1,1) process, Lévy process, stationarity, stochastic differential equation.

1. Introduction

ARCH/GARCH models were introduced by Engle (1982) and Bollerslev (1986) and their extensions have become the most popular tools in financial economics. Recently, financial data are treated mostly in continuous time. Continuous time processes are particularly appropriate for modelling irregularly-spaced and high frequency data. Also, continuous time processes are useful in financial applications such as option pricing. Various attempts have been made to model specific features that characterize time series of returns on financial assets in a continuous time model.

One of the main approaches for continuous time volatility model is an extension to the discrete time GARCH model by diffusion approximation studied by Nelson (1990), where two independent Brownian motions are used. In original GARCH model, however, jumps appear and a single source of randomness is sufficient. The continuous time stochastic volatility model of Barndorff-Nielsen and Shephard (2001) specifies the volatility as an Ornstein-Uhlenbeck type process driven by Lévy process and its price process is determined by using an independent Brownian motion as deriving noise, so it can exhibit jumps in the volatility but is still determined by two independent random processes. Klüppelberg *et al.* (2004) proposed a new approach to construct a continuous time model. This so-called COGARCH(1,1) model is based on a single background deriving Lévy process. As a noise process, increments of any Lévy processes replace the innovations in the discrete time GARCH model. It generalize the essential features of the discrete time GARCH(1,1) process in a direct way and

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many of stylized facts are captured by a single randomness like in the discrete time GARCH model. It is shown that the COGARCH(1,1) process can indeed be obtained as limit in law of a sequence of GARCH(1,1) models and thus the GARCH(1,1) can be embedded into a COGARCH(1,1) process (Kallsen and Vesenmayer, 2009; Maller *et al.*, 2008).

Although GARCH models are capable of capturing some important stylized features such as jumps, heavy-tailedness, volatility clustering and dependence without correlation, empirical observations show that the long range dependence and asymmetric phenomena, which cannot be modelled by a classical GARCH model are also important features to be considered. To represent the asymmetric phenomena, the discrete time exponential GARCH model is suggested by Nelson (1991) and its continuous time version the exponential GARCH (ECOGARCH) model is proposed in Haug and Czado (2007), where the stationarity, mixing and moment properties of the process are investigated. The ECOGARCH(p, q) process is defined by specifying the log-volatility process as a continuous time ARMA$(q, p-1)$ process, which is continuous time analogue of an ARMA$(q, p-1)$ process. CARMA process has been studied in the literature, e.g. Brockwell (2001).

As one other extension of the GARCH model, Ding *et al.* (1993) developed the augmented asymmetric power GARCH (APARCH) to represent the long memory dependence as well as the asymmetric property of the volatility processes. The APARCH(1,1) model is defined as follows:

$$Y_n = \varepsilon_n \sigma_n, \quad \sigma_n^\delta = \omega + \alpha(|Y_{n-1}| - \gamma Y_{n-1})^\delta + \beta \sigma_{n-1}^\delta,$$

where $\omega > 0, \alpha \geq 0, \beta \geq 0, |\gamma| < 1, \delta > 0$ and $(\varepsilon_n)_{n\in N}$ is a sequence of independent and identically distributed random variables with mean zero and $E|\varepsilon_n|^\delta < \infty$. The APARCH model nests the most symmetric and asymmetric GARCH models such as threshold GARCH($\delta =$ 1), GJR GARCH($\delta = 2$), nonlinear GARCH($\gamma = 0$) and classical GARCH ($\delta = 2$, $\gamma = 0$) (Hentschel, 1995; Lee and Kim, 2006; Park and Lee, 2007).

In this paper, we suggest the continuous time APARCH(1,1) model following the method adopted by Klüppelberg *et al.* (2004) and examine that how this process corresponds to the discrete time volatility equation and give its probabilistic properties. A model is suggested by iterating the volatility equation and replacing the noise variables ϵ_i by the jumps $\Delta L_t = L_t - L_{t-}$ of a Lévy process $L = (L_t)_{t\geq 0}$. We investigate that a given process is generalization of the discrete time APARCH(1,1) process and find their stochastic differential equation. Stationarity and moment conditions can be derived by a modification of the COGARCH(1,1) model.

2. Discrete time APGARCH(1,1) process

Consider the discrete time APARCH(1,1) process given by

$$Y_n = \varepsilon_n \sigma_n, \quad \sigma_n^\delta = \omega + \alpha(|Y_{n-1}| - \gamma Y_{n-1})^\delta + \beta \sigma_{n-1}^\delta, \tag{2.1}$$

where $\omega > 0, \alpha > 0, 0 < \beta < 1, |\gamma| < 1, \delta > 0$ and $(\varepsilon_n)_{n\in N}$ is a sequence of independent and identically distributed random variables with mean zero and $E|\varepsilon_n|^\delta < \infty$.

Theorem 2.1 Suppose that

$$E|\log(\beta + \alpha h(\varepsilon_1))| < \infty, \quad E\log(\beta + \alpha h(\varepsilon_1)) < 0, \tag{2.2}$$

where $h(x) = (|x| - \gamma x)^\delta$. Then $Y_n \xrightarrow{D} Y$ and $\sigma_n \xrightarrow{D} \sigma$ as $n \to \infty$ for finite random variables Y and σ.

Proof. The volatility process σ_n^δ in the equation (2.1) can be rewritten as

$$\sigma_n^\delta = \omega + (\alpha h(\varepsilon_{n-1}) + \beta)\sigma_{n-1}^\delta. \tag{2.3}$$

Note that σ_n and ε_n are independent and $h(x) \geq 0$ for any x. Applying Corollary 4.1 in Goldie and Maller (2000) yields that the assumptions in (2.2) guarantee the existence of the finite random variables σ and Y such that $\sigma_n \xrightarrow{D} \sigma$ and $Y_n \xrightarrow{D} Y = \sigma\varepsilon$ with σ and ε independent. □

Iterate (2.3) to get

$$\sigma_n^\delta = \omega \sum_{i=0}^{n-1} \Pi_{j=i+1}^{n-1}(\beta + \alpha h(\varepsilon_j)) + \sigma_0^\delta \Pi_{j=0}^{n-1}(\beta + \alpha h(\varepsilon_j)). \tag{2.4}$$

In Section 3, we define a continuous time APARCH(1,1) model by modifying the equation (2.4).

3. Continuous time APGARCH(1,1) process

Let $(L_t)_{t\geq 0}$ be a càdlàg Lévy process with jumps $\Delta L_t = L_t - L_{t^-}$, $t \geq 0$ defined on a probability space $(\Omega, \mathcal{F}, P)$. For each $t > 0$, the characteristic function of L_t can be written in the form

$$E(e^{i\theta L_t}) = exp\left(t(i\gamma_L\theta - \tau_L^2\frac{\theta^2}{2} + \int_{\Re}(e^{i\theta x} - 1 - i\theta x I_{|x|\leq 1})\Pi_L(dx))\right), \quad \theta \in R,$$

where the constants $\gamma_L \in R$ and $\tau_L^2 \geq 0$ and the measure Π_L on R form the characteristic triple of L. As usual, the Lévy measure Π_L is required to satisfy $\int_R \min(1, x^2)\Pi_L(dx) < \infty$. If $\int_R \min(1, |x|)\Pi_L(dx) < \infty$, then $\gamma_{L,0} = \gamma_L - \int_{[-1,1]} x\Pi_L(dx)$ is called a drift of L. We will only be interested in the situation where Π_L is nonzero.

Following Klüppelberg *et al.* (2004), we first define a càdlàg process $(X_t)_{t\geq 0}$ by

$$X_t = -t\log\beta - \sum_{0<s\leq t}\log(1 + \frac{\alpha}{\beta}h(\Delta L_s)), \quad t \geq 0 \tag{3.1}$$

where positive constants α, β and the function $h(\cdot)$ are given in section 2.

Suppose that $E|L_1|^\delta < \infty$. Then the Lévy measure Π_X of $(X_t)_{t\geq 0}$ is given by

$$\Pi_X(0,\infty) = 0, \qquad \Pi_X\{(-\infty, -x]\} = \Pi_L\{y : h(y) \geq \frac{\beta}{\alpha}(e^x - 1)\}.$$

Note that

$$\int_{[-1,1]} |x|\Pi_X(dx) = \int_{\{y:h(y)\leq\frac{\beta}{\alpha}(e-1)\}} \log(1 + \frac{\alpha}{\beta}h(y))\Pi_L(dy) \qquad < \infty,$$

whenever $\int_R h(y)\Pi_L(dy) < \infty$, which is equivalent to $E|L_1|^\delta < \infty$. Therefore, $(X_t)_{t\geq 0}$ in (3.1) is a spectrally negative Lévy process of bounded variation with drift $\gamma_{X,0} = -\log\beta$

and Gaussian component $\tau_X^2 = 0$ with Lévy measure Π_X given by $\Pi_X = \Pi_L \circ T^{-1}$ where $T(x) = -\log(1 + \frac{\alpha}{\beta}h(x)), \forall x \in R$.

Define a right continuous version of volatility process $(\sigma_t^\delta)_{t\geq 0}$analogously to (2.4) by

$$\sigma_t^\delta = \omega \int_0^t e^{-(X_t - X_{s-})} ds + \sigma_0^\delta e^{-X_t}, \quad t \geq 0 \tag{3.2}$$

where σ_0^δ is a positive finite random variable, independent of $(L_t)_{t\geq 0}$, and define the integrated continuous time APARCH(1,1) process $(G_t)_{t\geq 0}$ as a càdlàg process satisfying

$$G_t = \int_0^t \sigma_{s-} dL_s, \quad t \geq 0. \tag{3.3}$$

Theorem 3.1 $(\sigma_t^\delta)_{t\geq 0}$ satisfies the following stochastic differential equation:

$$d\sigma_t^\delta = \omega dt + \sigma_{t-}^\delta e^{X_{t-}} d(e^{-X_t}), \quad t \geq 0 \tag{3.4}$$

and we have that

$$\sigma_t^\delta = \sigma_0^\delta + \omega t + \log\beta \int_0^t \sigma_{s-}^\delta ds + \frac{\alpha}{\beta} \sum_{0<s\leq t} \sigma_{s-}^\delta h(\Delta L_s), \quad t \geq 0. \tag{3.5}$$

Proof. Set $K_t = t\log\beta, S_t = \Pi_{0<s\leq t}(1 + \frac{\alpha}{\beta}h(\Delta L_s))$ and $f(k, s) = e^k s$. Apply Itô's formula (Protter, 2004) to get that

$$\begin{aligned} e^{-X_t} &= 1 + \int_{0^+}^t e^{-X_{s-}} d(K_s) + \int_{0^+}^t e^{K_{s-}} d(S_s) \\ &\quad + \sum_{0<s\leq t} \{e^{-X_s} - e^{-X_{s-}} - e^{-X_{s-}}\Delta K_s - e^{K_s}\Delta S_s\} \\ &= 1 + \log\beta \int_{0^+}^t e^{-X_{s-}} ds + \frac{\alpha}{\beta} \sum_{0<s\leq t} e^{-X_{s-}} h(\Delta L_s). \end{aligned} \tag{3.6}$$

Adopt the integration by parts to yield

$$\begin{aligned} &e^{-X_t} \cdot \int_0^t e^{X_{s-}} ds \\ &= \int_{0^+}^t e^{-X_{s-}} d(\int_0^s e^{X_{y-}} dy) + \int_{0^+}^t (\int_0^s e^{X_{y-}} dy) d(e^{-X_s}) + [e^{-X_\cdot}, \int_0^\cdot e^{X_{s-}} ds]_t, \end{aligned}$$

and hence we have that

$$d(e^{-X_t} \cdot \int_0^t e^{X_{s-}} ds) = dt + (\int_0^t e^{X_{s-}} ds) d(e^{-X_t}). \tag{3.7}$$

Combining (3.2) and (3.7), we have the following stochastic differential equation (3.4)

$$\begin{aligned} d\sigma_t^\delta &= \omega d(e^{-X_t} \cdot \int_0^t e^{X_{s-}} ds) + \sigma_0^\delta d(e^{-X_t}) \\ &= \omega dt + (\omega \int_0^t e^{X_{s-}} ds + \sigma_0^\delta) d(e^{-X_t}) \\ &= \omega dt + \sigma_{t-}^\delta e^{X_{t-}} d(e^{-X_t}). \end{aligned} \tag{3.8}$$

Now use (3.6) and (3.8) to obtain that

$$\begin{aligned}\sigma_t^\delta &= \sigma_0^\delta + \omega t + \int_{0+}^t \sigma_{s-}^\delta e^{X_{s-}} d(e^{-X_s})\\ &= \sigma_0^\delta + \omega t + \int_{0+}^t \sigma_{s-}^\delta e^{X_{s-}} d\left(\log\beta \int_{0+}^s e^{-X_{u-}} du\right)\\ &\quad + \int_{0+}^t \sigma_{s-}^\delta e^{X_{s-}} d\left(\frac{\alpha}{\beta} \sum_{0<u\le s} e^{-X_{u-}} h(\Delta L_u)\right)\\ &= \sigma_0^\delta + \omega t + \log\beta \int_0^t \sigma_{s-}^\delta ds + \frac{\alpha}{\beta} \sum_{0<s\le t} \sigma_{s-}^\delta h(\Delta L_s),\end{aligned}$$

which completes the proof. □

Remark 1 From (2.3) we have that

$$\sigma_n^\delta = \sigma_0^\delta + n\omega + (\beta - 1)\sum_{i=1}^{n-1} \sigma_i^\delta + \alpha \sum_{i=1}^{n} h(\epsilon_i)\sigma_i^\delta. \tag{3.9}$$

Notice that the equation (3.9) is analogous to the equation (3.5)

Now recall that if the Laplace transform $E(e^{-cX_1}) < \infty$ for some $c > 0$, we define $\Psi(c) = \log E(e^{-cX_1})$ and we have that $E(e^{-cX_t}) = e^{t\Psi(c)}$ with

$$\Psi(c) = c\log\beta + \int_R ((1 + \frac{\alpha}{\beta}h(y))^c - 1)\Pi_L(dy), \quad c \ge 0. \tag{3.10}$$

In the next theorem, a sufficient condition for the strict stationarity of the suggested continuous time APARCH(1.1) given in (3.2) is obtained. The existence of the kth moments of the stationary distribution is also considered.

Theorem 3.1 Consider the σ_t^δ and G_t in (3.2) and (3.3).

(1) $(\sigma_t^\delta)_{t\ge0}$ and $(\sigma_t^\delta, G_t)_{t\ge0}$ are time homogeneous Markovian processes.

(2) $E(e^{-cX_t}) < \infty, \forall t > 0$ if and only if $E(|L_1|^{c\delta}) < \infty$.

(3) If $E|L_1|^\delta < \infty$ and $\Psi(1) < 0$, then

$$\int_R \log(1 + \frac{\alpha}{\beta}h(y))\Pi_L(dy) < -\log\beta, \tag{3.11}$$

and $\sigma_t^\delta \xrightarrow{D} \sigma_\infty^\delta$ as $t \to \infty$ for a finite random variable σ_∞^δ satisfying $\sigma_\infty^\delta \overset{\mathrm{D}}{=} \omega \int_0^\infty e^{-X_t} dt$. If $\sigma_0^\delta \overset{\mathrm{D}}{=} \sigma_\infty^\delta$ and σ_0^δ is independent of L_t, then $(\sigma_t^\delta)_{t\ge0}$ is strictly stationary and $(G_t)_{t\ge0}$ is a process with stationary increments.

(4) σ_∞^δ exists and has finite kth moments, $k \in N$ if and only if

$$\frac{1}{k}\int_R ((1 + \frac{\alpha}{\beta}h(y))^k - 1)\Pi_L(dy) < -\log\beta. \tag{3.12}$$

Proof. Since the proofs of the Theorem 3.2 can be obtained from those of Lemma 4.1, Theorem 3.2 and Theorem 4.1 in Klüppelberg *et al.* (2004) with simple modification, we only give sketches of the proofs of Theorem 3.2 (2)-(4).

(2) $E(e^{-cX_t}) < \infty$ for all $t \geq 0$ if and only if $\int_{|x|\geq 1} e^{-cx}\ \Pi_X(dx)$, which is equivalent to the finiteness of $\int_{\{y:h(y)\leq\frac{\beta}{\alpha}(e-1)\}} \log(1+\frac{\alpha}{\beta}h(y))\Pi_L(dy)$. Hence the conclusion follows.

(3) From (3.10), $\Psi(1) < 0$ implies the inequality (3.11), which is a sufficient condition for the existence of the finite limit of σ_t^δ as $t \to \infty$ (see Klüppelberg *et al.* (2004), Theorem 3.1).

(4) If $E|L_1|^{k\delta} < \infty$ and $\Psi(k) < 0$, then $E|L_1|^\delta < \infty$ and $\Psi(1) < 0$ and hence σ_∞^δ exists and $E(\sigma_\infty^{k\delta}) < \infty$. Note that (3.12) is equivalent to $E|L_1|^{k\delta} < \infty$ and $\Psi(k) < 0$. □

Remark 2 Compare the condition (3.11) for the stationarity of $(\sigma_t^\delta)_{t\geq 0}$ in (3.2) with the condition (2.2) for the stationarity of the corresponding discrete time APARCH(1.1) process $(\sigma_n)_{n\in N}$ in (2.1).

For more analogous results on moments and covariances of σ_t^δ and G_t to those of COGA-RCH(1,1), we refer to Klüppelberg *et al.* (2004, 2006) and Haug *et al.* (2007).

4. Conclusion

The discrete time asymmetric power GARCH (1,1) is a popular model for practitioners to capture the asymmetry and long range dependence of financial volatility in addition to many other stylized facts. Recently, however, financial data are treated mostly in continuous time. Continuous time processes are particularly appropriate for modelling irregularly-spaced and high frequency data. Also, continuous time processes are useful in financial applications such as option pricing. In this paper, a continuous time asymmetric power GARCH(1,1) model is suggested, based on a single background driving Lévy process. The model is defined following the approach in Klüppelberg *et al.* (2004). As mentioned in Remark 1 and Remark 2, the given model can be considered as an extension of the discrete time APARCH(1.1) to continuous time processes. The stochastic differential equation is derived and the strict stationarity and kth order moment conditions are also examined.

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