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A NOTE ON NIELSEN TYPE NUMBERS

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Abstract. The Reidemeister orbit set plays a crucial role in the Nielsen type theory of periodic orbits, such as the Reidemeister set does in Nielsen fixed point theory. In this paper, using Heath and You's methods on Nielsen type numbers, we show that these numbers can be evaluated by the set of essential orbit classes under suitable hypotheses, and obtain some formulas in some special cases.

1. Introduction

Nielsen fixed point theory has been extended to Nielsen type theory of periodic orbits [5, III.3]. Let $f : X \to X$ be a self map of compact connected ANR X. In Nielsen fixed point theory, the computation of the Nielsen number $N(f)$ often relies on our knowledge of the Reidemeister set, that is the set of Reidemeister conjugacy classes in the fundamental group of X. Our aim in this paper is to show that using the Reidemeister orbit sets (as defined in [7]), the Nielsen type number $N\Phi_n(f)$ (first defined in [5, III.4.7]) can be evaluated under certain conditions.

In [1, Theorem 5.1], the simple formua $N\Phi_m(f) = N(f^m)$ for all $m \mid n$ was derived under suitable conditions. But under less conditons, we have the formula

$$N\Phi_m(f) = d(\mathcal{E}O^{(m)}(f))$$

for all $m \mid n$, where $d(\mathcal{E}O^{(m)}(f))$ is the sum of the depths of essential m-orbit classes of f. Clearly, $N(f^m) \le d(\mathcal{E}O^{(m)}(f))$.

Let $\mathcal{E}IO^{(n)}(f)$ be the set of essentially irreducible n-orbit classes (see [6]). From the procedure in [5] and comments in [3], we derive the simple formula

$$N\Phi_{p^r}(f) = \sum_{m \mid p^r} d(\mathcal{E}IO^{(m)}(f)),$$

where p is a prime and r is a nonnegative integer.

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This paper consists of three sections. In Section 2, we redefine the Nielsen type number and obtain some formulas of Nielsen type numbers under certain conditions. In the last section also, we derive a formula in the special case.

2. Nielsen type numbers

Let X be a compact connected ANR. Let $f : X \to X$ be a map. We define the Nielsen relation on the fixed point set $\mathrm{Fix}(f)$. Two points $x, y \in \mathrm{Fix}(f)$ are related if there is a path c from x to y such that $f(c)$ is homotopic to c by a homotopy keeping the end points fixed. The set of fixed point classes will be denoted $\mathcal{F}P(f)$.

Let $n > 0$ be a given integer. Fixed point classes of the iterate $f^n : X \to X$ are called n-periodic point classes of f. Then f acts on the set $\mathcal{F}P(f^n)$ by $\mathbf{A}_{f^n} \mapsto f(\mathbf{A}_{f^n})$. The f-orbit of a class $\mathbf{A}_{f^n}$ is called an n-orbit class, denoted by $\mathbf{A}_f^{(n)}$. The set of n-orbit classes is denoted by $\mathcal{O}^{(n)}(f)$. The length of the n-orbit class $\mathbf{A}_f^{(n)}$ is the smallest integer $\ell > 0$ such that $\mathbf{A}_{f^n} = f^\ell(\mathbf{A}_{f^n})$. Clearly ℓ divides n because $\mathbf{A}_{f^n} = f^n(\mathbf{A}_{f^n})$. Standard fixed point index theory provides an integer index $\mathrm{ind}(\mathbf{A}_{f^n})$ for each periodic point class $\mathbf{A}_{f^n}$. A periodic point class $\mathbf{A}_{f^n}$ is essential if its index is nonzero. We let $\mathcal{E}(f^n)$ be the set of essential periodic point classes of f. Then $N(f^n)$ the Nielsen number of f^n is the cardinality of $\mathcal{E}(f^n)$. Also, f acts on $\mathcal{E}(f^n)$ by $\mathbf{A}_{f^n} \mapsto f(\mathbf{A}_{f^n})$. The f-orbit of an essential fixed point class $\mathbf{A}_{f^n}$ of f^n will be called an essential n-orbit class, denoted by $\mathbf{A}_f^{(n)}$. The set of essential n-orbit classes will be denoted by $\mathcal{E}O^{(n)}(f)$. We [7] defined the essential n-orbit number $EO^{(n)}(f)$ or simply $EO^{(n)}$ to be the cardinality of the set $\mathcal{E}O^{(n)}(f)$. This number is a homotopy invariant and it is a Nielsen type number in the general sense of [5].

On the other hand we recall that the Reidemeister operation on the fundamental group $\pi_1(X, x_0)$ is the left action of $\pi_1(X, x_0)$ on itself defined as follows. Choose a path w from the base point x_0 to $f(x_0)$ as the base path for f, the left action is defined by

$$\alpha \circ \gamma = \alpha\gamma w f^{-1}(\alpha) w^{-1}.$$

The set of orbits of the action will be called the Reidemeister set of f, denoted by $\mathcal{R}(f)$. Let $R(f)$ denote the cardinality of the Reidemeister set $\mathcal{R}(f)$. The Redemeiter class of $\gamma \in \pi_1(X, x_0)$ will be written $[\gamma]_f$.

Let $n > 0$ be a given integer. Note that $[\gamma]_{f^n} \mapsto [wf(\gamma)w^{-1}]_{f^n}$ defines an action on $\mathcal{R}(f^n)$. The f-orbit of a Reidemeister class $[\gamma]_{f^n}$ will be called a Reidemeister n-orbit, denoted by $[\gamma]_f^{(n)}$. The set of all such Reidemeister f-orbits is called the Reidemeister n-orbit set of f, denoted by $\mathcal{R}O^{(n)}(f)$. The length of the orbit $[\gamma]_f^{(n)}$ is the smallest integer $\ell > 0$ such that $[\gamma]_{f^n} =$

$[w^{(\ell)}f^{\ell}(\gamma)(w^{(\ell)})^{-1}]_{f^n}$, where $w^{(\ell)}$ stands for $wf(w)\cdots f^{\ell-1}(w)$. Clearly ℓ divides n. For $\ell \mid n$, we have the commutative diagram

$$\begin{array}{ccc} \mathcal{R}(f^{\ell}) & \xrightarrow{\iota_{\ell,n}} & \mathcal{R}(f^n) \\ \big\downarrow & & \big\downarrow \\ \mathcal{RO}^{(\ell)}(f) & \xrightarrow{\iota_{\ell,n}} & \mathcal{RO}^{(n)}(f) \end{array}$$

where the vertical maps are projections, and the horizontal maps are induced by the level-change function $\iota_{\ell,n} : \pi_1(X,x_0) \to \pi_1(X,x_0)$ defined (as in [2, Definition 1.9]) by

$$\iota_{\ell,n}(\beta) := \beta w^{(\ell)} f^{\ell}(\beta w^{(\ell)}) f^{2\ell}(\beta w^{(\ell)}) \cdots f^{n-\ell}(\beta)(w^{(n-\ell)})^{-1}.$$

We say that an f-orbit $[\alpha]_f^{(n)} \in \mathcal{RO}^{(n)}(f)$ is reducible to level h, if there exists a $[\beta]_f^{(h)} \in \mathcal{RO}^{(h)}(f)$ such that $\iota_{h,n}([\beta]_f^{(h)}) = [\alpha]_f^{(n)}$. The lowest level $d = d([\alpha]_f^{(n)})$ to which $[\alpha]_f^{(n)}$ reduces is its depth. Clearly, the length ℓ of the orbit $[\alpha]_f^{(n)}$ divides the depth d. A Reidemeister orbit $[\alpha]_f^{(n)} \in \mathcal{RO}^{(n)}(f)$ is said to have the full depth property if its depth equals its length, i.e., $d = \ell$ [7].

It is well known that every fixed point class of f is assigned a Reidemeister class in $\mathcal{R}(f)$, called its coordinate. We get an injection $\rho : \mathcal{FP}(f) \hookrightarrow \mathcal{R}(f)$ defined by $\rho(\mathbf{A}_f) := [cf(c^{-1})w^{-1}]_f$ for any path c from x_0 to a point x in $\mathbf{A}_f$. Thus we also get an injection $\rho : \mathcal{O}^{(n)}(f) \hookrightarrow \mathcal{RO}^{(n)}(f)$ defined by

$$\rho(\mathbf{A}_f^{(n)}) := [cf^n(c^{-1})(w^{(n)})^{-1}]_f^{(n)}$$

for any path c from x_0 to a point x in $\mathbf{A}_f^{(n)}$. The depth of an n-orbit class $\mathbf{A}_f^{(n)}$ is defined to be the depth of its coordinate $\rho(\mathbf{A}_f^{(n)})$ [7]. Thus for any positive integers ℓ and n with $\ell \mid n$, we have the commutative diagram

$$\begin{array}{ccc} \mathcal{O}^{(\ell)}(f) & \xrightarrow{\rho} & \mathcal{RO}^{(\ell)}(f) \\ {\scriptstyle i_{\ell,n}}\big\downarrow & & \big\downarrow{\scriptstyle \iota_{\ell,n}} \\ \mathcal{O}^{(n)}(f) & \xrightarrow{\rho} & \mathcal{RO}^{(n)}(f) \end{array}$$

where $i_{\ell,n} : \mathcal{O}^{(\ell)}(f) \to \mathcal{O}^{(n)}(f)$ is the function induced by $\mathrm{Fix}(f^{\ell}) \subset \mathrm{Fix}(f^n)$.

For any subset $\mathscr{A} \subset \bigcup_{\ell \mid n} \mathcal{RO}^{(\ell)}(f)$ we define $d(\mathscr{A}) = \sum_{[\alpha]_f^{(\ell)} \in \mathscr{A}} d([\alpha]_f^{(\ell)})$. For any subset $\mathscr{B} \subset \bigcup_{\ell \mid n} \mathcal{O}^{(\ell)}(f)$, by the observation above, we also define $d(\mathscr{B}) = \sum_{\mathbf{A}_f^{(\ell)} \in \mathscr{B}} d(\mathbf{A}_f^{(\ell)})$. The following definition is equivalent to definitions in [5, 3, 4].

Definition 2.1. A subset $\mathscr{A} \subset \bigcup_{m \mid n} \mathcal{RO}^{(m)}(f)$ is called a *reducing system of period n* if every essential Reidemeister f-orbit $[\alpha]_f^{(k)} \in \mathcal{RO}^{(k)}(f)$ with $k \mid n$, reduces to an element of $\mathscr{A}$, i.e., $[\alpha]_f^{(k)} = \iota_{\ell,k}([\beta]_f^{(\ell)})$ for some $[\beta]_f^{(\ell)} \in \mathscr{A}$.

$$N\Phi_n(f) = \min\{d(\mathscr{A}) \mid \mathscr{A} \text{ is a reducing system of period } n\}.$$

This minimal number is equal to the Nielsen type number $NF_n(f)$ (as defined in [5, III.4.7]). This number is a lower bound for the cardinality of the set of the n-periodic points (see [5, III.4.12]). The set of irreducible essential n-orbit classes is denoted by $\mathcal{IEO}^{(n)}(f)$. Let $IEO^{(n)}(f)$ or simply $IEO^{(n)}$ denote the cardinality of the set $\mathcal{IEO}^{(n)}(f)$. Note that $IEO^{(n)}(f) = (1/n)NP_n(f)$ (as defined in [5, III.4.7]) and $IEO^{(1)} = EO^{(1)} = N(f)$.

Essentially reducibility as defined in [1, Definition 4.1] is a property of orbits, we have:

Definition 2.2. We say that a map $f : X \to X$ is *essentially reducible* provided that for every essential k-orbit $[\alpha]_f^{(k)}$, if it reduces to some ℓ-orbit $[\beta]_f^{(\ell)}$ with $\ell \mid k$, then $[\beta]_f^{(\ell)}$ is essential.

Proposition 2.3. *If f is essentially reducible, then*

$$N\Phi_n(f) = \sum_{\ell \mid n} \ell \times IEO^{(\ell)}(f).$$

Proof. See [3, Theorem 4.2]. □

Proposition 2.4.

$$\mathcal{IEO}^{(k)}(f) = \mathcal{EO}^{(k)}(f) - \bigcup_{\substack{\ell \mid k \\ \ell < k}} i_{\ell,k}(i_{\ell,k}^{-1}(\mathcal{EO}^{(k)}(f))).$$

Proof. By definitions of $\mathcal{IEO}^{(k)}(f)$ and $\mathcal{EO}^{(k)}(f)$, it is clear. □

The following Lemma 2.5(3) is the same as Corollary 4.8 in [1], but we give a simple proof of it.

Lemma 2.5. *Suppose $f : X \to X$ is essentially reducible. Then we have the following properties*:

(1) $i_{\ell,k}^{-1}(\mathcal{EO}^{(k)}(f)) \subset \mathcal{EO}^{(\ell)}(f)$.

(2) $\mathcal{EO}^{(n)}(f) \subset \bigcup_{\ell \mid n} i_{\ell,n}(\mathcal{IEO}^{(\ell)}(f))$.

(3) *If $N(f^n) = R(f^n) \neq 0$, then $N(f^\ell) = R(f^\ell) \neq 0$ for every $\ell \mid n$.*

Proof. (1) If $\mathbf{A}_f^{(\ell)} \in i_{\ell,k}^{-1}(\mathcal{EO}^{(k)}(f))$, then there is an $\mathbf{A}_f^{(k)} \in \mathcal{EO}^{(k)}(f)$ such that $i_{\ell,k}(\mathbf{A}_f^{(\ell)}) = \mathbf{A}_f^{(k)}$. Clearly we have $\rho(\mathbf{A}_f^{(k)}) = \rho(i_{\ell,k}(\mathbf{A}_f^{(\ell)})) = \iota_{\ell,k}(\rho(\mathbf{A}_f^{(\ell)}))$. Since f is essentially reducible, $\rho(\mathbf{A}_f^{(\ell)})$ is essential. Thus $\mathbf{A}_f^{(\ell)} \in \mathcal{EO}^{(\ell)}(f)$.

(2) If $\mathbf{A}_f^{(n)} \in \mathcal{EO}^{(n)}(f)$ has depth d, then there exists an irreducible essential d-orbit $[\alpha]_f^{(d)} \in \mathcal{RO}^{(d)}(f)$ such that $\iota_{d,n}([\alpha]_f^{(d)}) = \rho(\mathbf{A}_f^{(n)})$ since f is essentially reducible. There exists an irreducible essential d-orbit class $\mathbf{A}_f^{(d)} \in \mathcal{IEO}^{(d)}(f)$

such that $\rho(\mathbf{A}_f^{(d)}) = [\alpha]_f^{(d)}$ because $[\alpha]_f^{(d)}$ is an irreducible essential d-orbit. Thus $\rho(i_{d,n}(\mathbf{A}_f^{(d)})) = \iota_{d,n}(\rho(\mathbf{A}_f^{(d)})) = \iota_{d,n}([\alpha]_f^{(d)}) = \rho(\mathbf{A}_f^{(n)})$ and so $i_{d,n}(\mathbf{A}_f^{(d)}) = \mathbf{A}_f^{(n)}$.

(3) If $N(f^n) = R(f^n) \neq 0$, then $\rho : \mathcal{E}O^{(n)}(f) \to \mathcal{R}O^{(n)}(f)$ is bijective. Thus any element of $\mathcal{R}O^{(\ell)}(f)$ is essential for every $\ell \mid n$, because f is essentially reducible and $\iota_{\ell,n}(\mathcal{R}O^{(\ell)}(f)) \subset \mathcal{R}O^{(n)}(f) = \rho(\mathcal{E}O^{(n)}(f))$. Therefore $\rho : \mathcal{E}O^{(\ell)}(f) \to \mathcal{R}O^{(\ell)}(f)$ is bijective, so $N(f^\ell) = R(f^\ell) \neq 0$. $\square$

Definition 2.6. We define $\mathcal{E}O^{(\ell,k)}(f) = i_{\ell,k}^{-1}(\mathcal{E}O^{(k)}(f))$. The cardinality of the set $\mathcal{E}O^{(\ell,k)}(f)$ will be denoted by $EO^{(\ell,k)}(f)$ or simply $EO^{(\ell,k)}$.

Note that by Lemma 2.5(1), if f is essentially reducible, then $EO^{(\ell,k)}(f) \leq EO^{(\ell)}(f)$.

Example 2.7. For any selfmap $f : X \to X$, by Proposition 2.4, we have

$$\mathcal{I}EO^{(6)}(f) = \mathcal{E}O^{(6)}(f) - (i_{3,6}(\mathcal{E}O^{(3,6)}(f)) \cup i_{2,6}(\mathcal{E}O^{(2,6)}(f)) \cup i_{1,6}(\mathcal{E}O^{(1,6)}(f))).$$

Moreover, if for every ℓ, k with $\ell \mid k$, $i_{\ell,k}$ is injective, then

$$IEO^{(6)} \geq EO^{(6)} - (EO^{(3,6)} + EO^{(2,6)} + EO^{(1,6)}).$$

Theorem 2.8. *If $f : X \to X$ is essentially reducible and for every m, n with $m \mid n$, $i_{m,n}$ is injective, then for a prime number p and a positive integer r, we have*

$$N\Phi_{p^r}(f) = N(f) + \sum_{1\leq \ell \leq r} p^\ell \times (EO^{(p^\ell)} - EO^{(p^{\ell-1},p^\ell)}).$$

Proof. When f is essentially reducible, by Proposition 2.3, we have

$$N\Phi_{p^r}(f) = \sum_{0\leq \ell \leq r} p^\ell \times IEO^{(p^\ell)}.$$

If $\ell = 0$, then $IEO^{(1)} = N(f)$. For $\ell > 0$, there are inclusions

$$i_{1,p^\ell}(\mathcal{E}O^{(1,p^\ell)}(f)) \subset i_{p,p^\ell}(\mathcal{E}O^{(p,p^\ell)}(f)) \subset \cdots \subset i_{p^{\ell-1},p^\ell}(\mathcal{E}O^{(p^{\ell-1},p^\ell)}(f)).$$

Thus by Proposition 2.4, we have

$$\begin{aligned} \mathcal{I}EO^{(p^\ell)}(f) &= \mathcal{E}O^{(p^\ell)}(f) - \bigcup_{0\leq k<\ell} i_{p^k,p^\ell}(\mathcal{E}O^{(p^k,p^\ell)}(f)) \\ &= \mathcal{E}O^{(p^\ell)}(f) - i_{p^{\ell-1},p^\ell}(\mathcal{E}O^{(p^{\ell-1},p^\ell)}(f)). \end{aligned}$$

Since $i_{p^{\ell-1},p^\ell}$ is injective, we get the desired equality $IEO^{(p^\ell)} = EO^{(p^\ell)} - EO^{(p^{\ell-1},p^\ell)}$. $\square$

From the Definition 4.9 in [1], we have:

Definition 2.9. We say that a map $f : X \to X$ is *essentially reducible to the* GCD, if it is essentially reducible and whenever $\mathbf{A}_f^{(k)} \in \mathcal{E}O^{(k)}(f)$ reduces to both $\mathbf{A}_f^{(r)} \in \mathcal{E}O^{(r)}(f)$ and $\mathbf{A}_f^{(s)} \in \mathcal{E}O^{(s)}(f)$, then there is an $\mathbf{A}_f^{(\ell)} \in \mathcal{E}O^{(\ell)}(f)$ with $\ell = (r, s)$ to which both $\mathbf{A}_f^{(r)}$ and $\mathbf{A}_f^{(s)}$ reduce.

Lemma 2.10 ([1, Lemma 4.19])**.** *Suppose that* $f : X \to X$ *is essentially reducible to the* GCD *and* $i_{\ell,n}$ *is injective for every* $\ell \mid n$*. If* $r \neq s$ *with* $r, s \mid n$*, then*

$$i_{r,n}(\mathcal{I}EO^{(r)}(f)) \cap i_{s,n}(\mathcal{I}EO^{(s)}(f)) \cap \mathcal{E}O^{(n)}(f) = \emptyset.$$

Proof. Assume $\mathbf{A}_f^{(n)} \in i_{r,n}(\mathcal{I}EO^{(r)}(f)) \cap i_{s,n}(\mathcal{I}EO^{(s)}(f)) \cap \mathcal{E}O^{(n)}(f)$. Then $\mathbf{A}_f^{(n)}$ is essential and reduces to both $\mathbf{A}_f^{(r)} \in \mathcal{I}EO^{(r)}(f)$ and $\mathbf{A}_f^{(s)} \in \mathcal{I}EO^{(s)}(f)$. Since f is essentially reducible to the GCD, there exists an $\mathbf{A}_f^{(\ell)} \in \mathcal{E}O^{(\ell)}(f)$ to which both $\mathbf{A}_f^{(r)}$ and $\mathbf{A}_f^{(s)}$ reduce for $\ell = (r, s)$. Thus $\mathbf{A}_f^{(n)} = i_{r,n}(i_{\ell,r}(\mathbf{A}_f^{(\ell)})) = i_{r,n}(\mathbf{A}_f^{(r)})$ implies $i_{\ell,r}(\mathbf{A}_f^{(\ell)}) = \mathbf{A}_f^{(r)}$ because $i_{r,n}$ is injective. Howerever, since $\mathbf{A}_f^{(r)}$ is irreducible, we have $\ell = r$. Similary $\ell = s$. This is in controdiction to $r \neq s$. ☐

Recall [1, Definition 1.1] that a map $f : X \to X$ is weakly Jiang provided that either $N(f) = 0$ or else $N(f) = R(f)$.

Theorem 2.11. *Suppose that* $f : X \to X$ *is essentially reducible to the* GCD *and* i_{ℓ_1,ℓ_2} *is injective for every* $\ell_1 \mid \ell_2 \mid n$*. If* $f : X \to X$ *is such that* $N(f^n) = R(f^n) \neq 0$*, then for all* $m \mid n$*,*

$$N\Phi_m(f) = d(\mathcal{E}O^{(m)}(f)).$$

Proof. When f is essentially reducible and $N(f^n) = R(f^n)$, by Lemma 2.5(3), we have $N(f^m) = R(f^m)$ for all $m \mid n$. Thus we need only prove the theorem for $m = n$. For every $\ell \mid n$, it is clear that $i_{\ell,n}(\mathcal{I}EO^{(\ell)}(f))$ belongs to $\mathcal{E}O^{(n)}(f)$ since $R(f^n) = N(f^n)$. Thus Lemma 2.5(2) and Lemma 2.10 tell us

$$\mathcal{E}O^{(n)}(f) = \bigcup_{\ell \mid n} i_{\ell,n}(\mathcal{I}EO^{(\ell)}(f)),$$

where summands are disjoint. By definition of depth and $i_{\ell,n}$ is injective for every $\ell \mid n$, we have

$$\begin{aligned} d(\mathcal{E}O^{(n)}(f)) &= \sum_{\ell \mid n} d(i_{\ell,n}(\mathcal{I}EO^{(\ell)}(f))) \\ &= \sum_{\ell \mid n} \ell \times IEO^{(\ell)}(f) \\ &= N\Phi_n(f) \end{aligned}$$

the last equality follows from Proposition 2.3. ☐

The next result is the same as [1, Theorem 5.1].

Corollary 2.12. *Under the conditions of Theorem 2.11, and if every Reidemeister f-orbit has the full depth property, then for all $m \mid n$,*

$$N\Phi_m(f) = N(f^m).$$

Proof. It is clear that

$$d(\mathcal{E}O^{(m)}(f)) = \sum_{\mathbf{A}_f^{(m)} \in \mathcal{E}O^{(m)}(f)} d(\mathbf{A}_f^{(m)}) = \sum \ell(\mathbf{A}_f^{(m)}) = N(f^m).$$

□

Example 2.13. Consider the antipodal map on the sphere $S^2 = \{(x,y,z) \mid x^2+y^2+z^2=1\}$, given by $f(x,y,z) = (-x,-y,-z)$. By [5, II.4.1], we have $N(f^2) = R(f^2) = 1$ because the Lefschetz number of f^2 is $L(f^2) = 2$. The fundamental group of S^2 is trivial, and so the essential 2-orbit $[0]_f^{(2)} \in \mathcal{R}O_f^{(2)}$ is reduced to level one which is inessential. Thus $N\Phi_2(f) = 1$.

3. Alternative approach on Nielsen type numbers

Recall [6, Definition in Section 2.2] that an n-orbit class $\mathbf{A}_f^{(n)} \in \mathcal{O}^{(n)}(f)$ and all n-periodic point classes contained in it are essentially irreducible if it is essential and it does not reduce to any essential m-orbit class for any $m < n$. The set of essetially irreducible n-orbit classes will be denoted by $\mathcal{E}IO^{(n)}(f)$. Let $EIO^{(n)}(f)$ or simply $EIO^{(n)}$ denote the cardinality of the set $\mathcal{E}IO^{(n)}(f)$. Note that $EIO^{(1)} = EO^{(1)} = N(f) = IEO^{(1)}$. Thus we have:

Proposition 3.1. $\mathcal{I}EO^{(n)}(f) \subset \mathcal{E}IO^{(n)}(f) \subset \mathcal{E}O^{(n)}(f)$.

From comments in [3], we have:

Proposition 3.2.

$$\sum_{\ell \mid n} d(\mathcal{I}EO^{(\ell)}(f)) \le N\Phi_n(f) \le \sum_{\ell \mid n} d(\mathcal{E}IO^{(\ell)}(f)).$$

Proof. First inequality follows from Proposition 2.2. For the last inequality, it suffices to show that

$$\bigcup_{\ell \mid n} \rho(\mathcal{E}IO^{(\ell)}(f))$$

is a reducing system of period n.

(Case1) If an essential ℓ-orbit $[\alpha]_f^{(\ell)} \in \mathcal{R}O^{(\ell)}(f)$ with $\ell \mid n$ is irreducible, then by Proposition 3.1, we have $[\alpha]_f^{(\ell)} \in \rho(\mathcal{E}IO^{(\ell)}(f))$.

(Case2) Suppose an essential k-orbit $[\alpha]_f^{(k)} \in \mathcal{R}O^{(k)}(f)$ with $k \mid n$ has depth $d < k$. If it does not reduce to an essential ℓ-orbit for $d \le \ell < k$, then it belongs to $\rho(\mathcal{E}IO^{(k)}(f))$. On the other hand, if $[\alpha]_f^{(k)}$ reduces to an essential ℓ-orbit $[\beta]_f^{(\ell)} \in \mathcal{R}O^{(\ell)}(f)$ with $d \le \ell < k$ and it does not reduce to an essential r-orbit for $d \le r < \ell$, then it reduces to $[\beta]_f^{(\ell)} \in \rho(\mathcal{E}IO^{(\ell)}(f))$. □

Let $\mathscr{A}$ be a reducing system of period n. For every essential k-orbit $[\alpha]_f^{(k)}$ with $k \mid n$, we define $C_{[\alpha]_f^{(k)}} = \{[\beta]_f^{(\ell)} \in \mathscr{A} \mid \iota_{\ell,k}([\beta]_f^{(\ell)}) = [\alpha]_f^{(k)}\}$. Note that by Definition 2.1, it is nonempty as a subset of $\mathscr{A}$.

Lemma 3.3. *Suppose $n = p^r$ for a prime p and a positive integer r. If $[\alpha]_f^{(s)}, [\beta]_f^{(t)} \in \cup_{k|n}\rho(\mathcal{E}IO^{(k)}(f))$ with $[\alpha]_f^{(s)} \neq [\beta]_f^{(t)}$, then $C_{[\alpha]_f^{(s)}} \cap C_{[\beta]_f^{(t)}} = \emptyset$.*

Proof. Assume that $[\gamma]_f^{(\ell)} \in C_{[\alpha]_f^{(s)}} \cap C_{[\beta]_f^{(t)}}$. This means that $[\alpha]_f^{(s)} = \iota_{\ell,s}([\gamma]_f^{(\ell)})$ and $\iota_{\ell,t}([\gamma]_f^{(\ell)}) = [\beta]_f^{(t)}$. If $s = t$, then $[\alpha]_f^{(s)} = [\beta]_f^{(t)}$. On the other hand, if $s < t$, then $s \mid t$ since $n = p^r$. Thus we have $[\beta]_f^{(t)} = \iota_{\ell,t}([\gamma]_f^{(\ell)}) = \iota_{s,t} \circ \iota_{\ell,s}([\gamma]_f^{(\ell)}) = \iota_{s,t}([\alpha]_f^{(s)})$. This is in contradiction to $[\beta]_f^{(t)}$ is essentially irreducible. $\square$

From the procedure in [5] and comments in [3], we have:

Theorem 3.4. *For a prime p and a nonnegative integer r,*

$$N\Phi_{p^r}(f) = \sum_{\ell|p^r} d(\mathcal{E}IO^{(\ell)}(f)).$$

Proof. By Proposition 3.2, it suffices to show that $d(\cup_{k|n}\mathcal{E}IO^{(k)}(f))$ is minimal for $n = p^r$. Let $\mathscr{A}$ be a reducing system of period n and $[\alpha]_f^{(k)} \in \cup_{k|n}\rho(\mathcal{E}IO^{(k)}(f))$ be given. We define a function $\phi : \cup_{k|n}\rho(\mathcal{E}IO^{(k)}(f)) \to \mathscr{A}$ by $\phi([\alpha]_f^{(k)}) = r(C_{[\alpha]_f^{(k)}})$, where $r(C_{[\alpha]_f^{(k)}})$ is the representative of the set $C_{[\alpha]_f^{(k)}}$, and it is injective by Lemma 3.3. Thus we have $d(\cup_{k|n}\mathcal{E}IO^{(k)}(f)) \leq d(\mathscr{A})$ [3, Lemma 3.6]. $\square$

Example 3.5. Consider the flip map on S^1. Here for $r \geq 0$, we have

$$N\Phi_{p^r}(f) = \sum_{\ell|p^r} d(\mathcal{E}IO^{(\ell)}(f)) = EIO^{(1)} = N(f) = 2.$$

References

[1] P. R. Heath and E. Keppelmann, *Fibre techniques in Nielsen periodic point theory on nil and solvmanifolds. I*, Topology Appl. **76** (1997), no. 3, 217–247.

[2] P. R. Heath, R. Piccinini, and C. You, *Nielsen-type numbers for periodic points. I*, Topological fixed point theory and applications (Tianjin, 1988), 88–106, Lecture Notes in Math., 1411, Springer, Berlin, 1989.

[3] P. R. Heath and C. You, *Nielsen-type numbers for periodic points. II*, Topology Appl. **43** (1992), no. 3, 219–236.

[4] J. Jezierski, *Wecken's theorem for periodic points*, Topology **42** (2003), no. 5, 1101–1124.

[5] B. Jiang, *Lectures on Nielsen Fixed Point Theory*, Contemporary Mathematics, 14. American Mathematical Society, Providence, R.I., 1983.

[6] ———, *Estimation of the number of periodic orbits*, Pacific J. Math. **172** (1996), no. 1, 151–185.

[7] B. Jiang, S. H. Lee, and M. H. Woo, *Reidemeister orbit sets*, Fund. Math. **183** (2004), no. 2, 139–156.

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