Commun. Korean Math. Soc. **25** (2010), No. 2, pp. 273–282
DOI 10.4134/CKMS.2010.25.2.273

INTUITIONISTIC FUZZY θ-CLOSURE AND θ-INTERIOR

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Abstract. The concept of intuitionistic fuzzy θ-interior operator is introduced and discussed in intuitionistic fuzzy topological spaces. As applications of this concept, intuitionistic fuzzy strongly θ-continuous, intuitionistic fuzzy θ-continuous, and intuitionistic fuzzy weakly continuous functions are characterized in terms of intuitionistic fuzzy θ-interior operator.

1. Introduction

As a generalization of fuzzy sets, the concept of intuitionistic fuzzy set was introduced by Atanassov [1]. Recently, Çoker and his colleagues [2, 3, 4] introduced intuitionistic fuzzy topological space using intuitionistic fuzzy sets. Mukherjee introduced the concepts of fuzzy θ-closure operator in [9] and the notions of fuzzy θ-continuous and fuzzy weakly continuous functions in [8]. Hanafy et al. introduced and investigated intuitionistic fuzzy θ-closure operator, intuitionistic fuzzy strongly θ-continuous, intuitionistic fuzzy θ-continuous and intuitionistic fuzzy weakly continuous functions in [6]. In this paper, we define intuitionistic fuzzy θ-interior operator and study the properties of intuitionistic fuzzy θ-interior operator in intuitionistic fuzzy topological spaces. As applications of this concept, intuitionistic fuzzy strongly θ-continuous, intuitionistic fuzzy θ-continuous, and intuitionistic fuzzy weakly continuous functions are characterized in terms of intuitionistic fuzzy θ-interior operator.

2. Preliminaries

Let X be a nonempty set and I the unit interval $[0, 1]$. An *intuitionistic fuzzy set* (IFS for short) A is an object having the form

$$A = \{\langle x, \mu_A(x), \gamma_A(x)\rangle : x \in X\},$$

where the functions $\mu_A : X \to I$ and $\gamma_A : X \to I$ denote the degree of membership and the degree of nonmembership, respectively, and $\mu_A + \gamma_A \leq 1$.

Received June 24, 2009.
2000 *Mathematics Subject Classification.* 54A40.
Key words and phrases. intuitionistic fuzzy θ-interior, intuitionistic fuzzy strongly θ-continuous, intuitionistic fuzzy θ-continuous, intuitionistic fuzzy weakly continuous.

Sometimes we denote $A = (\mu_A, \gamma_A)$ for simplicity. Let $I(X)$ denote the set of all intuitionistic fuzzy sets in X.

Obviously, every fuzzy set μ_A in X is an intuitionistic fuzzy set of the form $\{\langle x, \mu_A(x), 1 - \mu_A(x)\rangle : x \in X\}$.

Definition 2.1 ([1])**.** Let X be a nonempty set and the IFSs A and B be of the form $A = \{\langle x, \mu_A(x), \gamma_A(x)\rangle : x \in X\}$, $B = \{\langle x, \mu_B(x), \gamma_B(x)\rangle : x \in X\}$. Then

(1) $A \leq B$ if and only if $\mu_A(x) \leq \mu_B(x)$ and $\gamma_A(x) \geq \gamma_B(x)$ for all $x \in X$,
(2) $A = B$ if and only if $A \leq B$ and $B \leq A$,
(3) $A^c = \{\langle x, \gamma_A(x), \mu_A(x)\rangle : x \in X\}$,
(4) $A \cap B = \{\langle x, \mu_A \wedge \mu_B(x), \gamma_A \vee \gamma_B(x)\rangle : x \in X\}$,
(5) $A \cup B = \{\langle x, \mu_A \vee \mu_B(x), \gamma_A \wedge \gamma_B(x)\rangle : x \in X\}$,
(6) $0_\sim = \{\langle x, \tilde{0}, \tilde{1}\rangle : x \in X\}$ and $1_\sim = \{\langle x, \tilde{1}, \tilde{0}\rangle : x \in X\}$.

Definition 2.2 ([2])**.** Let X and Y be two nonempty sets, and let $f : X \to Y$ be a function.

(1) If $B = \{\langle y, \mu_B(y), \gamma_B(y)\rangle : y \in Y\}$ is an IFS in Y, then *the preimage of* B under f, denoted by $f^{-1}(B)$, is the IFS in X defined by

$$f^{-1}(B) = \{\langle x, f^{-1}(\mu_B)(x), f^{-1}(\gamma_B)(x)\rangle : x \in X\}.$$

(2) If $A = \{\langle x, \lambda_A(x), \delta_A(x)\rangle : x \in X\}$ is an IFS in X, then *the image of* A under f, denoted by $f(A)$, is the IFS in Y defined by

$$f(A) = \{\langle y, f(\lambda_A)(y), (1 - f(1 - \delta_A))(y)\rangle : y \in Y\},$$

where

$$f(\lambda_A)(y) = \begin{cases} \sup\limits_{x \in f^{-1}(y)} \lambda_A(x) & \text{if } f^{-1}(y) \neq \phi \\ 0 & \text{otherwise,} \end{cases}$$

$$(1 - f(1 - \delta_A))(y) = \begin{cases} \inf\limits_{x \in f^{-1}(y)} \lambda_A(x) & \text{if } f^{-1}(y) \neq \phi \\ 1 & \text{otherwise.} \end{cases}$$

Theorem 2.3 ([2])**.** *Let A and A_j $(j \in J)$ be IFSs in X, B and B_j $(j \in K)$ IFSs in Y. Let $f : X \to Y$ be a function. Then*

(1) $A_1 \leq A_2 \Rightarrow f(A_1) \leq f(A_2)$,
(2) $B_1 \leq B_2 \Rightarrow f^{-1}(B_1) \leq f^{-1}(B_2)$,
(3) $A \leq f^{-1}(f(A))$ *(If f is injective, then $A = f^{-1}(f(A))$)*,
(4) $f(f^{-1}(B)) \leq B$ *(If f is surjective, then $B = f(f^{-1}(B))$)*,
(5) $f^{-1}(\bigcup B_j) = \bigcup f^{-1}(B_j)$,
(6) $f^{-1}(\bigcap B_j) = \bigcap f^{-1}(B_j)$,
(7) $f(\bigcup A_j) = \bigcup f(A_j)$,
(8) $f(\bigcap A_j) \leq \bigcap f(A_j)$, *(If f is injective, then $f(\bigcap A_j) = \bigcap f(A_j)$)*,
(9) $f^{-1}(\tilde{1}) = \tilde{1}$, *if f is surjective,*
(10) $f(\tilde{0}) = \tilde{0}$,
(11) $f(A)^c \leq f(A^c)$, *if f is surjective,*

(12) $f^{-1}(B^c) = f^{-1}(B)^c$.

Definition 2.4 ([2]). An *intuitionistic fuzzy topology* (IFT for short) on a nonempty set X is a family $\mathcal{T}$ of IFSs in X which satisfies the following axioms:

(1) $0_\sim, 1_\sim \in \mathcal{T}$.
(2) $G_1 \cap G_2 \in \mathcal{T}$ for any $G_1, G_2 \in \mathcal{T}$.
(3) $\bigcup G_i \in \mathcal{T}$ for any arbitrary $\{G_i : i \in J\} \leq \mathcal{T}$.

In this case the pair $(X, \mathcal{T})$ is called an *intuitionistic fuzzy topological space* (IFTS for short) and any IFS in $\mathcal{T}$ is known as an *intuitionistic fuzzy open set* (IFOS for short) in X.

Definition 2.5 ([2]). Let $(X, \mathcal{T})$ be an IFTS and $A = \langle x, \mu_A, \lambda_A \rangle$ an IFS in X. Then the *intuitionistic fuzzy interior of* A and the *intuitionistic fuzzy closure of* A are defined by

$$\mathrm{cl}(A) = \bigcap\{K \mid A \leq K, K^c \in \mathcal{T}\}$$

and

$$\mathrm{int}(A) = \bigcup\{G \mid G \leq A, G \in \mathcal{T}\}.$$

Theorem 2.6 ([2]). *For any IFS A in $(X, \mathcal{T})$, we have*

$$\mathrm{cl}(A^c) = (\mathrm{int}(A))^c \quad \textit{and} \quad \mathrm{int}(A^c) = (\mathrm{cl}(A))^c.$$

Definition 2.7 ([3, 4]). Let $\alpha, \beta \in [0, 1]$ and $\alpha + \beta \leq 1$. An *intuitionistic fuzzy point* (IFP for short) $x_{(\alpha,\beta)}$ of X is an IFS in X defined by

$$x_{(\alpha,\beta)}(y) = \begin{cases} (\alpha, \beta) & \text{if} \quad y = x, \\ (0, 1) & \text{if} \quad y \neq x. \end{cases}$$

In this case, x is called the *support* of $x_{(\alpha,\beta)}$, α the *value* of $x_{(\alpha,\beta)}$ and β the *nonvalue* of $x_{(\alpha,\beta)}$. An IFP $x_{(\alpha,\beta)}$ is said to *belong* to an IFS $A = (\mu_A, \gamma_A)$ in X, denoted by $x_{(\alpha,\beta)} \in A$, if $\alpha \leq \mu_A(x)$ and $\beta \geq \gamma_A(x)$.

Remark 2.8. If we consider an IFP $x_{(\alpha,\beta)}$ as an IFS, then we have the relation $x_{(\alpha,\beta)} \in A$ if and only if $x_{(\alpha,\beta)} \leq A$.

Definition 2.9 ([3, 4]). Let $x_{(\alpha,\beta)}$ be an IFP in X and $U = (\mu_U, \gamma_U)$ an IFS in X. Suppose further that α and β are real numbers between 0 and 1. The IFP $x_{(\alpha,\beta)}$ is said to be *properly contained* in U if and only if $\alpha < \mu_U(x)$ and $\beta > \gamma_U(x)$.

Definition 2.10 ([4]). (1) An IFP $x_{(\alpha,\beta)}$ is said to be *quasi-coincident* with the IFS $U = \langle x, \mu_U, \gamma_U \rangle$, denoted by $x_{(\alpha,\beta)} q U$, if and only if $\alpha > \gamma_U(x)$ or $\beta < \mu_U(x)$.

(2) Let $U = (\mu_U, \gamma_U)$ and $V = (\mu_V, \gamma_V)$ be two IFSs in X. Then U and V are said to be *quasi-coincident*, denoted by UqV, if and only if there exists an element $x \in X$ such that $\mu_U(x) > \gamma_V(x)$ or $\gamma_U(x) < \mu_V(x)$.

The word 'not quasi-coincident' will be abbreviated as $\widetilde{q}$.

Proposition 2.11 ([4]). *Let U, V be IFSs and $x_{(\alpha,\beta)}$ an IFP in X. Then*

(1) $U\tilde{q}V^c \iff U \leq V$,
(2) $UqV \iff U \nleq V^c$,
(3) $x_{(\alpha,\beta)} \leq U \iff x_{(\alpha,\beta)}\tilde{q}U^c$,
(4) $x_{(\alpha,\beta)}qU \iff x_{(\alpha,\beta)} \nleq U^c$.

Definition 2.12 ([4]). Let $(X, \mathcal{T})$ be an IFTS and $x_{(\alpha,\beta)}$ an IFP in X. An IFS A is called a *neighborhood* (*q-neighborhood*, respectively) of $x_{(\alpha,\beta)}$, if there exists an IFOS U in X such that $x_{(\alpha,\beta)} \in U \leq A$ ($x_{(\alpha,\beta)}qU \leq A$, respectively). The family of all neighborhoods (q-neighborhoods, respectively) of $x_{(\alpha,\beta)}$ will be denoted by $N(x_{(\alpha,\beta)})(N^q(x_{(\alpha,\beta)})$, respectively).

3. Intuitionistic fuzzy θ-closure and θ-interior

In this section, we study some properties of intuitionistic fuzzy θ-interior.

Definition 3.1 ([6]). An IFP $x_{(\alpha,\beta)}$ is said to be *intuitionistic fuzzy θ-cluster point* of an IFS U if and only if $\mathrm{cl}(A)qU$ for each q-neighborhood A of $x_{(\alpha,\beta)}$. The set of all intuitionistic fuzzy θ-cluster points of U is called the *intuitionistic fuzzy θ-closure* of U and denoted by $\mathrm{cl}_\theta(U)$. An IFS U will be called *intuitionistic fuzzy θ-closed* (IFθCS for short) if and only if $U = \mathrm{cl}_\theta(U)$. The complement of an IFθCS is called an intuitionistic fuzzy θ-open set (IFθOS for short).

Remark 3.2. Usually, the complement of a fuzzy set A is defined by $1 - A$, but the complement of an intuitionistic fuzzy set $A = \langle x, \mu_A, \gamma_A\rangle$ is defined by $A^c = \langle x, \gamma_A, \mu_A\rangle$. So

$$1 - A = \langle x, 1 - \mu_A, 1 - \gamma_A\rangle \neq \langle x, \gamma_A, \mu_A\rangle = A^c.$$

Moreover, although A is an intuitionistic fuzzy set, the set $1 - A$ is not necessarily an IFS. In [6], Hanafy defined the intuitionistic fuzzy θ-interior of U by

$$\mathrm{int}_\theta(U) = 1 - \mathrm{cl}_\theta(1 - U).$$

This definition could be misunderstood because of the expression $1-U$. So we rephrase the definition of intuitionistic fuzzy θ-interior as follows.

Definition 3.3. Let $(X, \mathcal{T})$ be an IFTS and U an IFS in X. The *intuitionistic fuzzy θ-interior* of U is denoted and defined by

$$\mathrm{int}_\theta(U) = (\mathrm{cl}_\theta(U^c))^c.$$

From the above definition, we have the following relations:
(1) $\mathrm{cl}_\theta(U^c) = (\mathrm{int}_\theta(U))^c$,
(2) $(\mathrm{cl}_\theta(U))^c = \mathrm{int}_\theta(U^c)$.

Lemma 3.4. *Let U, V and A be IFSs in an IFTS $(X, \mathcal{T})$. If $Aq(U \cup V)$, then AqU or AqV.*

Proof. Suppose that $A\tilde{q}U$ and $A\tilde{q}V$. Then $A \leq U^c$ and $A \leq V^c$. Thus $A \leq U^c \cap V^c = (U \cup V)^c$. Hence $A\tilde{q}(U \cup V)$. □

Theorem 3.5. *Let U and V be two IFSs in an IFTS $(X, \mathcal{T})$. Then we have the following*:

(1) $\mathrm{cl}_\theta(0_\sim) = 0_\sim$,
(2) $U \leq \mathrm{cl}_\theta(U)$,
(3) $U \leq V \Rightarrow \mathrm{cl}_\theta(U) \leq \mathrm{cl}_\theta(V)$,
(4) $\mathrm{cl}_\theta(U) \cup \mathrm{cl}_\theta(V) = \mathrm{cl}_\theta(U \cup V)$,
(5) $\mathrm{cl}_\theta(U \cap V) \leq \mathrm{cl}_\theta(U) \cap \mathrm{cl}_\theta(V)$.

Proof. (1) Obvious.

(2) Suppose that there is an IFP $x_{(\alpha,\beta)}$ in X such that $x_{(\alpha,\beta)} \notin \mathrm{cl}_\theta(U)$ and $x_{(\alpha,\beta)} \in U$. Then there is a q-neighborhood A of $x_{(\alpha,\beta)}$ such that $\mathrm{cl}(A)\widetilde{q}U$. Thus $A \leq U^c$. Since A is a q-neighborhood of $x_{(\alpha,\beta)}$, there is an IFOS V such that $x_{(\alpha,\beta)}qV \leq A$. Since $A \leq U^c$, we have $x_{(\alpha,\beta)}qU^c$, and hence $x_{(\alpha,\beta)} \not\leq U$. On the other hand we have $x_{(\alpha,\beta)} \leq U$, because $x_{(\alpha,\beta)} \in U$. It is a contradiction.

(3) Let $x_{(\alpha,\beta)}$ be an IFP in X such that $x_{(\alpha,\beta)} \notin \mathrm{cl}_\theta(V)$. Then there is a q-neighborhood A of $x_{(\alpha,\beta)}$ such that $\mathrm{cl}(A)\widetilde{q}V$. Since $U \leq V$, we have $\mathrm{cl}(A)\widetilde{q}U$. Therefore $x_{(\alpha,\beta)} \notin \mathrm{cl}_\theta(U)$.

(4) Since $U \leq U \cup V$, $\mathrm{cl}_\theta(U) \leq \mathrm{cl}_\theta(U \cup V)$. Similarly, $\mathrm{cl}_\theta(V) \leq \mathrm{cl}_\theta(U \cup V)$. Hence $\mathrm{cl}_\theta(U) \cup \mathrm{cl}_\theta(V) \leq \mathrm{cl}_\theta(U \cup V)$. On the other hand, take any $x_{(\alpha,\beta)} \in \mathrm{cl}_\theta(U \cup V)$. Then for any q-neighborhood A of $x_{(\alpha,\beta)}$, $\mathrm{cl}(A)q(U \cup V)$. By Lemma 3.4, $\mathrm{cl}(A)qU$ or $\mathrm{cl}(A)qV$. Therefore $x_{(\alpha,\beta)} \in \mathrm{cl}_\theta(U)$ or $x_{(\alpha,\beta)} \in \mathrm{cl}_\theta(V)$. Hence $\mathrm{cl}_\theta(U \cup V) \leq \mathrm{cl}_\theta(U) \cup \mathrm{cl}_\theta(V)$.

(5) Since $U \cap V \leq U$, $\mathrm{cl}_\theta(U \cap V) \leq \mathrm{cl}_\theta(U)$. Similarly, $\mathrm{cl}_\theta(U \cap V) \leq \mathrm{cl}_\theta(V)$. Therefore $\mathrm{cl}_\theta(U \cap V) \leq \mathrm{cl}_\theta(U) \cap \mathrm{cl}_\theta(V)$. □

Remark 3.6. For an IFS A in an IFTS $(X, \mathcal{T})$, intuitionistic fuzzy θ-closure $\mathrm{cl}_\theta(A)$ is not necessarily an IFθCS, and hence $\mathrm{cl}_\theta(\mathrm{cl}_\theta(A)) \neq \mathrm{cl}_\theta(A)$, which is shown in the following example. Thus cl_θ operator does not satisfies the Kuratowski closure axioms.

Example 3.7. Let $X = \{a, b, c\}$ and $U = \langle(\frac{a}{0.5}, \frac{b}{0.6}, \frac{c}{0.2}), (\frac{a}{0.4}, \frac{b}{0.3}, \frac{c}{0.4})\rangle, V = \langle(\frac{a}{0.4}, \frac{b}{0.5}, \frac{c}{0.1}), (\frac{a}{0.6}, \frac{b}{0.5}, \frac{c}{0.4})\rangle$. Then the family $\mathcal{T} = \{\underline{0}, \underline{1}, U, V\}$ of IFSs of X is an IFT on X. Let $A = \langle(\frac{a}{0.3}, \frac{b}{0.3}, \frac{c}{0.0}), (\frac{a}{0.7}, \frac{b}{0.5}, \frac{c}{0.4})\rangle$ be an IFS in X. Then $a_{(0.8,0.1)} \notin \mathrm{cl}_\theta(A)$ and $a_{(0.6,0.4)} \in \mathrm{cl}_\theta(A)$. But $a_{(0.8,0.1)} \in \mathrm{cl}_\theta(a_{(0.6,0.4)}) \leq \mathrm{cl}_\theta(\mathrm{cl}_\theta(A))$. Hence $\mathrm{cl}_\theta(\mathrm{cl}_\theta(A)) \neq \mathrm{cl}_\theta(A)$.

Remark 3.8 ([6]). For any IFS U in IFTS $(X, \mathcal{T})$, $\mathrm{cl}(U) \leq \mathrm{cl}_\theta(U)$. Moreover $\mathrm{cl}(U) = \mathrm{cl}_\theta(U)$ for an IFOS. Thus for any IFS U in IFTS $(X, \mathcal{T})$,

$$\begin{aligned}\mathrm{cl}_\theta(U) &= \bigcap\{\mathrm{cl}_\theta(A) \mid A \in \mathcal{T}, U \leq A\} \\ &= \bigcap\{\mathrm{cl}(A) \mid A \in \mathcal{T}, U \leq A\}.\end{aligned}$$

So, in an intuitionistic fuzzy regular space $(X, \mathcal{T})$, every IFCS is an IFθCS and hence for any IFS U in X, $\mathrm{cl}_\theta(U)$ is an IFθCS.

Clearly, U is an IFθOS if and only if $\mathrm{int}_\theta(U) = U$. Also we have following properties for the interior operator.

Theorem 3.9. *Let U and V be two IFSs in an IFTS $(X, \mathcal{T})$. Then we have the following*:

(1) $\mathrm{int}_\theta(1_\sim) = 1_\sim$,
(2) $\mathrm{int}_\theta(U) \leq U$,
(3) $U \leq V \Rightarrow \mathrm{int}_\theta(U) \leq \mathrm{int}_\theta(V)$,
(4) $\mathrm{int}_\theta(U \cap V) = \mathrm{int}_\theta(U) \cap \mathrm{int}_\theta(V)$,
(5) $\mathrm{int}_\theta(U) \cup \mathrm{int}_\theta(V) \leq \mathrm{int}_\theta(U \cup V)$.

Proof. (1) Obvious.

(2) Let $x_{(\alpha,\beta)} \in \mathrm{int}_\theta(U)$. From the fact that $\mathrm{int}_\theta(U) = (\mathrm{cl}_\theta(U^c))^c = \langle x, \gamma_{\mathrm{cl}_\theta(U^c)}, \mu_{\mathrm{cl}_\theta(U^c)}\rangle$, we have $\alpha \leq \gamma_{\mathrm{cl}_\theta(U^c)}(x)$ and $\beta \geq \mu_{\mathrm{cl}_\theta(U^c)}(x)$. Since $U^c \leq \mathrm{cl}_\theta(U^c)$, we have $\mu_{U^c} \leq \mu_{\mathrm{cl}_\theta(U^c)}$ and $\gamma_{U^c} \geq \gamma_{\mathrm{cl}_\theta(U^c)}$. Thus $\alpha \leq \gamma_{U^c}(x) = \mu_U(x)$ and $\beta \geq \mu_{U^c}(x) = \gamma_U(x)$. Hence $x_{(\alpha,\beta)} \in U$.

(3) Let $U \leq V$. Then $U^c \geq V^c$. By Theorem 3.5, $\mathrm{cl}_\theta(U^c) \geq \mathrm{cl}_\theta(V^c)$. Thus $(\mathrm{cl}_\theta(U^c))^c \leq (\mathrm{cl}_\theta(V^c))^c$. Hence $\mathrm{int}_\theta(U) = (\mathrm{cl}_\theta(U^c))^c \leq (\mathrm{cl}_\theta(V^c))^c = \mathrm{int}_\theta(V)$.

(4) $\mathrm{int}_\theta(U \cap V) = (\mathrm{cl}_\theta((U \cap V)^c))^c = (\mathrm{cl}_\theta(U^c \cup V^c))^c = (\mathrm{cl}_\theta(U^c) \cup \mathrm{cl}_\theta(V^c))^c = (\mathrm{cl}_\theta(U^c))^c \cap (\mathrm{cl}_\theta(V^c))^c = \mathrm{int}_\theta(U) \cap \mathrm{int}_\theta(V)$.

(5) Since $U \leq U \cup V$, we have $\mathrm{int}_\theta(U) \leq \mathrm{int}_\theta(U \cup V)$. Since $V \leq U \cup V$, we have $\mathrm{int}_\theta(V) \leq \mathrm{int}_\theta(U \cup V)$. Therefore $\mathrm{int}_\theta(U) \cup \mathrm{int}_\theta(V) \leq \mathrm{int}_\theta(U \cup V)$. □

Corollary 3.10. *For an IFS U*, $\mathrm{int}_\theta(U) \leq \mathrm{int}(U)$.

Proof. Let U be an IFS. Then U^c is an IFS. Thus $\mathrm{cl}(U^c) \leq \mathrm{cl}_\theta(U^c)$ by [6, Theorem 3.3 (ii)]. Hence $\mathrm{int}_\theta(U) = (\mathrm{cl}_\theta(U^c))^c \leq (\mathrm{cl}(U^c))^c = \mathrm{int}(U)$. □

Theorem 3.11. *If U is an IFCS in an IFTS $(X, \mathcal{T})$, then* $\mathrm{int}_\theta(U) = \mathrm{int}(U)$.

Proof. Let U be an IFCS. Then U^c is an IFOS. Thus $\mathrm{cl}(U^c) = \mathrm{cl}_\theta(U^c)$ by [6, Theorem 3.6]. Hence $\mathrm{int}_\theta(U) = (\mathrm{cl}_\theta(U^c))^c = (\mathrm{cl}(U^c))^c = \mathrm{int}(U)$. □

Theorem 3.12. *Let U be an IFS in an IFTS $(X, \mathcal{T})$. Then*

$$\begin{aligned}\mathrm{int}_\theta(U) &= \bigvee\{\mathrm{int}_\theta(A) \mid A^c \in \mathcal{T}, A \leq U\} \\ &= \bigvee\{\mathrm{int}(A) \mid A^c \in \mathcal{T}, A \leq U\}.\end{aligned}$$

Proof. Using [6, Theorem 3.15], we have

$$\begin{aligned}\mathrm{int}_\theta(U) = (\mathrm{cl}_\theta(U^c))^c &= (\bigwedge\{\mathrm{cl}_\theta(B) \mid B \in \mathcal{T}, U^c \leq B\})^c \\ &= \bigvee\{(\mathrm{cl}_\theta(B))^c \mid B \in \mathcal{T}, U^c \leq B\} \\ &= \bigvee\{\mathrm{int}_\theta(B^c) \mid B \in \mathcal{T}, U^c \leq B\}.\end{aligned}$$

Let $A = B^c$. Then

$$\mathrm{int}_\theta(U) = \bigvee\{\mathrm{int}_\theta(A) \mid A^c \in \mathcal{T}, A \leq U\}.$$

The second equality holds from Theorem 3.11. □

Corollary 3.13. *For an IFS U in an IFTS $(X, \mathcal{T})$*, $\mathrm{int}_\theta(U)$ *is an IFOS.*

Remark 3.14. For an IFS U in an IFTS $(X, \mathcal{T})$, $\text{int}_\theta(U)$ is not necessarily IFθOS.

4. Characterizations for some types of functions

Hanafy et al. already characterized some types of functions by intuitionistic fuzzy θ-closure. Here, we will characterize an intuitionistic fuzzy strongly θ-continuous, intuitionistic fuzzy θ-continuous, and intuitionistic fuzzy weakly continuous functions in terms of intuitionistic fuzzy θ-interior.

Lemma 4.1. *Let $f : (X, \mathcal{T}) \to (Y, \mathcal{T}')$ be a function and U, V be an IFSs. If UqV, then $f(U)qf(V)$.*

Proof. Suppose that $f(U)\widetilde{q}f(V)$. Then $f(U) \leq (f(V))^c$. Since $U \leq f^{-1}(f(U))$, we have $U \leq f^{-1}(f(U)) \leq f^{-1}((f(V))^c)$. Thus we have $U\widetilde{q}(f^{-1}((f(V))^c))^c = f^{-1}(((f(V))^c)^c) = f^{-1}(f(V))$. Since $V \leq f^{-1}(f(V))$ and $U\widetilde{q}f^{-1}(f(V))$, we have $U\widetilde{q}V$. □

Recall that a function $f : (X, \mathcal{T}) \to (Y, \mathcal{T}')$ is said to be *intuitionistic fuzzy strongly θ-continuous* if and only if for each IFP $x_{(\alpha,\beta)}$ in X and $V \in N^q(f(x_{(\alpha,\beta)})$, there exists $U \in N^q(x_{(\alpha,\beta)})$ such that $f(\text{cl}(U)) \leq V$ (See [6]).

Theorem 4.2. *Let $f : (X, \mathcal{T}) \to (Y, \mathcal{T}')$ be a function. Then the following statements are equivalent*:

(1) *f is an intuitionistic fuzzy strongly θ-continuous function.*
(2) $f(\text{cl}_\theta(U))) \leq \text{cl}(f(U))$ *for each IFS U in X.*
(3) $\text{cl}_\theta(f^{-1}(V)) \leq f^{-1}(\text{cl}(V))$ *for each IFS V in Y.*
(4) *$f^{-1}(V)$ is an IFθCS in X for each IFCS V in Y.*
(5) *$f^{-1}(V)$ is an IFθOS in X for each IFOS V in Y.*
(6) $f^{-1}(\text{int}(V)) \leq \text{int}_\theta(f^{-1}(V))$ *for each IFS V of Y.*

Proof. (1) ⇔ (2) ⇔ (3) ⇔ (4) ⇔ (5). See [6].

(3) ⇒ (6). Let V be an IFS in Y. Then V^c is an IFS in Y. Since f is an intuitionistic fuzzy strongly θ-continuous function, by the hypothesis, $\text{cl}_\theta(f^{-1}(V^c)) \leq f^{-1}(\text{cl}(V^c))$. Thus

$$\begin{aligned} f^{-1}(\text{int}(V)) &= f^{-1}((\text{cl}(V^c))^c) = (f^{-1}(\text{cl}(V^c)))^c \\ &\leq (\text{cl}_\theta(f^{-1}(V^c)))^c = (\text{cl}_\theta((f^{-1}(V))^c))^c = \text{int}_\theta(f^{-1}(V)). \end{aligned}$$

(6) ⇒ (3). Let V be an IFS in Y. Then V^c is an IFS in Y. By the hypothesis, $f^{-1}(\text{int}(V^c)) \leq \text{int}_\theta(f^{-1}(V^c))$. Thus

$$\begin{aligned} \text{cl}_\theta(f^{-1}(V)) &= (\text{int}_\theta((f^{-1}(V))^c))^c = (\text{int}_\theta(f^{-1}(V^c)))^c \\ &\leq (f^{-1}(\text{int}(V^c)))^c = f^{-1}((\text{int}(V^c))^c) = f^{-1}(\text{cl}(V)). \end{aligned}$$ □

Theorem 4.3. *Let $f : (X, \mathcal{T}) \to (Y, \mathcal{T}')$ be a bijection. Then the following statements are equivalent*:

(1) *f is an intuitionistic fuzzy strongly θ-continuous function.*

(2) $f^{-1}(\text{int}(V)) \leq \text{int}_\theta(f^{-1}(V))$ *for each IFS* V *of* Y.
(3) $\text{int}(f(U)) \leq f(\text{int}_\theta(U))$ *for each IFS* U *in* X.

Proof. By Theorem 4.2, it suffices to show that (2) is equivalent to (3).

(2) $\Rightarrow$ (3). Let U be an IFS in X. Then $f(U)$ is an IFS in Y. By the hypothesis, $f^{-1}(\text{int}(f(U))) \leq \text{int}_\theta(f^{-1}(f(U)))$. Since f is one-to-one,

$$f^{-1}(\text{int}(f(U))) \leq \text{int}_\theta(f^{-1}(f(U))) = \text{int}_\theta(U).$$

Since f is onto,

$$\text{int}(f(U)) = f(f^{-1}(\text{int}(f(U)))) \leq f(\text{int}_\theta(U)).$$

(3) $\Rightarrow$ (2). Let V be an IFS in Y. Then $f^{-1}(V)$ is an IFS in X. By the hypothesis, $\text{int}(f(f^{-1}(V))) \leq f(\text{int}_\theta(f^{-1}(V)))$. Since f is onto,

$$\text{int}(V) \leq f(\text{int}_\theta(f^{-1}(V))).$$

Since f is one-to-one,

$$f^{-1}(\text{int}(V)) \leq f^{-1}(f(\text{int}_\theta(f^{-1}(V))) = \text{int}_\theta(f^{-1}(V)). \qquad \square$$

Recall that function $f : (X, \mathcal{T}) \to (Y, \mathcal{T}')$ is said to be an *intuitionistic fuzzy θ-continuous* if and only if for each IFP $x_{(\alpha,\beta)}$ in X and $V \in N^q(f(x_{(\alpha,\beta)}))$, there exists $U \in N^q(x_{(\alpha,\beta)})$ such that $f(\text{cl}(U)) \leq \text{cl}(V)$ (See [6]).

Theorem 4.4 ([6])**.** *Let $f : (X, \mathcal{T}) \to (Y, \mathcal{T}')$ be a function. Then the following statements are equivalent*:

(1) f *is an intuitionistic fuzzy θ-continuous function.*
(2) $f(\text{cl}_\theta(U))) \leq \text{cl}_\theta(f(U))$ *for each IFS* U *in* X.
(3) $\text{cl}_\theta(f^{-1}(V)) \leq f^{-1}(\text{cl}_\theta(V))$ *for each IFS* V *in* Y.
(4) $\text{cl}_\theta(f^{-1}(V)) \leq f^{-1}(\text{cl}(V))$ *for each IFOS* V *in* Y.
(5) $f^{-1}(\text{int}_\theta(V)) \leq \text{int}_\theta(f^{-1}(V))$ *for each IFS* V *of* Y.

Proof. (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4). See [6].

(3) $\Rightarrow$ (5). Let V be an IFS in Y. Then V^c is an IFS in Y. Since f is an intuitionistic fuzzy θ-continuous function, by the hypothesis, $\text{cl}_\theta(f^{-1}(V^c)) \leq f^{-1}(\text{cl}_\theta(V^c))$. Thus

$$\begin{aligned} f^{-1}(\text{int}_\theta(V)) &= f^{-1}((\text{cl}_\theta(V^c))^c) = (f^{-1}(\text{cl}_\theta((V^c))))^c \\ &\leq (\text{cl}_\theta(f^{-1}(V^c)))^c = (\text{cl}_\theta((f^{-1}(V))^c))^c = \text{int}_\theta(f^{-1}(V)). \end{aligned}$$

(5) $\Rightarrow$ (3). Let V be an IFS in Y. Then V^c is an IFS in Y. By the hypothesis, $f^{-1}(\text{int}_\theta(V^c)) \leq \text{int}_\theta(f^{-1}(V^c))$. Thus

$$\begin{aligned} \text{cl}_\theta(f^{-1}(V)) &= (\text{int}_\theta((f^{-1}(V))^c))^c = (\text{int}_\theta(f^{-1}(V^c)))^c \\ &\leq (f^{-1}(\text{int}_\theta(V^c)))^c = f^{-1}((\text{int}_\theta(V^c))^c) = f^{-1}(\text{cl}_\theta(V)). \qquad \square \end{aligned}$$

Theorem 4.5. *Let $f : (X, \mathcal{T}) \to (Y, \mathcal{T}')$ be a bijection. Then the following statements are equivalent*:

(1) f *is an intuitionistic fuzzy θ-continuous function.*

(2) $f^{-1}(\mathrm{int}_\theta(V)) \leq \mathrm{int}_\theta(f^{-1}(V))$ *for each IFS* V *of* Y.
(3) $\mathrm{int}_\theta(f(U)) \leq f(\mathrm{int}_\theta(U))$ *for each IFS* U *in* X.

Proof. By Theorem 4.4, it suffices to show that (2) is equivalent to (3).

(2) $\Rightarrow$ (3). Let U be an IFS in X. Then $f(U)$ is an IFS in Y. By the hypothesis, $f^{-1}(\mathrm{int}_\theta(f(U))) \leq \mathrm{int}_\theta(f^{-1}(f(U)))$. Since f is one-to-one,

$$f^{-1}(\mathrm{int}_\theta(f(U))) \leq \mathrm{int}_\theta(f^{-1}(f(U))) = \mathrm{int}_\theta(U).$$

Since f is onto,

$$\mathrm{int}_\theta(f(U)) = f(f^{-1}(\mathrm{int}(f(U)))) \leq f(\mathrm{int}(U)).$$

(3) $\Rightarrow$ (2). Let V be an IFS in Y. Then $f^{-1}(V)$ is an IFS in X. By the hypothesis, $\mathrm{int}_\theta(f(f^{-1}(V))) \leq f(\mathrm{int}_\theta(f^{-1}(V)))$. Since f is onto,

$$\mathrm{int}_\theta(V) = \mathrm{int}_\theta(f(f^{-1}(V))) \leq f(\mathrm{int}_\theta(f^{-1}(V))).$$

Since f is one-to-one,

$$f^{-1}(\mathrm{int}_\theta(V)) \leq f^{-1}(f(\mathrm{int}_\theta(f^{-1}(V)))) = \mathrm{int}_\theta(f^{-1}(V)). \qquad \square$$

Recall that function $f : (X, \mathcal{T}) \to (Y, \mathcal{T}')$ is said to be an *intuitionistic fuzzy weakly continuous* if and only if for each IFOS V in Y, $f^{-1}(V) \leq \mathrm{int}(f^{-1}(\mathrm{cl}(V)))$ (See [6]).

Theorem 4.6 ([6])**.** *Let* $f : (X, \mathcal{T}) \to (Y, \mathcal{T}')$ *be a function. Then the following statements are equivalent*:

(1) f *is an intuitionistic fuzzy weakly continuous function.*
(2) $f(\mathrm{cl}(U))) \leq \mathrm{cl}_\theta(f(U))$ *for each IFS* U *in* X.
(3) $\mathrm{cl}(f^{-1}(V)) \leq f^{-1}(\mathrm{cl}_\theta(V))$ *for each IFS* V *in* Y.
(4) $\mathrm{cl}(f^{-1}(V)) \leq f^{-1}(\mathrm{cl}(V))$ *for each IFOS* V *of* Y.
(5) $f^{-1}(\mathrm{int}_\theta(V)) \leq \mathrm{int}(f^{-1}(V))$ *for each IFS* V *of* Y.

Proof. (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4). See [6].

(3) $\Rightarrow$ (5). Let V be an IFS in Y. Then V^c is an IFS in Y. Since f is an intuitionistic fuzzy weakly continuous function, by the hypothesis, $\mathrm{cl}(f^{-1}(V^c)) \leq f^{-1}(\mathrm{cl}_\theta(V^c))$. Thus

$$\begin{aligned} f^{-1}(\mathrm{int}_\theta(V)) &= f^{-1}((\mathrm{cl}_\theta(V^c))^c) = (f^{-1}(\mathrm{cl}_\theta((V^c))))^c \\ &\leq (\mathrm{cl}(f^{-1}(V^c)))^c = (\mathrm{cl}((f^{-1}(V))^c))^c = \mathrm{int}(f^{-1}(V)). \end{aligned}$$

(5) $\Rightarrow$ (3). Let V be an IFS in Y. Then V^c is an IFS in Y. By the hypothesis, $f^{-1}(\mathrm{int}_\theta(V^c)) \leq \mathrm{int}(f^{-1}(V^c))$. Thus

$$\begin{aligned} \mathrm{cl}(f^{-1}(V)) &= (\mathrm{int}((f^{-1}(V))^c))^c = (\mathrm{int}(f^{-1}(V^c))^c) \\ &\leq (f^{-1}(\mathrm{int}_\theta(V^c)))^c = f^{-1}((\mathrm{int}_\theta(V^c))^c) = f^{-1}(\mathrm{cl}_\theta(V)). \qquad \square \end{aligned}$$

Theorem 4.7. *Let* $f : (X, \mathcal{T}) \to (Y, \mathcal{T}')$ *be a bijection. Then the following statements are equivalent*:

(1) f *is an intuitionistic fuzzy weakly continuous function.*

(2) $f^{-1}(\text{int}_\theta(V)) \le \text{int}(f^{-1}(V))$ *for each IFS* V *of* Y.
(3) $\text{int}_\theta(f(U)) \le f(\text{int}(U))$ *for each IFS* U *in* X.

Proof. By Theorem 4.6, it suffices to show that (2) is equivalent to (3).

(2) $\Rightarrow$ (3). Let U be an IFS in X. Then $f(U)$ is an IFS in Y. By the hypothesis, $f^{-1}(\text{int}_\theta(f(U))) \le \text{int}(f^{-1}(f(U)))$. Since f is one-to-one,

$$f^{-1}(\text{int}_\theta(f(U))) \le \text{int}(f^{-1}(f(U))) = \text{int}(U).$$

Since f is onto,

$$\text{int}_\theta(f(U)) = f(f^{-1}(\text{int}(f(U)))) \le f(\text{int}(U)).$$

(3) $\Rightarrow$ (2). Let V be an IFS in Y. Then $f^{-1}(V)$ is an IFS in X. By the hypothesis, $\text{int}_\theta(f(f^{-1}(V))) \le f(\text{int}(f^{-1}(V)))$. Since f is onto,

$$\text{int}_\theta(V) \le f(\text{int} f^{-1}(V)).$$

Since f is one-to-one,

$$f^{-1}(\text{int}_\theta(V)) \le f^{-1}(f(\text{int}(f^{-1}(V)))) = \text{int}(f^{-1}(V)). \qquad \square$$

References

[1] K. T. Atanassov, *Intuitionistic fuzzy sets*, Fuzzy Sets and Systems **20** (1986), no. 1, 87–96.

[2] D. Çoker, *An introduction to intuitionistic fuzzy topological spaces*, Fuzzy Sets and Systems **88** (1997), no. 1, 81–89.

[3] ———, *An introduction to fuzzy subspaces in intuitionistic fuzzy topological spaces*, J. Fuzzy Math. **4** (1996), no. 4, 749–764.

[4] D. Çoker and M. Demirci, *On intuitionistic fuzzy points*, Notes IFS **1** (1995), no. 2, 79–84.

[5] I. M. Hanafy, *Intuitionistic fuzzy functions*, International Journal of Fuzzy Logic and Intelligent Systems **3** (2003), no. 2, 200–205.

[6] I. M. Hanafy, A. M. Abd El-Aziz, and T. M. Salman, *Intuitionistic fuzzy θ-closure operator*, Bol. Asoc. Mat. Venez. **13** (2006), no. 1, 27–39.

[7] S. J. Lee and E. P. Lee, *Fuzzy r-preopen sets and fuzzy r-precontinuous maps*, Bull. Korean Math. Soc. **36** (1999), no. 1, 91–108.

[8] M. N. Mukherjee and S. P. Sinha, *On some near-fuzzy continuous functions between fuzzy topological spaces*, Fuzzy Sets and Systems **34** (1990), no. 2, 245–254.

[9] ———, *Fuzzy Θ-closure operator on fuzzy topological spaces*, Internat. J. Math. Math. Sci. **14** (1991), no. 2, 309–314.

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