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# MANIFOLDS WITH TRIVIAL HOMOLOGY GROUPS IN SOME RANGE AS CODIMENSION-$K$ FIBRATORS

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Abstract. Approximate fibrations provide a useful class of maps. Fibrators give instant detection of maps in this class, and PL fibrators do the same in the PL category. We show that rational homology spheres with some additional conditions are codimension-$k$ PL fibrators and PL manifolds with trivial homology groups in some range can be codimension-$k$ ($k > 2$) PL fibrators.

## 1. Introduction

Approximate fibrations form a very useful class of maps, because of the presence of several associated sequences displaying computable homotopical and homological relationships involving the domain, image and typical fiber. As a consequence, we seek to identify homotopy types by means of which a proper map defined on an arbitrary manifold of a given dimension can be recognized as an approximate fibration, simply because all point preimages have the specified homotopy type. More precisely, the goal is to present closed $n$-manifolds $N$ which force proper maps $p : M \to B$ to be approximate fibrations, when $M$ is a connected $(n + k)$-manifold and each $p^{-1}(b)$ has the homotopy type of $N$. Such a manifold $N$ is called a codimension-$k$ fibrator.

To explain what all this means and to limit the focus somewhat, we begin by presenting the notation and fundamental terminology to be employed throughout: Assume all manifolds are orientable. $M$ is a connected $(n + k)$-manifold and $p : M \to B$ is a proper map of $M$ to a space $B$ such that each $p^{-1}(b)$ has the homotopy type of a closed, connected $n$-manifold. Such a map $p$ will be called a *codimension-$k$* map. When $N$ is a fixed PL $n$-manifold, $M$ is a PL manifold, $B$ is a polyhedron, and $p : M \to B$ is a PL map, then $p$ is said to be *$N$-like* if each $p^{-1}(b)$ collapses to an $n$-complex homotopy equivalent to $N$. We call $N$ a *codimension-$k$ PL fibrator* if, for every PL $(n + k)$-manifold $M$ and

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$N$-like PL map $p : M \to B$, $p$ is an approximate fibration. Finally, if $N$ is a codimension-$k$ PL fibrator for all $k > 0$, we simply call $N$ a *PL fibrator*.

We will call a PL $n$-manifold $N$ a *codimension-k PL m-fibrator* if every codimension-$k$ $N$-like PL map $p : M^{n+k} \to B$ onto a $k$-manifold $B$ is an approximate fibration, and we call $N$ simply a *PL m-fibrator* if it is a codimension-$k$ PL $m$-fibrator for all $k > 0$.

In this paper, we show that rational homology spheres with some additional conditions are codimension-$k$ PL fibrators and PL manifolds $N$ with trivial homology groups in some range can be codimension-$k$ ($k > 2$) PL fibrators.

## 2. Partially acyclic manifolds as codimension-$k$ fibrators

Throughout this paper, the symbols $\approx$ and $\cong$ denote homeomorphism and isomorphism in that order, and homology groups will be computed with integer coefficients unless specified.

We begin by presenting the notation and fundamental terminology to be employed throughout.

A group $G$ is said to be: *hopfian* if each epimorphism $G \to G$ is an isomorphism; *cohopfian* if each monomorphism $G \to G$ is an isomorphism; and *normally cohopfian* if each monomorphism $G \to G$ with image a normal subgroup of $G$ is an isomorphism.

A group $G$ is *sparsely Abelian* if it contains no nontrivial normal subgroup $A$ such that $G/A$ is isomorphic to a normal subgroup of $\Gamma$. Groups $G$ that are both sparsely Abelian and normally cohopfian have the useful feature that every homomorphism $G \to G$ with, at worst, Abelian kernel necessarily is an automorphism. For brevity a group $G$ which is both normally cohopfian and sparsely Abelian will be said to have Property NCSA.

The (absolute) degree of a map is computed with integer coefficents and is understood to be a nonnegative number. Explicitly, a map $f : N \to N'$ between closed, orientable $n$-manifolds is said to have *degree d* if there are choices of generators $\gamma \in H_n(N;\mathbb{Z}), \gamma' \in H_n(N';\mathbb{Z})$ such that $f_*(\gamma) = d\gamma'$, where $d \geq 0$ is an integer. A closed, orientable manifold $N$ is said to be *hopfian* if every degree 1 map $N \to N$ which induces an isomorphism at the fundamental group level is a homotopy equivalence. As a result, when $\pi_1(N)$ is a hopfian group, $N$ is a hopfian manifold if and only if all degree 1 maps $N \to N$ are homotopy equivalences.

A codimension-$k$ map $p : M^{n+k} \to B$ is said to have Property $\mathcal{R}\cong$ if for each $x \in B$, a retraction $\mathcal{R} : p^{-1}(U) \to p^{-1}(x)$ defined on some open neighborhood $U$ of $x$ in $B$ induces an isomorphism $(\mathcal{R}|p^{-1}(y))_* : \pi_1(p^{-1}(y)) \to \pi_1(p^{-1}(x))$ for all $y \in U$.

A codimension-$k$ map $p : M^{n+k} \to B$ is said to have Property $\mathcal{R}_i^{\cong}$ if for each $x \in B$, a retraction $\mathcal{R} : p^{-1}(U) \to p^{-1}(x)$ defined on some open neighborhood $U$ of $x$ in $B$ induces an isomorphism $(\mathcal{R}|p^{-1}(y))_* : H_i(p^{-1}(y)) \to H_i(p^{-1}(x))$ for all $y \in U$. The *continuity set* of $p$ consists of all $x \in B$ equipped with such a neighborhood $U$ such that the associated $\mathcal{R} : p^{-1}(U) \to p^{-1}(x)$ restricts to an isomorphism $(\mathcal{R}|p^{-1}(y))_* : H_n(p^{-1}(y)) \to H_n(p^{-1}(x))$ for all $y \in U$. Establishing that $B$ equals the continuity set of $p$ is a cornerstone for showing an $N$-like map $p$ is an approximate fibration.

To show that rational homology spheres with some additional conditions are codimension $k$-fibrators, we begin with the following proposition and lemma.

**Proposition 2.1** ([6]). *Let $p : M^{n+k} \to \mathbb{R}^k$ be a codimension-k map from a $(n+k)$-manifold $M^{n+k}$ onto Euclidian k-space such that each fiber is homotopy equivalent to a closed n-manifold $N$. Suppose that p is an approximate fibration over $\mathbb{R}^k \setminus \mathbf{0}$. Then p has Property $\mathcal{R}^{\cong}_i$ for all $i < k-1$. Furthermore, if p has Property $\mathcal{R}^{\cong}_{k-1}$, then for all $y \in \mathbb{R}^k \setminus \mathbf{0}$, the degree of map $\mathcal{R}|p^{-1}(y) : p^{-1}(y) \to p^{-1}(\mathbf{0})$ is one.*

**Lemma 2.2.** *Let $N$ be a hopfian n-manifold such that $N$ is a codimension $(k-1)$ PL fibrator $(k \geq 3)$, $H_1(N)$ and $H_{k-1}(N)$ are finite. Then every $N$ like PL map $p : M^{n+k} \to B$ from a $(n+k)$-manifold $M$ to a k-manifold $B$, $k < n$, is an approximate fibration provided either*
(1) *$H_1(N) = 0$, or*
(2) *$H_{k-1}(N) = 0$, or*
(3) *$H_1(N)$ and $H_{k-1}(N)$ are relatively prime.*

*Proof.* Let $p : M \to B$ be a $N$-like PL map from a $(n+k)$-manifold $M$ to a $k$-manifold $B$, and $L$ and $S$ be the link and the star of an arbitrary vertex $v$ in $B$ respectively. It is enough to show that $p|S' : S' \to S$ is an approximate fibration. Since $B$ is a manifold and $k \geq 3$, $\pi_1(L) = 0$ and it is easy to check that $p$ has Property $R^{\cong}$ by the homotopy exact sequence of an approximate fibration $p|L' : L' \to L$.

First, the homology sequence of $(S', L')$ implies

$$\cdots \longrightarrow H_k(S', L') \longrightarrow H_{k-1}(L') \longrightarrow H_{k-1}(S') \longrightarrow H_{k-1}(S', L') \longrightarrow \cdots,$$

where the first term is $H_k(S', L') \cong H^n(N) \cong \mathbb{Z}$ and the last term is $H_{k-1}(S', L') \cong H^{n+1}(N) \cong \mathbf{0}$ by duality. So, $incl_* : H_{k-1}(L') \to H_{k-1}(S')$ is surjective. Now consider the following Wang homology sequence of $p|L'$

$$\cdots H_1(N) \to H_{k-1}(N) \to H_{k-1}(L') \to H_0(N) \to H_{k-2}(N) \to H_{k-2}(L') \cdots.$$

Here the last homomorphism is an isomorphism by Proposition 2.1. By the hypothesis, we have an injective map $incl_* : H_{k-1}(N) \to H_{k-1}(L')$. In case (2), the degree of map $\mathcal{R}|p^{-1}(y) : p^{-1}(y) \to p^{-1}(v)$ is one by Proposition 2.1. In case (1) or (3), $H_1(N)$ and $H_{k-1}(N)$ are finite. Then we have

$$H_{k-1}(L') \cong \mathbb{Z} \oplus j_*(H_{k-1}(p^{-1}(y))),$$

where $j : p^{-1}(y) \to L'$ denotes the inclusion. Since $j_*$ is injective, the composite map $incl_* : H_{k-1}(N) \to H_{k-1}(S')$ is an isomorphism and the degree of map $\mathcal{R}|p^{-1}(y) : p^{-1}(y) \to p^{-1}(v)$ is one by Proposition 2.1. Since $N$ is a hopfian manifold, $\mathcal{R}|p^{-1}(y) : p^{-1}(y) \to p^{-1}(v)$ is a homotopy equivalence and we have the conclusion [1]. □

We have the following results from Lemma 2.2.

**Theorem 2.3.** *Suppose a closed hopfian $n$-manifold $N$ is a codimension-2 PL fibrator, and $\beta_i(N) = 0$ for $i = 1, \ldots, k$. Then $N$ is a codimension-$k$ PL $m$-fibrator provided either*
(1) $H_1(N) = 0$*, or*
(2) $H_{k-1}(N) = 0$*, or*
(3) $H_1(N)$ *and* $H_{k-1}(N)$ *are relatively prime.*

*Proof.* Since $H_1(N)$ and $H_{k-1}(N)$ are finite, the conclusion follows from Lemma 2.2. □

**Theorem 2.4.** *Suppose a closed hopfian $n$-manifold $N$ is a codimension-2 PL fibrator, $\pi_1(N)$ has Property NCSA, and $\beta_i(N) = 0$ for $i = 1, \ldots, k$, where $k \leq 5$. Then $N$ is a codimension-$k$ PL fibrator provided either*
(1) $H_1(N) = 0$*, or*
(2) $H_{k-1}(N) = 0$*, or*
(3) $H_1(N)$ *and* $H_{k-1}(N)$ *are relatively prime.*

*Proof.* Suppose $p : M^{n+k} \to B$ is a PL $N$-like map. When $1 < k < 5$, Property NCSA ensures that $B$ is a manifold and the result follows from Lemma 2.2. Consider $k = 5$. The Vietoris-Begle theorem with $\mathbb{Q}$-coefficients then confirms that links $L \subset B$ are 1-connected $\mathbb{Q}$-homology $(k-1)$-spheres. As a result, $H_2(L; \mathbb{Z}) = 0$ (since the second homology of a 1-connected 4-manifold is free), and $L$ is the homotopy $(k-1)$-sphere. Thus $B$ is a simplicial homotopy $k$-manifold. Lemma 2.2 applies that $N$ is a codimension-$k$ PL fibrator . □

**Theorem 2.5.** *Suppose a closed hopfian 5-manifold $N$ is a codimension-2 PL fibrator, $\beta_1(N) = \beta_2(N) = 0$, and $\pi_1(N)$ has Property NCSA. Then $N$ is a codimension-5 PL fibrator.*

*Proof.* By Theorem 2.4, it suffices to check that any $N$-like PL map $p : M \to B$ has manifold image. For $k \leq 4$, $b$ is a manifold because of Property NCSA. We have $H_4(N) \cong H^1(N) = 0$ by duality. The $k = 5$ case relies upon [5, Lemma 5.12] and also the argument in the ealier Theorem 2.4 that any $N$-like PL map $p : M \to B$ has manifold image. □

**Corollary 2.6.** *Suppose a closed hopfian 5-manifold $N$ is $\beta_1(N) = \beta_2(N) = 0$, and $\pi_1(N)$ is hyperhopfian, normally cohopfian, and contains no nontrivial abelian normal subgroup. Then $N$ is a codimension-5 PL fibrator.*

*Proof.* $N$ is a codimension-2 fibrator [3] and $\pi_1(N)$ has Property NCSA. The conclusion follows from Theorem 2.4. □

Now we show that some manifolds with trivial homology groups in some range are codimension-$k$ PL fibrator.

**Lemma 2.7.** *Suppose a closed n-manifold N is a codimension-(k–1) PL fibrator, $H_i(N) \cong 0$ for $2 \leq i \leq t$, and $\pi_1(N)$ has Property NCSA. Suppose $p : M^{n+k} \to B$ is an N-like PL map defined on a manifold $M^{n+k}$, where $k \leq 2t + 2$. Then B is a k-manifold; furthermore, if $\chi(N) \neq 0, B$ is a $2t + 3$-manifold.*

*Proof.* It suffices to show that the link $L$ of an arbitrary vertex $v$ in $B$ is a homotopy ($k$-1)-sphere. Set $L' = p^{-1}(L)$. Since $p|L' : L' \to L$ is an approximate fibration, by hypothesis, $L$ must be a ($k$-1)-manifold [3, Theorem 5.4]. See [5, Lemma 2.1] about its being a sphere for the initial cases $k \leq 2$, and assume $k > 2$.

Since $\pi_1(N)$ has Property NCSA, $L$ is simply connected. Since $p$ is an approximate fibration over $L$, we have the following Serre exact sequence

$$H_2(p^{-1}(y)) \to H_2(L') \to H_2(L) \to H_1(p^{-1}(y)) \to H_1(L') \cdots ,$$

where $y \in L$, and we have an isomorphism $H_i(L') \to H_i(S')$ for $i \leq k - 2$ from the homology exact sequence of the pair $(S', L')$. Then $H_2(L) = 0$ because $H_2(N) = 0$. Therefore, $\pi_2(N) = 0$ by the Hurewicz theorem.

By the induction and repeating the Serre exact sequence, $H_i(L) = 0$ for $0 < i \leq t$. For $2 < k \leq 2t + 2$, $L$ is a homotopy sphere by the Poincaré duality and the Hurewicz theorem.

For $k = 2t + 3$, $L$ in $B$ is an $t$-connected ($2t$+2)-manifold. Routine computation indicates that $\chi(L) = 2$. As a result, $\beta_{t+1}(L) = 0$, and hence $H_{t+1}(L)$ is trivial, since $H_{t+1}(L) \cong H^{t+1}(L)$ has torsion isomorphic to $H_t(L) \cong 0$. Being a 1-connected homology ($2t$+2)-sphere, $L$ is a homotopy ($2t$+2)-sphere. Therefore, $B$ is a manifold. □

**Theorem 2.8.** *Suppose a closed hopfian n-manifold N is a codimension-2 fibrator, $\pi_1(N)$ has Property NCSA, and t is an integer such that $H_i(N) \cong 0$ for $2 \leq i \leq t$. Then N is a codimension-(t+1) PL fibrator.*

*Proof.* The argument proceeds by the induction. Assume $N$ is a codimension-$s$ PL fibrator, $2 \leq s < t + 1$, and consider an $N$-like PL map $p : M \to B$ defined on $M$, a ($n$+$s$+1)-manifold. Lemma 2.7 assures that $B$ is a manifold, and Proposition 2.1 promises that the degree of map $\mathcal{R}|p^{-1}(y) : p^{-1}(y) \to p^{-1}(v)$ is one for any vertex $v$. Since $N$ is a hopfian manifold and $\pi_1(N)$ has Property NCSA, $\mathcal{R}|p^{-1}(y) : p^{-1}(y) \to p^{-1}(v)$ is a homotopy equivalence and $N$ is a codimension-($s$+1) PL fibrator, as required [1]. □

**Theorem 2.9.** *Suppose a closed hopfian n-manifold N is a codimension-2 fibrator, $\pi_1(N)$ has Property NCSA, and $H_i(N) \cong 0$ for $2 \leq i \leq n - 2$ and $\beta_1(N) > 1$. Then N is a codimension-(2n–2) PL fibrator; furthermore, if $\chi(N) \neq 0, N$ is a codimension-(2n+1) PL fibrator.*

*Proof.* $N$ is a codimension $(n - 1)$-fibrator by Theorem 2.8. Assume that $N$ is a codimension $(k - 1)$ PL fibrator for $n \leq k \leq 2n - 1$. Suppose $p : M \to B$ is a $N$-like PL map from arbitrary $(n + k)$-manifold to a polyhedron. According to

Lemma 2.7, $B$ is a manifold. Concentrate on the case $k = n$; the case $k > n$ is similar and easier. The Serre exact sequence for an approximate fibration $p|L'$, where $L$ is an arbitrary link and $L' = p^{-1}(L)$, yields

$$H_{n-1}(L') \cong \mathbb{Z} \oplus j_*(H_{n-1}(p^{-1}(z)))$$

($j : p^{-1}(z) \to L'$ denoting inclusion). Furthermore, the long exact sequence for the pair $(S', L')$ reveals that the inclusion-induced

$$\phi_* : H_{n-1}(L') \to H_{n-1}(S')$$

is surjective, as the next term is trivial by duality. In view of the assumption that $\beta_1(N) = \beta_{n-1}(N) > 1$, there exists an indivisible element of $H_{n-1}(M) \cong H_{n-1}(p^{-1}(\mathbf{0}))$ in $\phi_* j_*(H_{n-1}(p^{-1}(z)))$ (Remark: $a \in H_{n-1}(M)$ is *indivisible* if $d{\cdot}a' = a$ for some $a' \in H_{n-1}(M)$ and $d \in \mathbb{Z}$ implies $d = \pm 1$). By Proposition 2.1 and universal coefficients,

$$j^*\phi^* : H^1(M)(\cong H^1(p^{-1}(\mathbf{0}))) \to H^1(p^{-1}(z))$$

is an isomorphism. From the Addendum to [5, Lemma 6.1], the degree of $j \circ \phi$ is one. Since $N$ is a hopfian manifold and $\pi_1(N)$ has Property NCSA, we have the conclusion. □

**Corollary 2.10.** *Suppose $N$ is a closed hopfian $2n$-manifold satisfying*: (i) $\pi_1(N)$ *is hopfian,* (ii) $\pi_1(N)$ *has Property NCSA,* (iii) $\beta_1(N) > 1$, *and* (iv) $H_i(N) = 0$ *for* $1 < i \leq n$. *Then $N$ is a codimension-*$(4n-1)$ *PL fibrator.*

*Proof.* It is easy to check that $\chi(N) = 2 - \beta_1(N) \neq 0$, and so $N$ is a codimension-2 fibrator. By the Poincaré duality, $H_i(N) = 0$ for $1 < i \leq 2n-2$, and so Theorem 2.9 implies that $N$ is a codimension-$(4n-1)$ PL fibrator. □

**Theorem 2.11.** *Suppose a closed hopfian $2n$-manifold $N$ is a codimension-2 fibrator, $\pi_1(N)$ has Property NCSA, and $H_i(N) \cong 0$ for $2 \leq i \leq n-1$ and $\beta_n(N) > 2$. Then $N$ is a codimension-*$(2n)$ *PL fibrator*; *furthermore, if $\chi(N) \neq 0, N$ is a codimension-*$(2n{+}1)$ *PL fibrator.*

*Proof.* According to Theorem 2.9, $N$ is a codimension-$n$ fibrator. Assume that $N$ is a codimension-$(k-1)$ PL fibrator for $n+1 \leq k \leq 2n$. Suppose $p : M \to B$ is a $N$-like PL map from arbitrary $(n+k)$-manifold to a polyhedron. According to Lemma 2.7, $B$ is a manifold. Concentrate on the case $k = n+1$; the case $k > n+1$ is similar and easier.

Working with the inclusion $\phi j : p^{-1}(z) \to S', z \in L$, described in Theorem 2.9, we repeat the analysis there, which indicates $H_n(S') \cong H_n(p^{-1}(\mathbf{0}))$ is isomorphic to either $\phi_* j_*(H_n(p^{-1}(z)))$ or $T \oplus \phi_* j_*(H_n(p^{-1}(z)))$, where $T$ is cyclic. By the Universal Coefficient Theorem for Cohomology, $H^n(p^{-1}(z))$ then is isomorphic to $j^*\phi^*(H^n(S'))$ or $T \oplus j^*\phi^*(H^n(S'))$. Consequently, there exists a choice of an indivisible $\xi \in H_n(p^{-1}(\mathbf{0}))$ for which the proof of Theorem 2.9 applies, that is, corresponding to $\xi$ is some $\nu \in H_n(p^{-1}(\mathbf{0}))$ with

$\xi = \mathcal{R}_*(\eta \frown \mathcal{R}^*(\nu))$. Just as before, the map $\mathcal{R} : p^{-1}(z) \to p^{-1}(\mathbf{0})$ has (absolute) degree 1. Since $N$ is a hopfian manifold, we have the conclusion as in the proof of Theorem 2.9. □

**Corollary 2.12.** *Suppose $N$ is a closed hopfian $2n$-manifold satisfying:* (i) $\pi_1(N)$ *is hopfian,* (ii) $\pi_1(N)$ *has Property NCSA,* (iii) $\beta_n(N) > 2$, *and* (iv) $H_i(N) = 0$ *for* $1 < i \leq n-1$. *Then $N$ is a codimension-$(2n+1)$ PL fibrator.*

*Proof.* It is easy to check that $\chi(N) = 2 - \beta_1(N) \neq 0$, and so $N$ is a codimension-2 fibrator. By Theorem 2.11, $N$ is a codimension-$(2n+1)$ PL fibrator. □

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