COMMENTS ON GENERALIZED R-KKM TYPE THEOREMS

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Abstract. Recently, some authors [3, 4, 11, 12, 15] adopted the concept of the so-called *generalized R-KKM maps* which are used to rewrite known results in the KKM theory. In the present paper, we show that those maps are simply KKM maps on G-convex spaces. Consequently, results on generalized R-KKM maps follow the corresponding previous ones on G-convex spaces.

1. Introduction

The KKM theory, originally called by the author [19], is nowadays the study of applications of various equivalent formulations or generalizations of the Knaster-Kuratowski-Mazurkiewicz theorem (simply, the KKM theorem) in 1929 [17]. In the last decade, the theory has been extensively studied for generalized convex spaces (simply, G-convex spaces) in a sequence of papers of the author; for details, see [21] and references therein.

Since the concept of G-convex spaces appeared in 1993 [24], a number of modifications or imitations of the concept have followed. Such examples are L-spaces due to Ben-El-Mechaiekh et al. [1], spaces having property (H) due to Huang [15], FC-spaces due to Ding ([7, 8, 9, 10] and many others), convexity structures satisfying the H-condition by Xiang et al. [27, 28], M-spaces and L-spaces due to González et al. [2, 14], simplicial spaces due to Kulpa et al. [18], GFC-spaces of Khanh et al. [16], and others. It is known that all of such examples are particular forms of G-convex spaces; see [22]. Recall that some of those authors tried to rewrite results on G-convex spaces by replacing $\Gamma(A)$ by $\phi_A(\Delta_n)$ everywhere and claimed to obtain generalizations without giving any justifications or proper examples. In fact, they obtained KKM type theorems or equivalents which can not be applicable even to the original KKM theorem [17] for $(\Delta_n \supset V; \mathrm{co})$ or to the celebrated Ky Fan lemma [13] for $(E \supset D; \mathrm{co})$, where E is a topological vector space.

Recently, there have appeared another authors in [3, 4, 11, 12, 15] who adopted the concept of the so-called *generalized R-KKM maps* which were used to rewrite known results in the KKM theory. In the present paper, we show that those maps are simply KKM maps on G-convex spaces and should be destroyed in order to preserve the elegancy of the G-convex space theory. Consequently, results on generalized R-KKM maps follow the corresponding previous ones on G-convex spaces.

Section 2 deals with definitions and examples of G-convex spaces. Moreover, spaces having a family $\{\phi_A\}_{A \in \langle D \rangle}$ or simply ϕ_A-*spaces* $(X, D; \{\phi_A\}_{A \in \langle D \rangle})$ are shown to be G-convex spaces. In Section 3, it is proved that two types of the so-called generalized relatively KKM (R-KKM) maps are simply KKM maps on G-convex spaces. Finally, Section 4 deals with Ding's generalized R-KKM type theorems, which are shown to be simple consequences of the corresponding ones in the G-convex space theory. Further some related matters are discussed.

The essential aim of this paper is to prevent unnecessary efforts to repeat trivial imitations of G-convex spaces and the KKM type theorems on them.

2. Generalized convex spaces

In this section, we follow mainly [21, 22] and references therein.

Definition. A *generalized convex space* or a *G-convex space* $(E, D; \Gamma)$ consists of a topological space E and a nonempty set D such that for each $A \in \langle D \rangle$ with the cardinality $|A| = n + 1$, there exist a subset $\Gamma(A)$ of E and a continuous function $\phi_A : \Delta_n \to \Gamma(A)$ such that $J \in \langle A \rangle$ implies $\phi_A(\Delta_J) \subset \Gamma(J)$.

Here, $\langle D \rangle$ denotes the set of all nonempty finite subsets of D, Δ_n the standard n-simplex with vertices $\{e_i\}_{i=0}^n$, and Δ_J the face of Δ_n corresponding to $J \in \langle A \rangle$; that is, if $A = \{a_0, a_1, \ldots, a_n\}$ and $J = \{a_{i_0}, a_{i_1}, \ldots, a_{i_k}\} \subset A$, then $\Delta_J := \mathrm{co}\{e_{i_0}, e_{i_1}, \ldots, e_{i_k}\}$. We may write $\Gamma_A := \Gamma(A)$. When $D \subset E$, the space is denoted by $(E \supset D; \Gamma)$. In case $E = D$, let $(E; \Gamma) := (E, E; \Gamma)$.

Example. The following are typical examples of G-convex spaces [21, 22]:
1. Any nonempty convex subset of a topological vector space.
2. Convex space in the sense of Lassonde.
3. C-spaces (or H-spaces) due to Horvath.
4. L-spaces due to Ben-El-Mechaiekh et al. [1].

We add particular subclasses or variants of G-convex spaces as follows:

Definition. A *space X having a family* $\{\phi_A\}_{A \in \langle D \rangle}$ or simply a *ϕ_A-space*

$$(X, D; \{\phi_A\}_{A \in \langle D \rangle})$$

consists of a topological space X, a nonempty set D, and a family of continuous functions $\phi_A : \Delta_n \to X$ (that is, singular n-simplexes) for $A \in \langle D \rangle$ with the cardinality $|A| = n + 1$.

Proposition 2.1. *A ϕ_A-space $(X, D; \{\phi_A\}_{A \in \langle D \rangle})$ can be made into a G-convex space $(X, D; \Gamma)$.*

Proof. This can be done at least in three ways.

(1) For each $A \in \langle D \rangle$, by putting $\Gamma_A := X$, we obtain a trivial G-convex space $(X, D; \Gamma)$.

(2) Let $\{\Gamma^\alpha\}_\alpha$ be the family of maps $\Gamma^\alpha : \langle D \rangle \multimap X$ giving a G-convex space $(X, D; \Gamma^\alpha)$. Note that, by (1), this family is not empty. Then, for each α and each $A \in \langle D \rangle$ with $|A| = n + 1$, we have

$$\phi_A(\Delta_n) \subset \Gamma_A^\alpha \quad \text{and} \quad \phi_A(\Delta_J) \subset \Gamma_J^\alpha \quad \text{for} \quad J \subset A.$$

Let $\Gamma := \bigcap_\alpha \Gamma^\alpha$, that is, $\Gamma_A = \bigcap_\alpha \Gamma_A^\alpha$. Then

$$\phi_A(\Delta_n) \subset \Gamma_A \quad \text{and} \quad \phi_A(\Delta_J) \subset \Gamma_J \quad \text{for} \quad J \subset A.$$

Therefore, $(X, D; \Gamma)$ is a G-convex space.

(3) Let $N \in \langle D \rangle$ with $|N| = n + 1$. For each $M \in \langle D \rangle$ with $N \subset M$, $M = \{a_0, \ldots, a_m\}$ and $N = \{a_{i_0}, \ldots, a_{i_n}\}$, there exists a subset $\phi_M(\Delta_n^M)$ of X such that $\Delta_n^M := \text{co}\{e_{i_j} \mid j = 0, \ldots, n\} \subset \Delta_m$. Now let

$$\Gamma_N = \Gamma(N) := \bigcup_{M \supset N} \phi_M(\Delta_n^M).$$

Then $\Gamma : \langle D \rangle \multimap X$ is well-defined and $(X, D; \Gamma)$ becomes a G-convex space: For each $A \in \langle D \rangle$ with $|A| = n + 1$, there exists a continuous map $\phi_A : \Delta_n \multimap \Gamma(A)$ such that $J \in \langle A \rangle$ implies $\phi_A(\Delta_J) \subset \Gamma(J)$. $\qquad\square$

Therefore, G-convex spaces and ϕ_A-spaces are essentially the same.

Example. The following are typical examples of ϕ_A-spaces:

1. Topological spaces having property (H) [15].
2. FC-spaces due to Ding [7, 8, 9, 10].
3. Convexity structures satisfying the H-condition by Xiang et al. [27, 28].
4. M-spaces and L-spaces due to González et al. [2, 14].
5. Simplicial spaces due to Kulpa et al. [18].
6. Recently, ϕ_A-spaces are called *generalized finitely continuous spaces* (simply, *GFC*-spaces) by Khanh et al. [16].

Some authors obtained KKM type theorems or equivalents which can not be applicable even to the original KKM theorem [17] for $(\Delta_n \supset V; \text{co})$, where V is the set of vertices of Δ_n and co is the convex hull operation in vector spaces, or to the celebrated Ky Fan lemma [13] for $(E \supset D; \text{co})$, where E is a topological vector space.

3. Generalized R-KKM maps

For a G-convex space $(X, D; \Gamma)$, a multimap $F : D \multimap X$ is called a *KKM map* if $\Gamma_A \subset F(A)$ for each $A \in \langle D \rangle$.

Recently, there have appeared authors of [3, 4, 11, 12, 15, 16] and others who tried to rewrite our works on G-convex spaces by replacing $\Gamma(A)$ by $\phi_A(\Delta_n)$ everywhere and claimed to obtain generalizations without giving any justifications or proper examples.

In 2003, the following appeared:

Definition ([3]). Let X be a nonempty set and Y be a topological space. $T : X \to 2^Y$ is said to be generalized relatively KKM (R-KKM) mapping if for any $N = \{x_0, x_1, \ldots, x_n\} \in \langle X \rangle$, there exists a continuous mapping $\phi_N : \Delta_n \to Y$ such that, for each $e_{i_0}, e_{i_1}, \ldots, e_{i_k}$,

$$\phi_N(\Delta_k) \subset \bigcup_{j=0}^{k} Tx_{i_j},$$

where Δ_k is the k-face of Δ_n with vertices $e_{i_0}, e_{i_1}, \ldots, e_{i_k}$.

The authors of [3] claimed that their definition unifies and extends a lot of similar definitions due to other authors, and that, applying their key result, they obtained new theorems which unify and extend many known results in recent literature. However, theirs are all disguised forms of known results and their applicability is doubtful; see [23].

In 2005, the authors of [12] obtained a necessary and sufficient condition for the open-valued KKM theorem and some of its routine consequences parallel to the corresponding ones in [3].

In the same year, in [15], its author followed the method in [3]. First of all, we note that, in these three papers [3, 12, 15], their authors concerned with maps having compactly closed (resp., open) values and peculiar closures ccl and interior cint. These are not practical, not general, and can be immediately replaced by ordinary ones by switching the relevant topology to its compactly generated extension; see [20].

In [15], a topological space Y is said to have property (H) if, for each $N = \{y_0, \ldots, y_n\} \in \langle Y \rangle$, there exists a continuous mapping $\varphi_N : \Delta_n \to Y$. Then the following is introduced:

Definition ([15]). Let X be a nonempty set and Y be a topological space with property (H). $T : X \to 2^Y$ is said to be a *generalized R-KKM mapping* if for each $\{x_0, \ldots, x_n\} \in \langle X \rangle$, there exists $N = \{y_0, \ldots, y_n\} \in \langle Y \rangle$ such that

$$\varphi_N(\Delta_k) \subset \bigcup_{j=0}^{k} Tx_{i_j}$$

for all $\{i_0, \ldots, i_k\} \subset \{0, \ldots, n\}$.

Adopting those concepts, in [15], the author obtained some known results or their modifications in the G-convex space theory in which we supplied already a large number of examples of such spaces [7]. It is noteworthy that the authors of [3, 4, 11, 12, 15] claimed to obtain generalizations of known results without giving any justifications or any single proper example.

Proposition 3.1. *Let X be a nonempty set and Y be a topological space with property* (H). *A generalized R-KKM map $T : X \to 2^Y$ is simply a KKM map for a G-convex space $(Y, X; \Gamma)$.*

Proof. In fact, let $A \in \langle X \rangle$ with $|A| = n + 1$. Then there corresponds an $N \in \langle Y \rangle$ with $|N| = n + 1$. Define $\Gamma : \langle X \rangle \multimap Y$ by $\Gamma_A := T(A)$ for each $A \in \langle X \rangle$. Then $(Y, X; \Gamma)$ becomes a G-convex space. In fact, for each A with $|A| = n + 1$, we have a continuous function $\phi_A := \varphi_N : \Delta_n \to T(A) =: \Gamma(A)$ such that $J \in \langle A \rangle$ implies $\phi_A(\Delta_J) \subset T(J) =: \Gamma(J)$. Moreover, note that $\Gamma_A \subset T(A)$ for each $A \in \langle X \rangle$ and hence $T : X \multimap Y$ is a KKM map on a G-convex space $(Y, X; \Gamma)$. $\square$

In a recent work [11], its author adopted the definition of a generalized R-KKM map as in [3]. Then we have the following:

Proposition 3.2. *Let X be a nonempty set and Y be a topological space as in [28]. A generalized R-KKM map $T : X \to 2^Y$ is simply a KKM map for a G-convex space $(Y, X; \Gamma)$.*

The author of [11] claimed as follows: "The above class of generalized R-KKM mappings includes those classes of KKM mappings, H-KKM mappings, G-KKM mappings, generalized G-KKM mappings, generalized S-KKM mappings, GLKKM mappings and GMKKM mappings defined in topological vector spaces, H-spaces, G-convex spaces, G-H-spaces, L-convex spaces and hyperconvex metric spaces, respectively, as true subclasses." This is inexact.

In view of this claim and Proposition 3.2, so many variants of KKM type theorems in [3, 4, 11, 12, 15] can be reduced to the ones in our G-convex space theory. We should recognize that, in the KKM theory on G-convex spaces, every argument is related to the finite intersection property of functional values of KKM maps having closed (resp., open) values, in other words, related to some $N \in \langle D \rangle$ in $(X, D; \Gamma)$.

4. Ding's generalized R-KKM type theorems

The following KKM theorem for G-convex spaces is known and its proof is just simple modification of the one in [20, 25]:

Theorem 4.1. *Let $(X, D; \Gamma)$ be a G-convex space and $F : D \multimap X$ a multimap such that*
    (1.1) F has closed (resp., open) values; and
    (1.2) F is a KKM map.
Then $\{F(z)\}_{z \in D}$ has the finite intersection property (More precisely, for each $N \in \langle D \rangle$ with $|N| = n + 1$, we have $\phi_N(\Delta_n) \cap \bigcap_{z \in N} F(z) \neq \emptyset$).
    Further, if
    (1.3) $\bigcap_{z \in M} \overline{F(z)}$ is compact for some $M \in \langle D \rangle$,
then we have $\bigcap_{z \in D} \overline{F(z)} \neq \emptyset$.

Proof. Let $N = \{z_0, z_1, \ldots, z_n\}$. Since F is a KKM map, for each vertex e_i of Δ_n, we have $\phi_N(e_i) \in F(z_i)$ for $0 \leq i \leq n$. Then $e_i \mapsto \phi_N^{-1} F(z_i)$ is a closed (resp., open) valued map such that $\Delta_k = \mathrm{co}\{e_{i_0}, e_{i_1}, \ldots, e_{i_k}\} \subset \bigcup_{j=0}^{k} \phi_N^{-1} F(z_{i_j})$

for each face Δ_k of Δ_n. Therefore, by the original KKM theorem and its 'open' version, $\Delta_n \supset \bigcap_{i=0}^n \phi_N^{-1} F(z_i) \neq \emptyset$ and hence $\phi_N(\Delta_n) \cap \left(\bigcap_{z \in N} F(z) \right) \neq \emptyset$.

The second conclusion is clear. $\square$

Note that Theorem 4.1 generalizes the original KKM theorem [17] and the Ky Fan lemma [13]. But they are all equivalent.

During a quite long period, peoples used to adopt *compactly* closed (resp., open) values in Theorem 4.1 instead of closed (resp., open) values. Especially, in many of Ding's papers ([8, 9] for example), he used *the compact closure* and the *compact interior* of A, denoted by $\mathrm{ccl}(A)$ and $\mathrm{cint}(A)$. This misguided many followers. In our previous work [20], we suggested to destroy such inadequate terminology by replacing the original topology by its compactly generated extension.

Now, in [11], Ding uses *finitely* closed (resp., open) values in [11, Theorems and Corollaries in Section 3] as follows:

Definition ([11])**.** Let X be a nonempty set and Y be a topological space. A generalized R-KKM mapping $G : X \to 2^Y$ is said to be *finitely* closed-valued (resp., open-valued) if for each $N = \{x_0, \ldots, x_n\} \in \langle X \rangle$ and each $x \in N$, $\varphi_N(\Delta_n) \cap G(x)$ is closed (resp., open) in $\varphi_N(\Delta_n)$, $\varphi_N : \Delta_n \to Y$ is the map in Definition of generalized R-KKM mapping.

Note that the proof of Theorem 4.1 works even if F has finitely closed (resp., open) values. Therefore, from Theorem 4.1, we readily have the following in [11]:

Theorem 3.1 ([11])**.** *Let X be a nonempty set, Y be a topological space and $G : X \to 2^Y$ be a generalized R-KKM mapping with nonempty finitely closed values. Then for each $N = \{x_0, x_1, \ldots, x_n\} \in \langle X \rangle$, $\varphi_N(\Delta_n) \cap (\bigcap_{i=0}^n G(x_i)) \neq \emptyset$, where $\varphi_N : \Delta_n \to Y$ is the map in Definition of generalized R-KKM mapping.*

Theorem 3.2 ([11])**.** *Let X be a nonempty set, Y be a topological space and $G : X \to 2^Y$ be a generalized R-KKM mapping with nonempty finitely open values. Then for each $N = \{x_0, x_1, \ldots, x_n\} \in \langle X \rangle$, $\varphi_N(\Delta_n) \cap (\bigcap_{i=0}^n G(x_i)) \neq \emptyset$, where $\varphi_N : \Delta_n \to Y$ is the map in Definition of generalized R-KKM mapping.*

These generalize nothing and are not practical (in fact, they are not used in [11, Theorems and Corollaries in Section 4]). Precisely, the *finitely generated extension* of the original topology of Y is the smallest (coarsest) topology of Y containing all finitely closed (resp., open) subsets. If we replace the original topology of Y by its finitely generated extension, all finitely closed (resp., open) subsets becomes simply closed (resp., open) subsets in the new topology; see [25].

Another Proofs of [11, Theorems 3.1 and 3.2]. As in Propositions 3.1 and 3.2, we have a G-convex space $(Y, X; \Gamma)$ with $\Gamma_A := T(A)$ for each $A \in \langle X \rangle$. Now replace the topology of Y by its finitely generated extension. Then the conclusion follows immediately from Theorem 4.1. $\square$

Therefore, Ding's generalized R-KKM type theorems are simple consequences of the corresponding ones in the G-convex space theory.

For a multimap $F : D \multimap X$, we define a multimap $\overline{F} : D \multimap X$ by $\overline{F}(z) := \overline{F(z)}$ for all $z \in D$, where $\overline{}$ denotes the closure operator.

From the closed version of Theorem 4.1, we can deduce the following equivalent formulation:

Theorem 4.2. *Let $(X, D; \Gamma)$ be a G-convex space and $F : D \multimap X$ a map such that*

(2.1) $\bigcap_{z \in D} \overline{F(z)} = \bigcap_{z \in D} F(z)$ *(F is transfer closed-valued);*

(2.2) $\overline{F}$ *is a KKM map; and*

(2.3) $\bigcap_{z \in M} \overline{F(z)}$ *is compact for some $M \in \langle D \rangle$.*

Then we have $\bigcap_{z \in D} F(z) \neq \emptyset$.

Proof. The map $\overline{F} : D \multimap X$ is a KKM map with closed values. Hence, by Theorem 4.1, we have $\bigcap_{z \in D} \overline{F(z)} \neq \emptyset$. Then, by (2.1), we have $\bigcap_{z \in D} F(z) = \bigcap_{z \in D} \overline{F(z)} \neq \emptyset$. $\square$

Recall that Theorem 4.2 originates from Tian [26, Theorem 2]. Note that, if F is closed-valued, then (2.1) holds, but not conversely, and that Theorem 4.2 reduces to Theorem 4.1 if F has closed values. This is why peoples regard Theorem 4.2 generalize Theorem 4.1. However, as we have seen in the proof of Theorem 4.2, Theorems 4.1 (the closed case) and 4.2 are equivalent. Therefore, *now is the proper time to discard the "transfer" cases from the KKM theory.*

We consider another KKM type theorem in [11]:

Theorem 4.3 ([11])**.** *Let X be a nonempty set, Y be a topological space and $G : X \to 2^Y$ be a generalized R-KKM mapping with nonempty finitely closed values such that*

(i) $\mathrm{cl}_Y(\cap_{x \in A} G(x))$ *is compact for some $A \in \langle X \rangle$;*

(ii) *for each $N = \{x_0, x_1, \ldots, x_n\} \in \langle X \rangle$ with $A \subset N$,*

$$(\mathrm{cl}_Y(\bigcap_{x \in N} G(x))) \cap \varphi_N(\Delta_n) = (\bigcap_{x \in N} G(x)) \cap \varphi_N(\Delta_n),$$

where $\varphi_N : \Delta_n \to Y$ is the map in Definition of generalized R-KKM mapping. Then $\bigcap_{x \in X} G(x) \neq \emptyset$.

This can be restated by eliminating the artificial concept of generalized R-KKM map as follows:

Theorem 4.4. *Let $(Y, X; \Gamma)$ be a G-convex space and $G : X \multimap Y$ a KKM map with finitely closed values such that*

(i) $\mathrm{cl}_Y(\cap_{x \in A} G(x))$ *is compact for some $A \in \langle X \rangle$;*

(ii) *for each* $N = \{x_0, x_1, \ldots, x_n\} \in \langle X \rangle$ *with* $A \subset N$,

$$(\mathrm{cl}_Y(\bigcap_{x \in N} G(x))) \cap \varphi_N(\Delta_n) = (\bigcap_{x \in N} G(x)) \cap \varphi_N(\Delta_n),$$

where $\phi_N : \Delta_n \to Y$ *is the map in Definition of G-convex space.*
Then $\bigcap_{x \in X} G(x) \neq \emptyset$.

Similarly, other results in [11] can be stated without using artificial terminology and impractical assumptions.

Finally, we believe that reputed journals should clarify any incorrect statements in their publications. But some reviewers and editors help to hide such things by rejecting critical papers to them.

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