Commun. Korean Math. Soc. **25** (2010), No. 2, pp. 291–301
DOI 10.4134/CKMS.2010.25.2.291

THE LIMITING LOG GAUSSIANITY FOR AN EVOLVING BINOMIAL RANDOM FIELD

Sungyeun Kim, Wonbae Kim, and Jongsig Bae

Abstract. This paper consists of two main parts. Firstly, we introduce an evolving binomial process from a binomial stock model and consider various types of limiting behavior of the logarithm of the evolving binomial process. Among others we find that the logarithm of the binomial process converges weakly to a Gaussian process. Secondly, we provide new approaches for proving the limit theorems for an integral process motivated by the evolving binomial process. We provide a new proof for the uniform strong LLN for the integral process. We also provide a simple proof of the functional CLT by using a restriction of Bernstein inequality and a restricted chaining argument. We apply the functional CLT to derive the LIL for the IID random variables from that for Gaussian.

1. An evolving random walk process

1.1. An introduction and the main results

Let ξ be a random variable defined on a probability space $(\Omega, \mathcal{T}, P)$ with distribution function F, and let $\xi_1, \xi_2, \ldots$ be a sequence of independent copies of ξ. Let

$$F_n(x) := \frac{1}{n}\sum_{i=1}^{n} 1_{(-\infty,\ x]}(\xi_i)$$

be the empirical distribution of F. Let $f : \mathbb{R} \to \mathbb{R}$ be a given Borel measurable function. We consider

$$G_n(t) := \int_{-\infty}^{t} f(x)(F_n - F)(dx) \text{ for } t \in \mathbb{R}. \tag{1}$$

Received February 23, 2009.
2000 *Mathematics Subject Classification.* Primary 91B26, 91B99; Secondary 60F17, 60F99.
Key words and phrases. evolving binomial process, limiting log Gaussian property, uniform LLN, functional CLT, LIL.
This paper was financially supported by KRF 2007-354-C00007.

Let $\sigma > 0$. We consider an evolving random work process $\{Z_n(t) : t \in \mathbb{R}\}$ defined by

$$Z_n(t) := \left(1 + \frac{\sigma}{\sqrt{n}}\right)^{\frac{1}{2}(nt+nG_n(t))} \left(1 - \frac{\sigma}{\sqrt{n}}\right)^{\frac{1}{2}(nt-nG_n(t))}. \tag{2}$$

One of the main goal of the paper is to investigate the limiting behavior of the logarithm of the process Z_n.

Firstly, we state a uniform strong LLN for the process

$$\left\{ \frac{\log Z_n(t) + \frac{\sigma^2 t}{2}}{\sqrt{n}} : t \in \mathbb{R} \right\}.$$

Theorem 1.1. *Suppose* $\int |f| dF < \infty$. *Then*

$$\sup_{t \in \mathbb{R}} \frac{\log Z_n(t) + \frac{\sigma^2 t}{2}}{\sqrt{n}} \to 0 \textit{ almost surely.}$$

The convergence can be strengthened to convergence in the mean:

$$E\left(\sup_{t \in \mathbb{R}} \frac{\log Z_n(t) + \frac{\sigma^2 t}{2}}{\sqrt{n}} \right) \to 0.$$

Let $D(\mathbb{R})$ be the cadlag functions defined on $\mathbb{R}$. We equip the Skorohod metric on $D(\mathbb{R})$. We use the following weak convergence.

Definition. A sequence of $D(\mathbb{R})$-valued random functions $\{Y_n : n \geq 1\}$ converges in law to a $D(\mathbb{R})$-valued Borel measurable random function Y whose law concentrates on a separable subset of $D(\mathbb{R})$, denoted by $Y_n \Rightarrow Y$, if $Eg(Y) = \lim_{n\to\infty} Eg(Y_n)$ for all $g \in C(D(\mathbb{R}), || \cdot ||)$, where $C(D(\mathbb{R}), || \cdot ||)$ is the set of real bounded, continuous functions.

Secondly, we state a limiting Gaussian property for the process

$$\left\{ \log Z_n(t) + \frac{\sigma^2 t}{2} : t \in \mathbb{R} \right\}.$$

Theorem 1.2. *Suppose* $\int f^2 dF < \infty$. *Then*

$$\left\{ \log Z_n(t) + \frac{\sigma^2 t}{2} : t \geq 0 \right\} \Rightarrow Z \textit{ as random elements of } D(\mathbb{R}).$$

The limiting process $Z = \{Z(t) : t \in \mathbb{R}\}$ *is a mean zero Gaussian process with covariance*

$$cov(Z(s), Z(t)) = \int_{-\infty}^{s\wedge t} f^2(x)P(dx) - \int_{-\infty}^{s} f(x)P(dx) \int_{-\infty}^{t} f(x)P(dx).$$

The sample paths of Z *are continuous.*

Thirdly, for each fixed t, we state the LIL for the sequence of random variables

$$\left\{ \frac{\log Z_n(t) + \frac{\sigma^2 t}{2}}{\sqrt{2\log\log n}} : n \geq 3 \right\}.$$

Theorem 1.3. *Suppose that* $\int f^2 dF < \infty$. *Then for each fixed* t *the set of limit points of*

$$\left\{ \frac{\log Z_n(t) + \frac{\sigma^2 t}{2}}{\sqrt{2\log\log n}} : n \geq 3 \right\}$$

is, with probability 1, *the closed interval* $[-\sigma_t, \sigma_t]$, *where* σ_t^2 *is given by*

$$\sigma_t^2 := \int_{-\infty}^{t} f^2 dF.$$

1.2. Proof of the results

We begin by the following lemma:

Lemma 1.4.

$$\log Z_n(t) + \frac{1}{2}\sigma^2 t = \sigma\sqrt{n}G_n(t) + G_n(t)O(n^{-1/2}) + O(n^{-1/2}).$$

Proof. By the Taylor expansion, we have

$$\frac{1}{2}\log(1+x) + \frac{1}{2}\log(1-x) = -\frac{1}{2}x^2 + O(x^3).$$

Now we compute

$$\begin{aligned}
& \log Z_n(t) \\
= \; & \frac{1}{2}(nt + nG_n(t))\log\left(1 + \frac{\sigma}{\sqrt{n}}\right) + \frac{1}{2}(nt - nG_n(t))\log\left(1 - \frac{\sigma}{\sqrt{n}}\right) \\
= \; & nt\left[\frac{1}{2}\log\left(1 + \frac{\sigma}{\sqrt{n}}\right) + \frac{1}{2}\log\left(1 - \frac{\sigma}{\sqrt{n}}\right)\right] \\
& + nG_n(t)\left[\frac{1}{2}\log\left(1 + \frac{\sigma}{\sqrt{n}}\right) - \frac{1}{2}\log\left(1 - \frac{\sigma}{\sqrt{n}}\right)\right] \\
= \; & nt\left[-\frac{1}{2}\left(\frac{\sigma}{\sqrt{n}}\right)^2 + O(n^{-3/2})\right] + nG_n(t)\left[\frac{\sigma}{\sqrt{n}} + O(n^{-3/2})\right] \\
= \; & -\frac{1}{2}\sigma^2 t + \sigma\sqrt{n}G_n(t) + G_n(t)O(n^{-1/2}) + O(n^{-1/2}).
\end{aligned}$$

□

From the Lemma 1 we see that the proof of Theorems 1-3 will be completed if the limiting behaviors of the process $\{G_n(t) : t \in \mathbb{R}\}$, defined in (1), are established. We will consider this topics in the next section.

1.3. An evolving binomial process

We introduce a binomial model for a stock dynamics. Stock prices are assumed the following model. The initial stock price during the period under study is denoted S_0. At each time step, the stock price either goes up to by a factor of u or down by a factor of d. We visualize the model by tossing a coin at each time step, and say that the stock price moves up to uS_0 if the coin comes out heads(H), and down to dS_0 if it comes out tails(T).

Consider the probabilistic model for the experiment consisting of an infinite number of independent tosses of a coin when at each step the probability of falling head is 1/2. Let

$$\Omega = \{\omega : \omega = (\omega_1, \omega_2, \ldots), \omega_j = H \text{ or } T\}$$

be the sample space of the experiment. Consider the symmetric random walk defined by

$$\begin{aligned} M_0 &:= 0; \\ M_k &:= \xi_1 + \xi_2 + \cdots + \xi_k, \ k \geq 1, \end{aligned}$$

where

$$\xi_j(\omega) = \begin{cases} 1 & \text{if } \omega_j = H, \\ -1 & \text{if } \omega_j = T. \end{cases}$$

Consider the nth binomial model with the following parameters:

$$\begin{aligned} u_n &= 1 + \sigma n^{-1/2}, \ \sigma > 0; \\ d_n &= 1 - \sigma n^{-1/2}. \end{aligned}$$

Let $\nu_k(H)$ denote the number of H in the first k tosses, and let $\nu_k(T)$ denote the number of T in the first k tosses. Then

$$\nu_k(H) + \nu_k(T) = k,$$

$$\nu_k(H) - \nu_k(T) = M_k,$$

which implies,

$$\nu_k(H) = \frac{1}{2}(k + M_k),$$

$$\nu_k(T) = \frac{1}{2}(k - M_k).$$

We consider the evolving binomial process $Z_n = \{Z_n(t) : t \geq 0\}$ defined by

$$Z_n(t) := \left(1 + \frac{\sigma}{\sqrt{n}}\right)^{\frac{1}{2}\left([nt] + M_{[nt]}\right)} \left(1 - \frac{\sigma}{\sqrt{n}}\right)^{\frac{1}{2}\left([nt] - M_{[nt]}\right)}, \tag{3}$$

$[x]$ denotes the integral part of x. Observe that the identity

$$\log Z_n(t) + \frac{1}{2}\sigma^2 t = \sigma \left(\frac{1}{\sqrt{n}} \sum_{j=1}^{[nt]} \xi_j \right) + \left(\frac{1}{n} \sum_{j=1}^{[nt]} \xi_j \right) O(n^{-1/2}) + O(n^{-1/2}).$$

Applying the CLT and LLN we see that the finite dimensional distributions of

$$\left\{\log Z_n(t) + \frac{1}{2}\sigma^2 t : t \geq 0\right\}$$

converges to a multivariate normal random variable. More specifically, if $0 < t_1 < \cdots < t_k$, then the random vector

$$\left(\log Z_n(t_1) + \frac{1}{2}\sigma^2 t_1,\ \ldots,\ \log Z_n(t_k) + \frac{1}{2}\sigma^2 t_k\right)$$

converges in distribution to $\mathcal{N}_k(0, V)$, where the variance-covariance matrix V is given by

$$V = \begin{pmatrix} \sigma^2 t_1 & \sigma^2 t_1 & \cdots & \sigma^2 t_1 \\ & \sigma^2 t_2 & \cdots & \sigma^2 t_2 \\ & & \ddots & \vdots \\ Sym & & & \sigma^2 t_k \end{pmatrix}.$$

Consider the process $Z = \{Z(t) : t \geq 0\}$, known as a geometric Brownian motion, given by

$$Z(t) := \exp\left(\sigma B(t) - \frac{1}{2}\sigma^2 t\right), \tag{4}$$

where $B = \{B(t) : t \geq 0\}$ is a Brownian motion. Applying Theorem 2 we obtain the following:

Corollary 1.5. *As $n \to \infty$,*

$$Z_n \Rightarrow Z,$$

as random elements of $D[0, \infty)$.

1.4. The limits of binomial options

In this subsection we deal with the following model for a financial market $\mathcal{M}$, in which a security is traded continuously during the time period $[0, T]$. We refer to the asset as the stock and model the evolution of the price-per-share $Z(t)$ of the stock at time t by the linear stochastic differential equation

$$dZ(t) = Z(t)[\sigma dB(t)],\ Z(0) = z \in (0, \infty). \tag{5}$$

See, for example, Karatzas [3].

We suggest the binomial evolving process on $[0, T]$ given by $\{Z_n(t) : t \in [0, T]\}$, where

$$Z_n(t) := \left(1 + \frac{\sigma}{\sqrt{n}}\right)^{\frac{1}{2}\left([nt] + M_{[nt]}\right)} \left(1 - \frac{\sigma}{\sqrt{n}}\right)^{\frac{1}{2}\left([nt] - M_{[nt]}\right)}. \tag{6}$$

Observe that

$$Z(t) := \exp\left\{\sigma B(t) - \frac{1}{2}\sigma^2 t\right\}$$

solves the stochastic differential equation (5).

Since the restriction mapping from $[0, \infty)$ into $[0, T]$ is continuous, we can apply the continuous mapping theorem to Corollary 1.5 to get:

Corollary 1.6. *As* $n \to \infty$,

$$Z_n \Rightarrow Z \text{ as random elements of } D[0,T],$$

equipped with the Skorohod metric.

The following results show that the various options based on the evolving process converge weakly to those of continuous stock marketing model.

Corollary 1.7. *Suppose that* $\{Z_n(t) : t \in [0,T]\}$ *is given by* (6).

(i) *European Call Option*

$$Y_n := (Z_n(T) - q)^+ \Rightarrow Y := (Z(T) - q)^+ .$$

(ii) *European Put Option*

$$Y_n := (q - Z_n(T))^+ \Rightarrow Y := (q - Z(T))^+ .$$

(iii) *Path Dependent Option*

$$Y_n := \left(\max_{T-\delta \le t \le T} Z_n(t) - q \right)^+ \Rightarrow Y := \left(\max_{T-\delta \le t \le T} Z(t) - q \right)^+ .$$

(iv) *Asian Option*

$$Y_n := \left(\frac{1}{T} \int_0^T Z_n(s) ds - q \right)^+ \Rightarrow Y := \left(\frac{1}{T} \int_0^T Z(s) ds - q \right)^+ .$$

Proof. The mapping $H : B[0, \infty) \to \mathbb{R}$ defined by

$$Hx = (x(T) - q)^+$$

is continuous. Continuous mapping theorem completes the proof of (i). The proofs of others are similar. □

2. Limit theorems for an integral process

Let ξ be a random variable defined on a probability space $(\Omega, \mathcal{T}, P)$ with the distribution function F, and let $\xi_1, \xi_2, \ldots$ be a sequence of independent copies of ξ. Suppose that $f : \mathbb{R} \to \mathbb{R}$ be a Borel measurable function with $\int f^2 dF < \infty$. Consider a metric d on $\mathbb{R}$ by

$$d^2(s,t) := \left| \int_s^t f^2(x) F(dx) \right| .$$

Consider the integral processes defined by

$$G_n(t) := \int_{-\infty}^t f(x)(\mathbf{P}_n - P)(dx) \quad \text{for } t \in \mathbb{R}, \tag{7}$$

$$U_n(t) := \sqrt{n} G_n(t) \quad \text{for } t \in \mathbb{R}, \tag{8}$$

$$U_n := \frac{\sqrt{n}\int f(x)(\mathbf{P}_n - P)(dx)}{\sqrt{2\log\log n}} \quad \text{for } n \geq 3, \tag{9}$$

where $\mathbf{P}_n(\cdot) = n^{-1}\sum_{i=1}^{n}\delta_{\xi_i}(\cdot)$ denotes the empirical measure and P denotes the measure induced by the distribution F.

The other goal of the paper is to provide new proofs for the various limit theorems for the integral processes.

We firstly prove a uniform strong LLN for the process $\{G_n(t) : t \in \mathbb{R}\}$ under the assumption of $\int |f| dF < \infty$. Secondly, we prove a functional CLT for the process $\{U_n(t) : t \in \mathbb{R}\}$ under the assumption of $\int f^2 dF < \infty$. The main tools are a restriction of Bernstein inequality and a restricted chaining argument. Thirdly, we will use the functional CLT to prove the LIL for the sequence of random variables $\{U_n : n \geq 3\}$.

Firstly, we state a uniform strong LLN for the process

$$\left\{G_n(t) := \int_{-\infty}^{t} f(x)(\mathbf{P}_n - P)(dx) : t \in \mathbb{R}\right\}.$$

Theorem 2.1. *Suppose* $\int |f| dF < \infty$. *Then*

$$\sup_{t\in\mathbb{R}}\left|\int_{-\infty}^{t} f(x)(\mathbf{P}_n - P)(dx)\right| \to 0 \text{ almost surely.}$$

The convergence can be strengthened to convergence in the mean:

$$E\left(\sup_{t\in\mathbb{R}}\left|\int_{-\infty}^{t} f(x)(\mathbf{P}_n - P)(dx)\right|\right) \to 0.$$

Secondly, we state the functional CLT for the integral process

$$\left\{U_n(t) := \sqrt{n}\int_{-\infty}^{t} f(x)(\mathbf{P}_n - P)(dx) : t \in \mathbb{R}\right\}.$$

Establishing a functional CLT means showing that $\mathcal{L}(U_n(t) : t \in \mathbb{R}) \to \mathcal{L}(Z(t) : t \in \mathbb{R})$, where the processes are considered as random elements of $D(\mathbb{R})$, the cadlag functions defined on $\mathbb{R}$, equipped with the Skorohod metric on $D(\mathbb{R})$. Z is a Gaussian process whose sample paths are continuous.

Theorem 2.2. *Suppose* $\int f^2 dF < \infty$. *Then*

$$U_n \Rightarrow Z \text{ as random elements of } D(\mathbb{R}).$$

The limiting process $Z = \{Z(t) : t \in \mathbb{R}\}$ *is a mean zero Gaussian process with covariance*

$$cov(Z(s), Z(t)) = \int_{-\infty}^{s\wedge t} f^2(x)P(dx) - \int_{-\infty}^{s} f(x)P(dx)\int_{-\infty}^{t} f(x)P(dx).$$

The sample paths of Z *are continuous.*

Thirdly, we state the LIL for the sequence of random variables

$$\left\{U_n := \frac{\sqrt{n}\int f(x)(\mathbf{P}_n - P)(dx)}{\sqrt{2\log\log n}} : n \geq 3\right\}.$$

Theorem 2.3. *Suppose that* $\int f^2 dF < \infty$. *Then the set of limit points of*

$$\left\{ \frac{\sqrt{n} \int f(x)(\mathbf{P}_n - P)(dx)}{\sqrt{2 \log \log n}} : n \geq 3 \right\}$$

is, with probability 1, *the closed interval* $[-\sigma_f, \sigma_f]$, *where* σ_f^2 *is given by*

$$\sigma_f^2 := \int f^2 dF.$$

2.1. A proof for the uniform LLN

In this subsection we provide a proof of the unform strong LLN for the process $\{G_n(t) : t \in \mathbb{R}\}$.

Proof of Theorem 2.1. Let $\epsilon > 0$. Assume $f \geq 0$ without loss of generality. Choose a grid of points $-\infty = t_0 < t_1 < \cdots < t_{k(\epsilon)} = \infty$ with the property that $f 1_{(-\infty, t_i]} \leq f 1_{(-\infty, t]} \leq f 1_{(-\infty, t_{i+1}]}$ and

$$\int_{t_i}^{t_{i+1}} f(x) F(dx) < \epsilon.$$

Then, for each $t \in \mathbb{R}$, there is t_i such that

$$\begin{aligned} G_n(t) &= \int_{-\infty}^{t} f(x) F_n(dx) - \int_{-\infty}^{t} f(x) F(x) \\ &= \int_{-\infty}^{t} f(x) F_n(dx) - \int_{-\infty}^{t_{i+1}} f(x) F(dx) \\ &\quad + \int_{-\infty}^{t_{i+1}} f(x) F(dx) - \int_{-\infty}^{t} f(x) F(dx) \\ &\leq \int_{-\infty}^{t_{i+1}} f(x)(F_n - F)(dx) + \int_{t_i}^{t_{i+1}} f(x) F(dx). \end{aligned}$$

Consequently,

$$\sup_{t \in \mathbb{R}} G_n(t) \leq \max_{0 \leq i \leq k(\epsilon)} \int f(x) 1_{(-\infty, t_{i+1}]}(x)(F_n - F)(dx) + \epsilon.$$

The right hand side converges almost surely to ϵ by the strong law of the large numbers. Combination with a similar argument for $\inf_{t \in \mathbb{R}} G_n(t)$ yields that

$$\limsup_{n \to \infty} \sup_{t \in \mathbb{R}} |G_n(t)| \leq \epsilon,$$

almost surely, for every $\epsilon > 0$. Take a sequence $\epsilon_m \downarrow 0$ to see that the limsup must actually be zero almost surely. The proof for the uniform strong LLN is completed. Now applying the LLN for convergence in the mean, we get the uniform LLN for the mean convergence. The proof is completed. □

2.2. A proof for the functional CLT

In this subsection we provide a new proof for the functional CLT for the process $\{U_n(t) : t \in \mathbb{R}\}$.

Proof of Theorem 2.2. Since the finite dimensional distributions converge by the multivariate CLT, the result is a consequence of Theorem 2.4: apply Theorem 10.2 of Pollard [7] to the process $\{U_n\}$ indexed by $\mathbb{R}$. □

For a function $\varphi : \mathbb{R} \to \mathbb{R}$, we let $||\varphi|| := \sup_{t\in\mathbb{R}} |\varphi(t)|$ denote the sup of $|\varphi|$ over $\mathbb{R}$. We also let $||\varphi||_r := \sup_{(r)} |\varphi(s) - \varphi(t)|$ denote the sup of $|\varphi(s) - \varphi(t)|$ over (r), where $(r) := \{(s, t) \in \mathbb{R} \times \mathbb{R} : d(s, t) < r\}$.

The following tightness result will be used to prove the functional CLT.

Theorem 2.4. *Suppose $\int f^2 dF < \infty$. Then given $\epsilon > 0$ there exists $r > 0$ for which*

$$\limsup_{n\to\infty} P\{||U_n||_r > 5\epsilon\} < 3\epsilon\ .$$

Proof. The result can be obtained from Ossiander [5]. □

2.3. A proof for the LIL

In this subsection we provide a proof of the LIL for the sequence of random variables $\{U_n : n \geq 3\}$.

Let $\mathbf{W}$ be a Gaussian random variable which has mean zero and variance

$$\sigma_f^2 := \int f^2 dF.$$

We use the following restatement of Theorem 2.2 in the proof of Theorem 2.3. See Theorem 1.3 of Dudley and Philipp [1].

Theorem 2.5. *Suppose that $\int f^2 dF < \infty$. Then there exists a sequence $\mathbf{W}_1, \mathbf{W}_2, \ldots$ IID copies of $\mathbf{W}$ with*

$$\bar{\mathbf{W}}_n = \frac{\mathbf{W}_1 + \mathbf{W}_2 + \cdots + \mathbf{W}_n}{n}$$

such that

$$\sqrt{n} \sup_{t\in\mathbb{R}} \Big| \int_{-\infty}^{t} f(x)(\mathbf{P}_n - P - \bar{\mathbf{W}}_n)(dx) \Big| \to^P 0 \ \textit{as}\ n \to \infty.$$

The $\mathbf{W}_i$'s can also be chosen such that, with probability 1 for some measurable Y_n

$$\sqrt{n} \Big| \int f(x)(\mathbf{P}_n - P - \bar{\mathbf{W}}_n)(dx) \Big| \leq Y_n = o(\sqrt{\log\log n}).$$

Consider $\{\tilde{\mathbf{W}}_n\}$ defined by

$$\tilde{\mathbf{W}}_n = \sqrt{n} \int f(x) \bar{\mathbf{W}}_n(dx). \tag{10}$$

Then $\{\tilde{\mathbf{W}}_n\}$ is a Gaussian process with mean zero and the variance is given by

$$\operatorname{var}(\tilde{\mathbf{W}}_n) = \sigma_f^2.$$

The following corollary is easily followed from the above Theorem 2.5.

Corollary 2.6. *Suppose that $\int f^2 dF < \infty$. Then there exists a sequence $\{\tilde{\mathbf{W}}_n : n \geq 1\}$ of copies of a Gaussian random variable $\mathbf{W}$ defined on $(\Omega, \mathcal{T}, P)$ such that $|U_n - \tilde{\mathbf{W}}_n| \to^P 0$ as $n \to \infty$. The $\mathbf{W}_i$'s can also be chosen such that, with probability 1 for some measurable Y_n, $|U_n - \tilde{\mathbf{W}}_n| \leq Y_n = o(\sqrt{\log\log n})$.*

Proof. Define $\tilde{\mathbf{W}}_n$ as in (10) whose existence is guaranteed by Theorem 2.5. Observe that $U_n \Rightarrow \mathbf{W}$ as a random elements of $D(\mathbb{R})$. Observe also that

$$|U_n - \tilde{\mathbf{W}}_n| = \sqrt{n} \left| \int f d(\mathbf{P}_n - P - \bar{\mathbf{W}}_n) \right|.$$

Since the Gaussian random variable $\mathbf{W}$ and $\tilde{\mathbf{W}}$ have the same mean and variance, they have the same distribution. Theorem 2.2 implies the results. □

In the proof of Theorem 2.3 we will use the following LIL for the Gaussian random variables whose proof appears in Theorem 7.21 in Karr [4].

Proposition 2.7 ([4, Theorem 7.21 of Karr])**.** *Let $\{\mathbf{W}_i : i \geq 1\}$ be a sequence of IID copies of a Gaussian random variable $\mathbf{W}$, defined on $(\Omega, \mathcal{T}, P)$, with $E\mathbf{W} = 0$ and $E\mathbf{W}^2 < \infty$. Then $\mathbf{W}$ satisfies the LIL. That is, the set of limit points of*

$$\left\{ \frac{\sum_{i=1}^n \mathbf{W}_i}{\sqrt{2n \log\log n}} : n \geq 3 \right\}$$

is, with probability 1, the closed interval

$$\left[-\left(E\mathbf{W}^2\right)^{1/2}, \left(E\mathbf{W}^2\right)^{1/2} \right].$$

Proof of Theorem 2.3. Let $\mathbf{W}$ be a Gaussian random variable with mean zero and variance is

$$\operatorname{var}(\mathbf{W}) = \int f^2 dF. \tag{11}$$

Apply Corollary 2.6 to choose a sequence $\{\mathbf{W}_i : i \geq 1\}$ of IID copies of $\mathbf{W}$ and a measurable sequence Y_n's such that, with probability 1,

$$\frac{|U_n - \tilde{\mathbf{W}}_n|}{\sqrt{2\log\log n}} \leq Y_n = o(1). \tag{12}$$

Notice that

$$\tilde{\mathbf{W}}_n = \frac{1}{\sqrt{n}} \sum_{i=1}^n \mathbf{W}_i$$

By Proposition 2.7, $\mathbf{W}$ satisfies the LIL. That is, the set of limit points of

$$\left\{ \frac{\sum_{i=1}^n \mathbf{W}_i}{\sqrt{2n \log\log n}} : n \geq 3 \right\}$$

is, with probability 1, the closed interval $[-\left(E\mathbf{W}^2\right)^{1/2}, \left(E\mathbf{W}^2\right)^{1/2}]$. This, together with (11) and (12), completes the proof of Theorem 2.3. □

References

[1] R. M. Dudley and W. Philipp, *Invariance principles for sums of Banach space valued random elements and empirical processes*, Z. Wahrsch. Verw. Gebiete **62** (1983), no. 4, 509–552.

[2] E. Giné and J. Zinn, *Some limit theorems for empirical processes*, Ann. Probab. **12** (1984), no. 4, 929–998.

[3] I. Karatzas, *Lectures on the Mathematics of Finance*, CRM Monograph Series, 8. American Mathematical Society, Providence, RI, 1997.

[4] A. F. Karr, *Probability*, Springer-Verlag, New York, 1993.

[5] M. Ossiander, *A central limit theorem under metric entropy with L_2 bracketing*, Ann. Probab. 15 (1987), no. 3, 897–919.

[6] D. Pollard, *Convergence of Stochastic Processes*, Springer series in Statistics, Springer-Verlag, New York, 1984.

[7] ———, *Empirical Processes*, Regional conference series in Probability and Statistics 2, Inst. Math. Statist., Hayward, CA, 1990.

[8] A. W. van der Vaart and J. A. Wellner, *Weak Convergence and Empirical Processes*, Springer series in Statistics, Springer-Verlag, New York, 1996.

Sungyeun Kim
Institute of Basic Science
Daejin University
Pochen 487-711, Korea
E-mail address: sykim@daejin.ac.kr

Wonbae Kim
Department of Mathemaics
Daejin University
Pochen 487-711, Korea
E-mail address: wbkim@daejin.ac.kr

Jongsig Bae
Department of Mathematics
Sungkyundwan University
Suwon 440-746, Korea
E-mail address: jsbae@skku.edu