Journal of the Korean Data & Information Science Society
2010, **21**(6), 1305–1310

한국데이터정보과학회지

The CUSUM test for stochastic volatility models[†]

Moosup Kim[1]· Sangyeol Lee [2]

[12]Department of Statistics, Seoul National University

Received 17 August 2010, revised 13 October 2010, accepted 20 October 2010

Abstract

In this paper, we consider a change point test for stochastic volatility models. By considering the relation between moments of the logarithms of squared returns and the parameters, we construct the cusum test to detect changes of the parameters. We also carry out a simulation study and verify that the proposed test is more powerful than the cusum test proposed by Kokoszka and Leipus (2000).

Keywords: Change point test, cusum test, stochastic volatility model.

1. Introduction

The stochastic volatility (SV) model is one of the most interesting and prominent conditional volatility models in financial time series together with GARCH models proposed by Bollerslev (1986). The change point test has received much attention since financial time series frequently experience changes in underlying models. In this study, we focus on the change point detection problem for SV models.

For decades, several authors have investigated change point tests for ARCH-type models, which are also applicable to the SV models. For instance, see Inclan and Tiao (1994), Kokoszka and Leipus (2000), and Andreou and Ghysels (2002). Their study focused on the cusum test of squared returns. However, such cusum tests are not sufficient to reflect the change of parameters in SV models, since the second moment of returns does not totally correspond to the parameters in the model. Thus, by noting that the parameters correspond to the vector consisting of the mean, variance, and first-order autocovariance of the logarithm of squared returns, we construct a cusum test which examines the change of the corresponding sample moments. It turns out that the cusum test statistic is easy to calculate and its asymptotic null distribution does not depend upon the error distribution, while most

† This research was supported by Mid-career Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education, Science and Technology (grant number 2010-0000374).

1 Corresponding author: Department of Statistics, Seoul National University, Seoul 151-747, Korea. E-mail: mooseob@hanmail.net

2 Department of Statistics, Seoul National University, Seoul 151-747, Korea.

estimation procedures require intensive calculations and depend on the error distribution. For general viewpoint of the cusum test, we refer to Im and Cho (2009), Lee *et al.* (2010), Na *et al.* (2010), and Park and Lee (2007).

The organization of this paper is as follows. In Section 2, we introduce the cusum test and the change point estimator based on it. In Section 3, we demonstrate the performance of our test through a simulation study. In Section 4, we address the asymptotic null distribution of the cusum test and provide the proof of the result.

2. Main result

We consider an SV model:

$$\begin{cases} r_t &= \xi_t e^{h_t/2} \\ h_t &= \alpha + \beta h_{t-1} + \sigma \epsilon_t \end{cases} \quad \text{for each } t \in \mathbb{Z}, \tag{2.1}$$

where $\alpha \in \mathbb{R}$, $|\beta| < 1$, $0 < \sigma < \infty$, and $\{\epsilon_t\}$ and $\{\xi_t\}$ are i.i.d. sequences of random variables, defined on a probability space $(\Omega, \mathcal{F}, P)$, with zero mean and unit variance. Moreover, we assume that $\{\xi_t\}$ and $\{\epsilon_t\}$ are independent. Here, $\{r_t\}$ and $\{e^{h_t}\}$ represent an observed strictly stationary solution of (2.1) and the unobserved conditional variances of $\{r_t\}$, respectively. Let $\boldsymbol{\theta} = (\alpha, \beta, \sigma)$.

Let $r_0, r_1, \cdots, r_n$ be observed returns. Suppose that one wishes to test

$$\mathcal{H}_0 : \boldsymbol{\theta} \text{ is constant in } \{r_0, r_1, \cdots, r_n\} \quad \text{vs.} \quad \mathcal{H}_1 : \text{not } \mathcal{H}_0.$$

Assume that $\mathcal{H}_0$ holds and $\xi_t \neq 0$ almost surely. Let $y_t := \log r_t^2 = h_t + \eta_t$, where $\eta_t := \log \xi_t^2$. We have from easy calculation that

$$\begin{aligned} \beta &= \frac{\mathrm{Cov}(h_t, h_{t-1})}{\mathrm{Var}(h_t)} = \frac{\mathrm{Cov}(y_t, y_{t-1})}{\mathrm{Var}(y_t) - \mathrm{Var}(\eta_t)}, \\ \alpha &= \mathrm{E}(h_t)(1-\beta) = \{\mathrm{E}(y_t) - \mathrm{E}(\eta_t)\}(1-\beta), \\ \sigma^2 &= \mathrm{Var}(h_t)(1-\beta^2) = \{\mathrm{Var}(y_t) - \mathrm{Var}(\eta_t)\}(1-\beta^2), \end{aligned}$$

i.e., all parameters are functionals of $\mathrm{E}(y_t)$, $\mathrm{Var}(y_t)$, $\mathrm{Cov}(y_t, y_{t-1})$, and vice verse. In other words, a one-to-one relation holds between the parameters and moments. From this viewpoint, it is reasonable to examine the moment change to test the hypotheses. Thus, we consider the cusum statistic $T_n := n^{-1} \max_{1 \le k \le n} \tilde{D}_k' \hat{\Sigma}_n^{-1} \tilde{D}_k$, where

$$\tilde{D}_k = \sum_{t=1}^{k} \tilde{W}_t - \frac{k}{n} \sum_{t=1}^{n} \tilde{W}_t, \quad \tilde{W}_t = [y_t - \bar{y},\, (y_t - \bar{y})^2,\, (y_t - \bar{y})(y_{t-1} - \bar{y})]', \quad \bar{y} = \frac{1}{n} \sum_{t=1}^{n} y_t$$

and $\hat{\Sigma}_n$ is a consistent and nonsingular estimate of

$$\Sigma := \lim_{n \to \infty} \frac{1}{n} \mathrm{Var}\left(\sum_{t=1}^{n} W_t \right),$$

where $W_t = [y_t - \mu,\, (y_t - \mu)^2,\, (y_t - \mu)(y_{t-1} - \mu)]'$ and $\mu := \mathrm{E}(y_1)$. Later, a condition of the existence of the limit will be presented. Here, we employ as the estimator of Σ

$$\hat{\Sigma}_n = \frac{1}{n}\sum_{t=1}^{n}(\tilde{W}_t - \bar{W})(\tilde{W}_t - \bar{W})' + \sum_{h=1}^{l_n}\frac{1}{n-h}\sum_{t=1}^{n-h}\{(\tilde{W}_t - \bar{W})(\tilde{W}_{t+h} - \bar{W})' + (\tilde{W}_{t+h} - \bar{W})(\tilde{W}_t - \bar{W})'\},$$

where $\{l_n\}$ is an increasing sequence of positive integers with $l_n = o(n)$ and $\bar{W} = \frac{1}{n}\sum_{t=1}^{n}\tilde{W}_t$. Actually, $\hat{\Sigma}_n$ can be seen as the sample moment with respect to Σ, where l_n indicates the order of truncation in finite approximation (4.1).

It can be proven that under mild conditions, T_n has a standard asymptotic null distribution (Theorem 4.1) of which critical values are available in Lee *et al.* (2003). Its proof is provided in Section 4.

When a single change point exists, we estimate it by

$$\hat{\tau}_n = n^{-1}\operatorname*{arg\,max}_{1\le k\le n}\tilde{D}_k'\,\hat{\Sigma}_n^{-1}\tilde{D}_k. \tag{2.2}$$

In Section 3, we will examine its performance in finite samples.

3. Simulation study

In this section, we evaluate the performance of T_n by a simulation study. Furthermore, we compare it with the cusum test of squared returns. First, we evaluate the sizes of the tests at nominal level 0.05. We consider several settings for the parameter $\boldsymbol{\theta}$ and the distribution of ξ_t, while the distribution of ϵ_t is fixed to be a standard normal distribution. We use the sample size $n = 1{,}000$ and $l_n = 10$ in the estimation of Σ. We reject $\mathcal{H}_0$ at nominal level 0.05 when $T_n > 3.004$. Table **3.1** presents their sizes obtained from 1,000 repetitions. Most of them are acceptable, although the cusum of squared returns reveals a slight size distortion in some cases such as $\boldsymbol{\theta} = (-0.706, 0.9, 0.135)$.

Table 3.1 The sizes of the tests at nominal level 0.05

	Test	T_n			CUSUM of squared returns		
	distribution of ξ_t	Normal	$t(10)$	$t(3)$	Normal	$t(10)$	$t(3)$
	$(-0.821, 0.9, 0.675)$	0.047	0.055	0.042	0.050	0.078	0.047
$\boldsymbol{\theta}$	$(-0.736, 0.9, 0.363)$	0.063	0.052	0.047	0.084	0.076	0.042
	$(-0.706, 0.9, 0.135)$	0.037	0.058	0.041	0.023	0.021	0.018

Next, we examine the power of the tests and calculate the MSE of the change point estimators. The change point estimator based on the cusum test of squared returns is defined in the same pattern as (2.2). Specifically, let $\{\epsilon_t^{(1)}\}$ and $\{\epsilon_t^{(2)}\}$ be mutually independent i.i.d. sequences. Assume that $\{h_t^{(1)}\}$ and $\{h_t^{(2)}\}$ are strictly stationary sequences satisfying

$$h_t^{(1)} = \alpha_1 + \beta_1 h_{t-1}^{(1)} + \sigma_1\epsilon_t^{(1)},$$
$$h_t^{(2)} = \alpha_2 + \beta_2 h_{t-1}^{(2)} + \sigma_2\epsilon_t^{(2)},$$

where $\alpha_1, \alpha_2 \in \mathbb{R}$, $|\beta_1| \vee |\beta_2| < 1$, $\sigma_1 \wedge \sigma_2 > 0$, $(\alpha_1, \beta_1, \sigma_1) \neq (\alpha_2, \beta_2, \sigma_2)$. Finally, we define an SV process with a change point:

$$r_t = \begin{cases} \xi_t e^{h_t^{(1)}/2} & t \leq [n\tau], \\ \xi_t e^{h_t^{(2)}/2} & t > [n\tau], \end{cases}$$

where $[n\tau]$ $(0 < \tau < 1)$ indicates the change point. A simulation study is carried out in diverse settings regarding the τ, the parameters, and the distribution of ξ_t. We consider 3 cases of parameter changes:

- Case 1 : $\boldsymbol{\theta}$ changes from $(-0.821, 0.9, 0.675)$ to $(-0.706, 0.9, 0.135)$ at $[n\tau]$,
- Case 2 : $\boldsymbol{\theta}$ changes from $(-0.821, 0.9, 0.675)$ to $(-0.736, 0.9, 0.363)$ at $[n\tau]$,
- Case 3 : $\boldsymbol{\theta}$ changes from $(-0.736, 0.9, 0.363)$ to $(-0.706, 0.9, 0.135)$ at $[n\tau]$.

Table **3.2** presents the powers of T_n and the cusum test of squared returns. As seen in the table, the former has good powers except the case of $\tau = 0.75$, while the latter produces very low powers. Table **3.3** presents the root MSE of the change point estimators. In general, the performance of a change point estimator depends on that of the relevant test statistic. From the result, we can see that $\hat{\tau}_n$ is more accurate than the other as T_n is more powerful than the cusum test of squared returns. All these results confirm that T_n is more suitable than the cusum test of squared returns.

Table 3.2 The powers of the tests at nominal level 0.05

Test		T_n			CUSUM of squared returns		
distribution of ξ_t		Normal	$t(10)$	$t(3)$	Normal	$t(10)$	$t(3)$
$\tau = 0.25$	Case 1	0.841	0.850	0.819	0.251	0.224	0.084
	Case 2	0.436	0.462	0.434	0.108	0.105	0.038
	Case 3	0.177	0.162	0.136	0.073	0.077	0.025
$\tau = 0.5$	Case 1	0.938	0.931	0.909	0.222	0.220	0.112
	Case 2	0.640	0.635	0.595	0.123	0.113	0.082
	Case 3	0.234	0.242	0.216	0.099	0.095	0.082
$\tau = 0.75$	Case 1	0.376	0.392	0.380	0.106	0.102	0.080
	Case 2	0.200	0.193	0.197	0.090	0.102	0.056
	Case 3	0.085	0.103	0.093	0.077	0.048	0.031

Table 3.3 The root MSE of the change point estimators

Estimator		$\hat{\tau}_n$			CUSUM of squared returns		
distribution of ξ_t		Normal	$t(10)$	$t(3)$	Normal	$t(10)$	$t(3)$
$\tau = 0.25$	Case 1	0.048	0.053	0.061	0.104	0.130	0.245
	Case 2	0.100	0.097	0.114	0.284	0.278	0.297
	Case 3	0.145	0.145	0.161	0.167	0.165	0.259
$\tau = 0.5$	Case 1	0.064	0.067	0.067	0.141	0.123	0.140
	Case 2	0.083	0.085	0.087	0.154	0.162	0.151
	Case 3	0.104	0.105	0.106	0.152	0.156	0.160
$\tau = 0.75$	Case 1	0.209	0.208	0.205	0.256	0.286	0.277
	Case 2	0.247	0.238	0.228	0.264	0.283	0.270
	Case 3	0.281	0.285	0.296	0.338	0.367	0.313

4. Asymptotic result and its proof

We assume that the following conditions hold:

A1 $\{h_t\}$ is strictly stationary and absolutely regular with exponential decay.

A2 Both ϵ_0 and η_0 have the finite moments of all orders.

A3 Σ is nonsingular.

Remark. **A1** has been vastly investigated (cf. Carrosco *et al.*, 2002). For the definition and the details of absolute regularity, we refer to Davydov (1973). **A2** is not only stringent but also includes the case that ξ_0 has a heavy tail such as the regularly varying tail. **A1-A2** imply the existence of Σ and

$$\Sigma = \mathrm{Var}(W_1) + \sum_{k=1}^{\infty} \{\mathrm{Cov}(W_1, W_{1+k}) + \mathrm{Cov}(W_{1+k}, W_1)\} \tag{4.1}$$

(cf. Billingsley, 1995, Theorem 27.4), since $\{y_t - \mu\}$ is strong mixing with exponential decaying rate and has finite moments of all orders. The nonsingularity of Σ depends on the parameters. If the distribution of (ξ_0, ϵ_0) is known, Σ is obtainable but its calculation is somewhat tedious.

In what follows, $B(t) = (B_1(t), B_2(t), B_3(t))'$ stands for a standard 3-dimensional Brownian motion and $B^\circ(t) = (B_1^\circ(t), B_2^\circ(t), B_3^\circ(t))' := B(t) - tB(1)$. Also, $D[0,1]$ represents the complete and separable metric space of the real-valued càdlàg functions defined on the interval $[0,1]$ endowed with the Skorohod metric (cf. Billingsley, 1999).

Theorem 4.1 Assume that $\mathcal{H}_0$ holds. Under **A1-A3**,

$$n^{-1} \max_{1 \le k \le n} \tilde{D}_k' \hat{\Sigma}_n^{-1} \tilde{D}_k \ \Rightarrow \sup_{0 \le s \le 1} \sum_{j=1}^{3} \left\{B_j^\circ(s)\right\}^2 .$$

Proof. Due to **A1-A2**, $\{y_t - \mu\}$ is strictly stationary and absolutely regular with exponential decaying rate (cf. Carrasco *et al.*, 2002) and has finite moments of all orders. Let λ be a column vector in $\mathbb{R}^3 \backslash \{\mathbf{0}\}$ and $\bar{\lambda} = \lambda' \Sigma^{-1/2}$. Then, we have

$$\lim_{n\to\infty} \frac{1}{n} \mathrm{Var}\left(\sum_{t=1}^{n} \bar{\lambda}' W_t\right) = \bar{\lambda}' \mathrm{Var}(W_1)\bar{\lambda} + 2\sum_{k=1}^{\infty} \bar{\lambda}' \mathrm{Cov}(W_1, W_{1+k})\bar{\lambda} = \bar{\lambda}'\Sigma\bar{\lambda} = \lambda'\lambda > 0,$$

(cf. Billingsley, 1995; Theorem 27.4). From the functional central limit theorem in Gallant (1987), Theorem 2, page 519, we obtain that for every $\lambda \in \mathbb{R}^3 \backslash \{\mathbf{0}\}$,

$$\frac{1}{\sqrt{n\lambda'\lambda}} \sum_{t=1}^{[ns]} \{\bar{\lambda}' W_t - \mathrm{E}(\bar{\lambda}' W_t)\} \Rightarrow B_1(s) \quad \text{in } D[0,1]. \tag{4.2}$$

Thus, by a standard argument, we can obtain that

$$\Sigma^{-1/2} \frac{1}{\sqrt{n}} \sum_{t=1}^{[ns]} \{W_t - \mathrm{E}(W_t)\} \Rightarrow B(s) \quad \text{in } D^3[0,1],$$

then, by the mapping theorem (cf. Billingsley, 1999)

$$\Sigma^{-1/2}\frac{1}{\sqrt{n}}D_{[ns]} \Rightarrow B^{\circ}(s) \quad \text{in } D^3[0,1],$$

where $D_k = \sum_{t=1}^{k} W_t - \frac{k}{n}\sum_{t=1}^{n} W_t$. Note that

$$\frac{1}{\sqrt{n}}\max_{1\le k\le n}|D_k - \tilde{D}_k| = o_P(1)$$

($|\boldsymbol{x}| := \sqrt{\boldsymbol{x}'\boldsymbol{x}}$ for $\boldsymbol{x} \in \mathbb{R}^3$), since

$$D_k - \tilde{D}_k = (\bar{y} - \mu)\left(0, 2\left(\sum_{t=1}^{k} y_t - \frac{k}{n}\sum_{t=1}^{n} y_t\right), \sum_{t=1}^{k} y_t - \frac{k}{n}\sum_{t=1}^{n} y_t + \sum_{t=1}^{k} y_{t-1} - \frac{k}{n}\sum_{t=1}^{n} y_{t-1}\right)',$$

and we obtain from (4.2) that

$$\frac{1}{\sqrt{n}}\max_{1\le k\le n}\left|\sum_{t=1}^{k} y_t - \frac{k}{n}\sum_{t=1}^{n} y_t\right| = O_P(1).$$

Hence, by the mapping theorem and the consistency of $\hat{\Sigma}_n$, the proof is completed. □

References

Andreou, E. and Ghysels, E. (2002). Detecting multiple breaks in financial market volatility dynamics. *Journal of Applied Econometrics*, **17**, 579-600.

Billingsley, P. (1995). *Probability and Measure*, 3rd Ed., Wiley, New York.

Billingsley, P. (1999). *Convergence of Probability Measures*, 2nd Ed., Wiley, New York.

Bollerslev, T. (1986). Generalized autoregressive conditional heteroskedasticity. *Journal of Econometrics*, **31**, 307-327.

Carrosco, M. and Chen, X. (2002) Mixing and moment properties of various GARCH and stochastic volatility models. *Econometric Theory*, **18**, 17-39.

Davydov, Y. A. (1973) Mixing conditions for Markov chains. *Theory of Probability and its Applications*, **XVIII**, 312-328.

Gallant, A. R. (1987) *Nonlinear Statistical Models*, Wiley, New York.

Im, C. and Cho, G. (2009) Multiparameter CUSUM charts with variable sampling intervals. *Journal of the Korean Data & Information Science Society*, **20**, 593-599.

Inclan, C. and Tiao, G. C. (1994) Use of cumulative sums of squares for restrospective detection of changes of variance. *Journal of the American Statistical Association*, **89**, 913-923.

Kokoszka, P. and Leipus, R. (2000). Change-point estimation in ARCH models. *Bernoulli*, **6**, 513-539.

Lee, S. Ha, J., Na, O. and Na, S. (2003). The cusum test for parameter change in time series models. *Scand. J. Statist*, **30**, 781-796.

Lee, S., Lee T. and Na, O. (2010). Cusum of squares test for discretely observed sample from diffusion processes. *Journal of the Korean Data & Information Science Society*, **21**, 179-183.

Na, O., Ko, B. and Lee, S. (2010) Cusum of squares test for discretely observed sample from multidimensional diffusion processes. *Journal of the Korean Data & Information Science Society*, **21**, 547-554.

Park, S. and Lee, S. (2007). Modelling KOSPI200 data based on GARCH(1,1) parameter change test. *Journal of the Korean Data & Information Science Society*, **18**, 11-16.