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C^2 DIFFEOMORPHISMS WITH THE INVERSE SHADOWING PROPERTY

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Abstract. Let f be a C^2-diffeomorphism on a closed surface which satisfies the Axiom A. Then f is in the C^2-interior of the set of all diffeomorphisms having the inverse shadowing property with respect to the class of the continuous methods if and only if f satisfies the strong transversality condition.

1. Introduction

The inverse shadowing property was introduced by Corless and Pilyugin in [1] as a "dual" notion of the shadowing property, and later Kloeden and Ombach defined this property using the notion of δ-method (see [2]). It is well known that the shadowing property and the inverse shadowing property are not equivalent. In fact, every shift homeomorphism has the shadowing property, but it does not have the inverse shadowing property (for more detail, see [4]). Moreover, we know that every pseudo-Anosov diffeomorphism on a compact surface has the inverse shadowing property, but it does not have the shadowing property.

In this paper, we characterize the C^2-interior of the set of all diffeomorphisms on a closed surface with the inverse shadowing property using the notion of strong transversality condition.

Let X be a compact metric space with metric d_0, and let $Z(X)$ denote the space of homeomorphisms on X with the C^0-metric d_0. Let $f \in Z(X)$. For $\delta > 0$, a sequence of points $\{x_i\}_{i=a}^{b}$ in X is called a δ-pseudo orbit of f if $d_0(f(x_i), x_{i+1}) < \delta$ for all $a \le i \le b-1$. Let $X^{\mathbb{Z}}$ be the space of all two sided sequences $\xi = \{x_n : n \in \mathbb{Z}\}$ with elements $x_n \in X$, endowed with the product topology. For a fixed $\delta > 0$, let $\Phi_f(\delta)$ denote the set of all δ-pseudo orbits of f. A mapping $\varphi : X \to \Phi_f(\delta) \subset X^{\mathbb{Z}}$ is said to be a δ-*method* for f if

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$\varphi(x)_0 = x$, where $\varphi(x)_0$ denotes the 0th component of $\varphi(x)$. Then each $\varphi(x)$ is a δ-pseudo orbit of f through x. For convenience, write $\varphi(x)$ for $\{\varphi(x)_k\}_{k\in\mathbb{Z}}$. We say that a map φ is called *a continuous δ-method* for f if φ is continuous. The set of all δ-methods for f will be denoted by $\mathcal{T}_0(f,\delta)$, and the set of all continuous δ-methods for f will be denoted by $\mathcal{T}_c(f,\delta)$. Every $g \in Z(X)$ with $d_0(f,g) < \delta$ induces a continuous δ-method $\varphi_g : X \to X^{\mathbb{Z}}$ for f by defining $\varphi_g(x) = \{g^k(x) : k \in \mathbb{Z}\}$, where d_0 is the C^0 metric. Denoted by $\mathcal{T}_h(f,\delta)$ the set of all continuous δ-methods φ_g for f which are induced by every $g \in Z(X)$ with $d_0(f,g) < \delta$.

Let M be a C^∞ closed manifold and $\mathrm{Diff}^r(M)(r \geq 1)$ be the space of C^r-diffeomorphisms of M endowed with C^r-topology. Denoted by d the distance on M induced from a Riemannian metric $\|\cdot\|$ on the tangent bundle TM.

Every $g \in \mathrm{Diff}(M)$ with $d_1(f,g) < \delta$ induces a continuous δ-method $\varphi_g : M \to M^{\mathbb{Z}}$ for f by defining $\varphi_g(x) = \{g^k(x) : k \in \mathbb{Z}\}$, where d_1 is the C^1 metric. Denote by $\mathcal{T}_d(f,\delta)$ the set of all continuous δ-methods φ_g for f which are induced by every $g \in \mathrm{Diff}(M)$ with $d_1(f,g) < \delta$. And so, we define the class Θ by

$$\Theta = \bigcup_{\delta>0} \mathcal{T}_\alpha(f,\delta),$$

where $\alpha = 0, c, h, d$. Then, we know that $\mathcal{T}_d(f) \subset \mathcal{T}_h(f) \subset \mathcal{T}_c(f) \subset \mathcal{T}_0(f)$, where $\mathcal{T}_\alpha(f) = \bigcup_{\delta>0} \mathcal{T}_\alpha(f,\delta), \alpha = 0, c, h, d$.

We say that f has *the inverse shadowing property with respect to the class* Θ if for any $\epsilon > 0$ there exists $\delta > 0$ such that for any δ-method $\varphi \in \Theta$ and for a point $x \in M$ there exists a point $y \in M$ such that

$$d(f^k(x), \varphi(y)_k) < \epsilon,\ k \in \mathbb{Z}.$$

Let Λ be a closed f-invariant subset of M. We say that $f|_\Lambda$ has the inverse shadowing property with respect to the class Θ if for any $\epsilon > 0$ there exists $\delta > 0$ such that for any δ-method $\varphi \in \Theta$ and for a point $x \in \Lambda$ there exists a point $y \in M$ such that

$$d(f^k(x), \varphi(y)_k) < \epsilon,\ k \in \mathbb{Z},$$

where $\varphi(y) = \{\varphi(y)_i\}_{i\in\mathbb{Z}} (\subset \Lambda)$ is a δ-pseudo orbit of f.

Note that f has the inverse shadowing property with respect to the class $\mathcal{T}_\alpha(f)$ if and only if f^n has the inverse shadowing property with respect to the class $\mathcal{T}_\alpha(f)$ for $n \in \mathbb{N} \setminus \{0\}$.

Let Λ be an invariant set for $f \in \mathrm{Diff}^r(M)$. We say that Λ is a *hyperbolic set* for f if there is a continuous splitting of the tangent bundle of M restricted to Λ: i.e., $T_\Lambda M$, which is Df-invariant splitting $E^s \oplus E^u$ and constants $C > 0$ and $\lambda \in (0,1)$ such that

$$\|D_x f^n|_{E^s_x}\| \leq C\lambda^n \ \text{ and } \ \|D_x f^{-n}|_{E^u_x}\| \leq C\lambda^n$$

for all $x \in \Lambda$ and $n \geq 0$.

We say that $f \in \text{Diff}^r(M)$ is *structurally stable* if there exists a C^r-neighborhood $\mathcal{U}(f)$ of f such that for each $g \in \mathcal{U}(f)$ there is a homeomorphism h of M satisfying $hf(x) = gh(x)$ for all $x \in M$. We say that $f \in \text{Diff}^r(M)$ satisfies *the strong transversality condition* when for every $x \in M$, the stable and unstable manifolds $W^s(x)$ and $W^u(x)$ are transverse at x. We denote by $\Omega(f)$ the set of non-wandering points of f. Let $f \in \text{Diff}^r(M)$ satisfies *Axiom A*. Then the non-wandering set $\Omega(f) = \overline{P(f)}$, and $\Omega(f)$ is hyperbolic, where $P(f)$ is the periodic set of f. A hyperbolic set Λ is called a *basic set* if there is a compact neighborhood U of Λ in M such that $\bigcap_{n\in\mathbb{Z}} f^n(U) = \Lambda$ and Λ is transitive for f. Since f satisfies Axiom A, we know that the non-wandering set $\Omega(f)$ is decomposed by finitely many closed, f-invariant and transitive subsets Λ_i, $1 \le i \le n$; i.e.,

$$\Omega(f) = \Lambda_1 \cup \Lambda_2 \cup \cdots \cup \Lambda_n,$$

where Λ_i is a basic set. Let Λ be a basic set for f. For $\epsilon > 0$, let $W^s_\epsilon(x) = \{y \in M : d(f^n(x), f^n(y)) < \epsilon \text{ for } n \ge 0\}$, and $W^u_\epsilon(x) = \{y \in M : d(f^n(x), f^n(y)) < \epsilon \text{ for } n \le 0\}$ be the *local stable set* and the *local unstable set* of $x \in \Lambda$, respectively. If Λ is hyperbolic, then there is $\epsilon_0 > 0$ such that for any $0 < \epsilon \le \epsilon_0$, the above sets are C^1-embedded submanifolds of M. The *stable manifold* $W^s(x)$ and the *unstable manifold* $W^u(x)$ of $x \in \Lambda$ are defined as follows: $W^s(x) = \{y \in M : d(f^i(x), f^i(y)) \to 0 \text{ as } i \to \infty\}$, and $W^u(x) = \{y \in M : d(f^i(x), f^i(y)) \to \infty \text{ as } i \to -\infty\}$, and we set $W^\sigma(\Lambda) = \bigcup_{x\in\Lambda} W^\sigma(x)$ ($\sigma = s, u$). A basic set Λ is called of *saddle type* if $0 < \text{dim}W^s(x) < \text{dimM}$ for $x \in \Lambda$.

In [6], Pilyugin proved that a structurally stable diffeomorphism has the inverse shadowing property with respect to classes of continuous methods. Also, he showed that any diffeomorphism belonging to the C^1-interior of the set of diffeomorphism with the inverse shadowing property with respect to classes of the continuous methods is structurally stable.

Let $f \in \text{Diff}^2(M^2)$ be satisfy Axiom A. In [7], Pilyugin and Sakai proved that f satisfies the C^0-transversality condition if and only if f has the inverse shadowing property with respect to a class of continuous methods. In [9], Sakai proved that for every diffeomorphism f on a surface satisfying Axiom A, f is in the C^2-interior of the set of all diffeomorphisms having the shadowing property if and only if f satisfies the strong transversality condition.

In this paper, we show that for every diffeomorphism $f \in \text{Diff}^2(M)$ on a surface satisfying Axiom A, f is in the C^2-interior of the set of all diffeomorphisms having the inverse shadowing property with respect to the class of continuous methods if and only if f satisfies the strong transversality condition.

Theorem A. *Let M be a C^∞ closed on a surface, and $f \in \text{Diff}^2(M)$ satisfy Axiom A. Then f is in the C^2-interior of the set of all diffeomorphisms having the inverse shadowing property with respect to the class of continuous method if and only if f satisfying the strong transversality condition.*

The "if" part of this theorem is proved in [8]. More precisely, let M be a closed surface and let $f \in \text{Diff}^2(M)$ satisfies Axiom A. If f satisfies the strong transversality condition, then f is structurally stable. Since f has the inverse shadowing property with respect to the class of the continuous method, choose a C^2-neighborhood $\mathcal{U}(f)$ of f such that for any $g \in \mathcal{U}(f)$, g has the inverse shadowing property with respect to the class of the continuous method. The "only if" part will proved by showing that if f is in the C^2-interior of the set of all diffeomorphisms having the inverse shadowing property with respect to the class of the continuous method.

2. Proof of Theorem A

Let M be a closed C^∞ surfaces, and let $f \in \mathrm{Diff}^2(M)$ satisfies Axiom A. Then the non-wandering set $\Omega(f)$ of f is a disjoint union of basic sets.

Proposition 1. *Let $\Lambda_i (i = 1, 2)$ be basic sets of $f \in \mathrm{Diff}^2(M)$ and suppose $x \in W^s(\Lambda_1) \cap W^u(\Lambda_2) \setminus \Lambda_1 \cup \Lambda_2$. If f is in the C^2-interior of the all diffeomorphisms having the inverse shadowing property with respect to the class of continuous methods, then $W^s(x)$ and $W^u(x)$ intersect transversely at x.*

Let M be a C^∞ closed surfaces. Then the notion of C^0-transversality condition between stable and unstable manifolds of basic sets Λ_i and Λ_j was introduced in [10] as follows. If there exists $x \in W^s(\Lambda_i) \cap W^u(\Lambda_j) \setminus \Lambda_i \cup \Lambda_j (i \neq j)$, then for $\epsilon > 0$ we denote by $C^s_\epsilon(x)$ the connected component of x in $W^s(x) \cap B_\epsilon(x)$. Similarly, $C^u_\epsilon(x) = W^u(x) \cap B_\epsilon(x)$ and let $B^+_\epsilon(x)$ and $B^-_\epsilon(x)$ be the components of $B_\epsilon(x) \setminus C^s_\epsilon(x)$, where $B_\epsilon(x) = \{y \in M : d(x, y) \leq \epsilon\}$. We say that $W^s(x)$ and $W^u(x)$ are C^0-*transversely at* x if $B^+_\epsilon(x) \cap C^u_\epsilon(x) \neq \phi$ and $B^-_\epsilon(x) \cap C^s_\epsilon(x) \neq \phi$ for any $\epsilon > 0$, where $\mathrm{dim}W^s(x) = 1$ and $\mathrm{dim}W^u(x) = 1$.

Let Λ be a basic set of $f \in \mathrm{Diff}^r(M)(r \geq 1)$. Since $\mathrm{dim}M = 2$, there is a locally f-invariant C^0-foliation with C^1-leaves defined in some neighborhood of Λ (see [5]). We use this foliation in the proof of the following lemma.

Lemma 2 ([7, Lemma 3.1])**.** *Let Λ_i $(i = 1, 2)$ be basic sets of $f \in \mathrm{Diff}^2(M)$, and suppose that $x \in W^s(p) \cap W^u(q) \setminus \Lambda_1 \cup \Lambda_2$ $(p \in \Lambda_1, q \in \Lambda_2)$. If f has the inverse shadowing property with respect to the class of continuous methods, then $W^s(p)$ and $W^u(q)$ intersect C^0-transversely at x.*

By Lemma 2, let $x \in W^s(p) \cap W^u(q)$. Then $W^s(p)$ and $W^u(q)$ intersect C^0-transversely at x. And so, they do not have a non-degenerate tangency at x.

If $f \in \mathrm{Diff}^r(M)(r \geq 1)$ satisfies Axiom A, then every basic sets are hyperbolic for f. And so $T_pM = E^s \oplus E^u$ for $p \in \Lambda$. Thus we can choose $\delta > 0$ and C^r maps $h_s : B^s_p(\delta) \to E^u_p$ and $h_u : B^u_p(\delta) \to E^s_p$ such that $W^s_\delta(p) = \mathrm{graph}(h_s)$ with $h_s(0) = 0$ and $Dh_s(0) = 0$. and $W^u_\delta(p) = \mathrm{graph}(h_u)$ with $h_u(0) = 0$ and $Dh_u(0) = 0$, where $B^\sigma_p(\delta) = \{v \in E^\sigma_p : \|v\| \leq \delta\}$. And so, we can take a C^r-diffeomorphism $\varphi : B_\delta(p)(= B^s_p(\delta) \oplus B^u_p(\delta)) \to T_pM$ such that $\varphi(W^s_\delta(p)) \subset E^s_p$. If we put $\epsilon = \delta/2$, then $\varphi(C^s_\epsilon(x)) \subset E^s_p(\delta)$. Therefore, we get the following lemma.

Lemma 3 ([9, Lemma 2])**.** *Let* Λ_i $(i = 1, 2)$ *be basic set of* $f \in \mathrm{Diff}^r(M)(r \geq 2)$ *and suppose that* $x \in W^s(p) \cap W^u(q) \setminus \Lambda_1 \cup \Lambda_2$ $(p \in \Lambda_1, q \in \Lambda_2)$*. Then there are* $\epsilon > 0$ *and a* C^r *diffeomorphism* $h_x : B_\epsilon(x) \to \mathbb{R}^2 = \{(v, w)|v, w \in \mathbb{R}\}$ *such that* $h_x(x) = (0, 0)$ *and* $h(C^s_\epsilon(x)) \subset v$*-axis.*

Proof of Proposition 1. We shall prove that if there is a C^2-neighborhood $\mathcal{U}(f)$ of $f \in \mathrm{Diff}^2(M)$ such that for any $g \in \mathcal{U}(f)$, g has the inverse shadowing property with respect to the the class of continuous methods, then $T_xM = T_xW^s(x, g) + T_xW^u(x, g)$, where $x \in W^s(p) \cap W^u(q) \setminus \Lambda_1 \cup \Lambda_2$. To get a contradiction. Let us assume that there exist points $p,\ q \in \Omega(f)$ and $y \in W^s(p) \cap W^u(q)$ such that y is not transversal. By Lemma 3, there are $\epsilon > 0$ and a C^r-diffeomorphism $h_y : B_\epsilon(y) \to \mathbb{R}^2$ such that $h_y(y) = (0, 0)$ and $h(C^s_\epsilon(y)) \subset v$-axis, where $B_\epsilon(y)(= B^s_\epsilon(y) \oplus B^u_\epsilon(y))$ is a neighborhood of y and $B^s_\epsilon(y) \subset T_yW^s(p)$ and $B^u_\epsilon(y) \subset T_yW^u(q)$. Since y is not transversal, there exists a segment $\langle v\rangle = T_yW^s(p) \cap T_yW^u(q)$, where $\langle v\rangle = \{tv : t \in \mathbb{R}\}$. Thus we can choose $\epsilon_0 > 0$ and a C^2-function $\gamma : [-\epsilon_0, \epsilon_0] \to \mathbb{R}$ such that $\mathrm{graph}(\gamma) \subset h_y(C^u_\delta(y))$ and $(0, \gamma(0)) = h_y(y) = (0, 0)$ for some $\delta > 0$.

Since $W^s(p)$ and $W^u(q)$ are C^0-non-transversal at y we have $\gamma'(0) = 0$ and $\gamma''(0) = 0$. Choose $\epsilon_1 > 0$ and g C^2-nearby f such that $g(z) = id$ for $z \in B_{\epsilon_1}(y)$. Then we get $g(W^s(p) \cap B_{\epsilon_1/4}(y)) \subset W^u(q)$ and $g_1 = g^{-1} \circ f \in \mathcal{U}(f)$. Consequently, $W^u(q, g_1) \cap B_{\epsilon_1}(y) = W^s(p, g_1) \cap B_{\epsilon_1}(y)$. For $g_1 \in \mathcal{U}(f)$, $W^u(q, g_1)$ and $W^s(p, g_1)$ are not transversal at y. And so by Lemma 2, this is a contradiction. This completes the proof of Proposition 1. □

End of the proof of Theorem A. We show that if f is in the C^2-interior of the set of all diffeomorphisms having the inverse shadowing property with respect to class Θ, then f satisfies the strong transversality condition. Since M is a C^∞ closed surface and $f \in \mathrm{Diff}^2(M)$ satisfy Axiom A, we know that the non-wandering set $\Omega(f)$ is decomposed by a finite union of basic sets Λ_i : i.e., $\Omega(f) = \Lambda_1 \cup \cdots \cup \Lambda_n$. Thus we can consider Λ_i and Λ_j, for $0 \leq i, (\neq) j \leq n$. Suppose that f is in the C^2-interior of the set of all diffeomorphisms having the inverse shadowing property with respect to the class of the continuous methods. Then by Proposition 1, we can get that f satisfies the strong transversality condition, This completes the proof of Theorem A. □

Form [9] and our result, we get the following fact. Note that if $f \in \mathrm{Diff}^r(M)$ is structurally stable, then f is in the C^2-interior of the set of all diffeomorphisms having the shadowing property (or inverse shadowing property with respect to the class of continuous methods).

Corollary. *Let* M *be a* C^∞ *closed surfaces and* $f \in \mathrm{Diff}^2(M)$ *satisfy Axiom A. Then* f *is in the* C^2*-interior of the set of all diffeomorphisms having the inverse shadowing property with respect to the class of continuous methods if and only if* f *is in the* C^2*-interior of the set of all diffeomorphisms having the shadowing property.*

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