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αm-OPEN SETS AND αM-CONTINUOUS FUNCTIONS

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Abstract. In this paper, we introduce the notions of αm-open sets and αM-continuous functions and investigate some properties of such concepts.

1. Introduction

In [4], Popa and Noiri introduced the notion of minimal structure which is a generalization of a topology on a given nonempty set. And they introduced the notion of M-continuous functions as functions defined between minimal structures. They showed that the M-continuous functions on minimal structures have properties similar to those of continuous functions between topological spaces.

In this paper, we introduce the notions of αm-open sets, α-interior and α-closure operators in minimal structures. We investigate some basic properties of such notions. Also we introduce the notion of αM-continuous functions and study characterizations of αM-continuous functions by using the α-interior and α-closure operators.

2. Preliminaries

Definition 2.1 ([1, 4]). A subfamily m_X of the power set $P(X)$ of a nonempty set X is called a *minimal structure* on X if $\emptyset \in m_X$ and $X \in m_X$. By (X, m_X), we denote a nonempty set X with a minimal structure m_X on X. Simply we call (X, m_X) a minimal structure on X. Set $M(x) = \{U \in m_X : x \in U\}$.

Definition 2.2 ([1, 4]). Let (X, m_X) be a minimal structure. For a subset A of X, the closure of A and the interior of A are defined as the following:

(1) $mInt(A) = \cup\{U : U \subseteq A, U \in m_X\}$.

(2) $mCl(A) = \cap\{F : A \subseteq F, X - F \in m_X\}$.

Theorem 2.3 ([1, 4]). *Let (X, m_X) be a minimal structure and $A \subseteq X$.*

(1) *$X = mInt(X)$ and $\emptyset = mCl(\emptyset)$.*

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(2) $mInt(A) \subseteq A$ *and* $A \subseteq mCl(A)$.
(3) *If* $A \in m_X$, *then* $mInt(A) = A$ *and if* $X - F \in m_X$, *then* $mCl(F) = F$.
(4) *If* $A \subseteq B$, *then* $mInt(A) \subseteq mInt(B)$ *and* $mCl(A) \subseteq mCl(B)$.
(5) $mInt(mInt(A)) = mInt(A)$ *and* $mCl(mCl(A)) = mCl(A)$.
(6) $mCl(X - A) = X - mInt(A)$ *and* $mInt(X - A) = X - mCl(A)$.

Definition 2.4 ([4]). Let (X, m_X) and (Y, m_Y) be two minimal structures. Then $f : X \to Y$ is said to be *M-continuous* if for $x \in X$ and $V \in M(f(x))$, there is $U \in M(x)$ such that $f(U) \subseteq V$.

3. αm-open sets and αM-continity

Definition 3.1. Let (X, m_X) be a minimal structure and $A \subset X$. A subset A of X is called an *αm-open set* if $A \subseteq mInt(mCl(mInt(A)))$. The complement of an αm-open set is called an *αm-closed set*. The family of all αm-open sets in X will be denoted by $\alpha M(X)$.

Remark 3.2. Let (X, τ) be a topological space and $A \subseteq X$. A is called an α-open set [3] if $A \subseteq int(cl(int(A)))$. If the minimal structure m_X is a topology, clearly an αm-open set is α-open.

From Definition of 3.1, obviously the following statements are obtained:

Lemma 3.3. *Let* (X, m_X) *be a minimal structure. Then*
(1) *Every m-open set is* αm*-open.*
(2) *A is an* αm*-closed set if and only if* $mCl(mInt(mCl(A))) \subseteq A$.

Theorem 3.4. *Let* (X, m_X) *be a minimal structure. Any union of* αm*-open sets is* αm*-open.*

Proof. Let A_i be an αm-open set for $i \in J$. From Definition 3.1 and Theorem 2.3(4), it follows

$$A_i \subseteq mInt(mCl(mInt(A_i))) \subseteq mInt(mCl(mInt(\cup A_i))).$$

This implies $\cup A_i \subseteq mInt(mCl(mInt(\cup A_i)))$. Hence $\cup A_i$ is an αm-open set. □

Remark 3.5. Let (X, m_X) be a minimal structure. The intersection of any two αm-open sets may not be αm-open set as shown in the next example.

Example 3.6. Let $X = \{a, b, c\}$ and $m_X = \{\emptyset, \{a, b\}, \{a, c\}, X\}$ a minimal structure in X. Then obviously $\{a, b\}$ and $\{a, c\}$ are α-m open sets. But $\{a\}$ is not αm-open because of $mInt(mCl(mInt(\{a\}))) = \emptyset$. Thus the intersection of two αm-open sets $\{a, b\}$ and $\{a, c\}$ is not αm-open.

Definition 3.7. Let (X, m_X) be a minimal structure. For a subset A of X, the α-closure of A and the α-interior of A, denoted by $\alpha mCl(A)$ and $\alpha mInt(A)$, respectively, are defined as the following:

$$\alpha mCl(A) = \cap\{F : A \subseteq F, F \text{ is } \alpha m\text{-closed in } X\},$$
$$\alpha mInt(A) = \cup\{U : U \subseteq A, U \text{ is } \alpha m\text{-open in } X\}.$$

Theorem 3.8. *Let (X, m_X) be a minimal structure and $A \subset X$. Then*
(1) $\alpha mInt(A) \subseteq A$.
(2) *If $A \subseteq B$, then $\alpha mInt(A) \subseteq \alpha mInt(B)$.*
(3) *A is αm-open if and only if $\alpha mInt(A) = A$.*
(4) $\alpha mInt(\alpha mInt(A)) = \alpha mInt(A)$.
(5) $\alpha mCl(X - A) = X - \alpha mInt(A)$ *and* $\alpha mInt(X - A) = X - \alpha mCl(A)$.

Proof. (1), (2) Obvious.
(3) It follows from Theorem 3.4.
(4) It follows from (3).
(5) For $A \subseteq X$,

$$\begin{aligned} X - \alpha mInt(A) &= X - \cup\{U : U \subseteq A, U \text{ is } \alpha m\text{-open}\} \\ &= \cap\{X - U : U \subseteq A, U \text{ is } \alpha m\text{-open}\} \\ &= \cap\{X - U : X - A \subseteq X - U, U \text{ is } \alpha m\text{-open}\} \\ &= \alpha mCl(X - A). \end{aligned}$$

Similarly, we have $\alpha mInt(X - A) = X - \alpha mCl(A)$. □

Theorem 3.9. *Let (X, m_X) be a minimal structure and $A \subset X$. Then*
(1) $A \subseteq \alpha mCl(A)$.
(2) *If $A \subseteq B$, then $\alpha mCl(A) \subseteq \alpha mCl(B)$.*
(3) *F is αm-closed if and only if $\alpha mCl(F) = F$.*
(4) $\alpha mCl(\alpha mCl(A)) = \alpha mCl(A)$.

Proof. It is similar to the proof of Theorem 3.8. □

Theorem 3.10. *Let (X, m_X) be a minimal structure and $A \subseteq X$. Then*
(1) *$x \in \alpha mCl(A)$ if and only if $A \cap V \neq \emptyset$ for every αm-open set V containing x.*
(2) *$x \in \alpha mInt(A)$ if and only if there exists an αm-open set U such that $U \subset A$.*

Proof. (1) Suppose there is an αm-open set V containing x such that $A \cap V = \emptyset$. Then $X - V$ is an αm-closed set such that $A \subseteq X - V$, $x \notin X - V$. This implies $x \notin \alpha mCl(A)$.

The reverse relation is obvious.

(2) Obvious. □

Definition 3.11. Let $f : X \to Y$ be a function between minimal structures (X, m_X) and (Y, m_Y). Then f is said to be *αM-continuous* if for each x and each m-open set V containing $f(x)$, there exists an αm-open set U containing x such that $f(U) \subseteq V$.

Every M-continuous function is αM-continuous but the converse may not be true.

Example 3.12. Let $X = \{a, b, c\}$. Consider two minimal structures defined as follows: $m_1 = \{\emptyset, \{a\}, X\}$, $m_2 = \{\emptyset, \{a, b\}, \{a, c\}, X\}$.

Let $f : (X, m_1) \to (X, m_2)$ be the identity function. Then f is αM-continuous but not M-continuous.

Remark 3.13. Let $f : X \to Y$ be an αM-continuous function between minimal structures (X, m_X) and (Y, m_Y). If the minimal structures (X, m_X) and (Y, m_Y) are topologies on X and Y, respectively, then f is α-continuous [2].

Theorem 3.14. *Let $f : X \to Y$ be a function on two minimal structures (X, m_X) and (Y, m_Y). Then the following statements are equivalent*:

(1) *f is αM-continuous.*
(2) *$f^{-1}(V)$ is an αm-open set for each m-open set V in Y.*
(3) *$f^{-1}(B)$ is an αm-closed set for each m-closed set B in Y.*
(4) *$f(\alpha mCl(A)) \subseteq mCl(f(A))$ for $A \subseteq X$.*
(5) *$\alpha mCl(f^{-1}(B)) \subseteq f^{-1}(mCl(B))$ for $B \subseteq Y$.*
(6) *$f^{-1}(mInt(B)) \subseteq \alpha mInt(f^{-1}(B))$ for $B \subseteq Y$.*

Proof. (1) ⇒ (2) Let V be an m-open set in Y and $x \in f^{-1}(V)$. By hypothesis, there exists an αm-open set U_x containing x such that $f(U) \subseteq V$. This implies $x \in U_x \subseteq f^{-1}(V)$ for all $x \in f^{-1}(V)$. Hence by Theorem 3.4, $f^{-1}(V)$ is αm-open.

(2) ⇒ (3) Obvious.

(3) ⇒ (4) For $A \subseteq X$,

$$\begin{aligned} f^{-1}(mCl(f(A))) &= f^{-1}(\cap\{F \subseteq Y : f(A) \subseteq F \text{ and } F \text{ is } m\text{-closed}\}) \\ &= \cap\{f^{-1}(F) \subseteq X : A \subseteq f^{-1}(F) \text{ and } F \text{ is } \alpha m\text{-closed}\} \\ &\supseteq \cap\{K \subseteq X : A \subseteq K \text{ and } K \text{ is } \alpha m\text{-closed}\} \\ &= \alpha mCl(A). \end{aligned}$$

Hence $f(\alpha mCl(A)) \subseteq mCl(f(A))$.

(4) ⇒ (5) For $A \subseteq X$, from (4), it follows

$$f(\alpha mCl(f^{-1}(A))) \subseteq mCl(f(f^{-1}(A))) \subseteq mCl(A).$$

Hence we get (5).

(5) ⇒ (6) For $B \subseteq Y$, from $mInt(B) = Y - mCl(Y - B)$ and (5), it follows:

$$\begin{aligned} f^{-1}(mInt(B)) &= f^{-1}(Y - mCl(Y - B)) \\ &= X - f^{-1}(mCl(Y - B)) \\ &\subseteq X - \alpha mCl(f^{-1}(Y - B)) \\ &= \alpha mInt(f^{-1}(B)). \end{aligned}$$

Hence (6) is obtained.

(6) ⇒ (1) Let $x \in X$ and V an m-open set containing $f(x)$. Then from (6) and Theorem 2.5, it follows $x \in f^{-1}(V) = f^{-1}(mInt(V)) \subseteq \alpha mInt(f^{-1}(V))$.

So from Theorem 3.10, we can say that there exists an αm-open set U containing x such that $x \in U \subseteq f^{-1}(V)$. Hence f is αM-continuous. □

Lemma 3.15. *Let (X, m_X) be a minimal structure and $A \subseteq X$. Then*
(1) $mCl(mInt(mCl(A))) \subseteq mCl(mInt(mCl(\alpha mCl(A)))) \subseteq \alpha mCl(A)$.
(2) $\alpha mInt(A) \subseteq mInt(mCl(mInt(\alpha mInt(A)))) \subseteq mInt(mCl(mInt(A)))$;

Proof. (1) For $A \subset X$, by Theorem 3.9, $\alpha mCl(A)$ is an αm-closed set. Hence from Lemma 3.3, we have

$$mCl(mInt(mCl(A))) \subseteq mCl(mInt(mCl(\alpha mCl(A)))) \subseteq \alpha mCl(A).$$

(2) It is similar to the proof of (1). □

Theorem 3.16. *Let $f : X \to Y$ be a function on two minimal structure (X, m_X) and (Y, m_Y). Then the following statements are equivalent*:
(1) *f is αM-continuous.*
(2) *$f^{-1}(V) \subseteq mInt(mCl(mInt(f^{-1}(V))))$ for each m-open set V in Y.*
(3) *$mCl(mInt(mCl(f^{-1}(F)))) \subseteq f^{-1}(F)$ for each m-closed set F in Y.*
(4) *$f(mCl(mInt(mCl(A)))) \subseteq mCl(f(A))$ for $A \subseteq X$.*
(5) *$mCl(mInt(mCl(f^{-1}(B)))) \subseteq f^{-1}(mCl(B))$ for $B \subseteq Y$.*
(6) *$f^{-1}(mInt(B)) \subseteq mInt(mCl(mInt(f^{-1}(B))))$ for $B \subseteq Y$.*

Proof. (1) ⇔ (2) It follows from Theorem 3.14 and definition of αm-open sets.

(1) ⇔ (3) It follows from Theorem 3.14 and Lemma 3.3.

(3) ⇒ (4) Let $A \subseteq X$. Then from Theorem 3.14 (4) and Lemma 3.15, it follows $mCl(mInt(mCl(A))) \subseteq \alpha mCl(A)) \subseteq f^{-1}(mCl(f(A)))$.

Hence $f(mCl(mInt(mCl(A)))) \subseteq mCl(f(A))$.

(4) ⇒ (5) Obvious.

(5) ⇒ (6) From (5) and Theorem 2.3, it follows:

$$\begin{aligned} f^{-1}(mInt(B)) &= f^{-1}(Y - mCl(Y - B)) \\ &= X - f^{-1}(mCl(Y - B)) \\ &\subseteq X - mCl(mInt(mCl(f^{-1}(Y - B)))) \\ &= mInt(mCl(mInt(f^{-1}(B)))). \end{aligned}$$

Hence, (6) is obtained.

(6) ⇒ (1) Let V be an m-open set in Y. Then by (6) and Theorem 2.3, we have $f^{-1}(V) = f^{-1}(mInt(V)) \subseteq mInt(mCl(mInt(f^{-1}(V))))$. This implies $f^{-1}(V)$ is an αm-open set. Hence by (2), f is αM-continuous. □

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