Commun. Korean Math. Soc. **25** (2010), No. 2, pp. 245–249
DOI 10.4134/CKMS.2010.25.2.245

LOCALLY CONFORMAL KÄHLER MANIFOLDS AND CONFORMAL SCALAR CURVATURE

Jaeman Kim

Abstract. We show that on a compact locally conformal Kähler manifold M^{2n} (dim $M^{2n} = 2n \geq 4$), M^{2n} is Kähler if and only if its conformal scalar curvature k is not smaller than the scalar curvature s of M^{2n} everywhere. As a consequence, if a compact locally conformal Kähler manifold M^{2n} is both conformally flat and scalar flat, then M^{2n} is Kähler. In contrast with the compact case, we show that there exists a locally conformal Kähler manifold with k equal to s, which is not Kähler.

1. Introduction

A locally conformal Kähler (lck) manifold $M^{2n} = (M^{2n}, J, g)$ (dim $M^{2n} = 2n \geq 4$) is a Hermitian manifold (i.e., the metric g of M^{2n} is compatible with complex structure J) whose metric g is locally conformal to a Kähler metric, which is equivalent to the existence of an 1-form θ (called Lee form) of M^{2n} such that $d\Omega = \theta \wedge \Omega$ and $d\theta = 0$ [10], [11], where Ω is the Kähler form of M^{2n}. From the viewpoint of conformal geometry, we can define a conformally well-behaved function (namely, conformal scalar curvature) k on M^{2n} associated with Weyl curvature tensor, which appeared in the literature of Hermitian geometry [1], [4], [5]. It is well known that the standard Hopf surface M^4 is a compact lck manifold with positive scalar curvature s and vanishing conformal scalar curvature k, which is not Kähler. In this note, we investigate a condition of the conformal scalar curvature k for a compact lck manifold M^{2n} to be Kähler and show that the sign of difference between k and s is a crucial condition for a compact lck manifold M^{2n} to be Kähler. More precisely, we prove the following:

Received July 10, 2009.
2000 *Mathematics Subject Classification.* 53A30, 53B35, 53C25, 53C55, 53C56.
Key words and phrases. compact locally conformal Kähler manifold, conformal scalar curvature, Kähler, conformally flat and scalar flat, a locally conformal Kähler manifold with k equal to s.
This study is supported by Kangwon National University.

Theorem 1.1. *Let $M^{2n} = (M^{2n}, J, g)$ be a compact lck manifold. Then M^{2n} is Kähler if and only if its conformal scalar curvature k is not smaller than the scalar curvature s of M^{2n} everywhere.*

As a consequence, we have:

Corollary 1.2. *If a compact lck manifold M^{2n} is conformally flat, and its scalar curvature vanishes, then M^{2n} is Kähler.*

Contrary to the compact case, we obtain the following:

Theorem 1.3. *Let $R^{2n}_+ = \{(x_1, x_2, \dots, x_{2n}) | x_{2n} > 0\}$ and J be the natural complex structure defined by $J\left(\frac{\partial}{\partial x_{2i-1}}\right) = \frac{\partial}{\partial x_{2i}}$, $J\left(\frac{\partial}{\partial x_{2i}}\right) = -\frac{\partial}{\partial x_{2i-1}}$. And a metric $g = (g_{ij})$ on R^{2n}_+ is given by $g_{ij} = x_{2n}^{\frac{4}{2n-2}} \delta_{ij}$. Then $M = (R^{2n}_+, J, g)$ is a lck manifold with $k = s$, which is not Kähler.*

2. Preliminaries

We shall denote by $M^{2n} = (M^{2n}, J, g)$ ($\dim M^{2n} = 2n \geq 4$) a lck manifold; by g its metric; by $\{U_\alpha\}$ an open covering of M^{2n} endowed with smooth functions f_α on U_α such that $\widetilde{g_\alpha} = e^{-f_\alpha} g$ are Kähler metrics; by J the complex structure; by ∇ the Levi-Civita connection of g; by θ the Lee form of M^{2n}, which satisfies $d\Omega = \theta \wedge \Omega$ and $d\theta = 0$, where Ω is the Kähler form of M^{2n}, i.e., $g(X, JY) = \Omega(X, Y)$. Note that M^{2n} is Kähler if and only if $\theta = 0$. The Riemannian curvature tensor R, the Ricci tensor Ric and the scalar curvature s are given by

$$R(X,Y)Z = [\nabla_X, \nabla_Y]Z - \nabla_{[X,Y]}Z,$$
$$Ric(X,Y) = \mathrm{Trace}\{Z \to R(Z,X)Y\},\ s = \mathrm{Trace}_g Ric.$$

Furthermore, the $*$-Ricci tensor and $*$-scalar curvature of (J, g) are given by

$$Ric^*(X,Y) = \mathrm{Trace}\{Z \to -JR(Z,X)JY\},\ s^* = \mathrm{Trace}_g Ric^*.$$

The Riemannian metric g induces a metric on the bundle $\bigwedge^2$ of 2-vectors on M^{2n} by

$$\langle X_k \wedge X_l, Y_p \wedge Y_q \rangle = \det(g(X_i, Y_j)).$$

Similarly, one can define a metric on the bundle $\bigwedge^2$ of 2-forms on M^{2n} by

$$\langle A \wedge B, C \wedge D \rangle = \langle A^\sharp \wedge B^\sharp, C^\sharp \wedge D^\sharp \rangle.$$

Here the symbol $\sharp$ is the natural isomorphism from 1-forms to vector fields. We also regard the curvature tensor as a (0,4)-tensor or an endomorphism of the 2-form bundle as follows:

$$R(X,Y,Z,V) = -g(R(X,Y)Z,V),\ \langle R(A \wedge B), C \wedge D \rangle = R(A^\sharp, B^\sharp, C^\sharp, D^\sharp).$$

The Riemannian curvature tensor R has the following well known $SO(2n)$-decomposition [2], [6], [7];

$$R = \frac{s}{4n(2n-1)} g \star g + \frac{1}{2n-2} Ric_o \star g + W, \tag{2.1}$$

where Ric_o is the traceless Ricci tensor and W is the Weyl curvature tensor. Here the symbol $\star$ is the Nomizu-Kulkarni product of symmetric (0,2)-tensors generating a curvature type tensor. Note that $Ric_o = 0$ if and only if M^{2n} is Einstein [2], [7], [8]; $W = 0$ if and only if M^{2n} is conformally flat [2], [6]. Now we define the conformal scalar curvature k of M^{2n} by

$$k = \frac{2n-1}{n-1}\langle W(\Omega), \Omega\rangle.$$

In particular, if $2n = 4$, then $k = 3\langle W^+(\Omega), \Omega\rangle$, which appeared in the literature of Hermitian geometry [1], [4]. Therefore, if M^{2n} is conformally flat, then the conformal scalar curvature k of M^{2n} is zero. Note that k has conformal weight -2, that is, if we replace g by $f^2 g$ for some non-vanishing function f, then k is replaced by $f^{-2}k$.

3. Proof of Theorem 1.1

Let $M^{2n} = (M^{2n}, J, g)$ be a compact lck manifold. From the relation (2.1) and $\langle (Ric_o \star g)(\Omega), \Omega\rangle = 0$, we obtain

$$\langle R(\Omega), \Omega\rangle = \frac{s}{4n(2n-1)} 2\langle\Omega, \Omega\rangle + \langle W(\Omega), \Omega\rangle.$$

Therefore, from $\langle\Omega, \Omega\rangle = n$ and the definitions of s^* and k, the above identity yields

$$\frac{s^*}{2} = \frac{s}{2(2n-1)} + \frac{n-1}{2n-1}k,$$

which implies

$$s - k = \frac{2n-1}{2n-2}(s - s^*). \tag{3.2}$$

Now suppose that M^{2n} is Kähler and hence s and s^* coincide; this is a consequence of the Kähler identity [2] $R(X,Y)(JZ) = J(R(X,Y)Z)$, which itself follows from the fact that $\nabla J = 0$. Therefore, the equation (3.2) implies $k = s$. Conversely, let a compact lck manifold M^{2n} satisfy $k \geq s$. And assume that $\widetilde{g} = e^{-f}g$ is the lck metric of g. Then one gets the well-known formula [3], [9]

$$\begin{aligned} e^f \widetilde{g}(\widetilde{R}(X,Y)Z, W) = g(R(X,Y)Z,W) &- \frac{1}{2}\{L(X,Z)g(Y,W) - L(Y,Z)g(X,W) \\ &+ L(Y,W)g(X,Z) - L(X,W)g(Y,Z)\} \\ &- \frac{||\theta||^2}{4}\{g(Y,Z)g(X,W) - g(X,Z)g(Y,W)\}, \end{aligned}$$

which yields

$$\begin{aligned} &g(R(X,Y)Z,W) - g(R(X,Y)JZ, JW) \\ = &\frac{1}{2}\{L(X,Z)g(Y,W) - L(Y,Z)g(X,W) + L(Y,W)g(X,Z) - L(X,W)g(Y,Z)\} \\ &+ \frac{||\theta||^2}{4}\{g(Y,Z)g(X,W) - g(X,Z)g(Y,W)\} \end{aligned}$$

$$- \frac{1}{2}\{L(X,JZ)g(Y,JW) - L(Y,JZ)g(X,JW) + L(Y,JW)g(X,JZ) - L(X,JW)g(Y,JZ)\}$$
$$- \frac{||\theta||^2}{4}\{g(Y,JZ)g(X,JW) - g(X,JZ)g(Y,JW)\},$$

where $L(X,Y) = (\nabla_X\theta)Y + \frac{1}{2}\theta(X)\theta(Y)$ and so $L(X,Y) = L(Y,X)$ since θ is closed. Now if we take $X = \frac{\partial}{\partial x_i}, Y = \frac{\partial}{\partial x_j}, Z = \frac{\partial}{\partial x_k}, W = \frac{\partial}{\partial x_l}$, where x_i $(i = 1,\dots,2n)$ are real coordinates on M^{2n}, and contract with $g^{il}g^{jk}$, and then use the identity (3.2), we get the following [3], [5], [9]:

$$s - k = \frac{2n-1}{2n-2}(2(n-1)\delta\theta + (n-1)^2||\theta||^2), \tag{3.3}$$

where $\delta\theta = -\mathrm{div}\theta$.
By integrating (3.3) over M^{2n}, we conclude that the condition of $k \geq s$ yields $\theta = 0$. Summing up the above arguments, we conclude that on a compact lck manifold, the Kähler condition is equivalent to that of $k \geq s$. This completes the proof of Theorem 1.1 and hence it is obvious that Corollary 1.2 holds because of $k = s$.

4. A lck manifold with k equal to s which is not Kähler

Let $R^{2n}_+ = \{(x_1, x_2, \dots, x_{2n})|x_{2n} > 0\}$ and J be the natural complex structure defined by $J\Big(\frac{\partial}{\partial x_{2i-1}}\Big) = \frac{\partial}{\partial x_{2i}}$, $J\Big(\frac{\partial}{\partial x_{2i}}\Big) = -\frac{\partial}{\partial x_{2i-1}}$. We define a Riemannian metric $g = (g_{ij})$ on R^{2n}_+ by $g_{ij} = x_{2n}^{\frac{4}{2n-2}}\delta_{ij}$. It is obvious that the metric g is compatible with complex structure J. With respect to $\{\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \dots, \frac{\partial}{\partial x_{2n}}\}$, we set $\nabla_{\frac{\partial}{\partial x_i}}\frac{\partial}{\partial x_j} = \sum_{k=1,\dots,2n}\Gamma^k_{ij}\frac{\partial}{\partial x_k}$ for $1 \leq i,j \leq 2n$. Then the Christoffel symbol Γ^k_{ij} of metric g is obtained as follows:

$$\begin{aligned}
\Gamma^1_{1(2n)} &= \Gamma^1_{(2n)1} = \Gamma^2_{2(2n)} = \Gamma^2_{(2n)2} = \cdots \\
&= \Gamma^{2n-1}_{(2n-1)2n} = \Gamma^{2n-1}_{2n(2n-1)} = \frac{2}{2n-2}(\frac{1}{x_{2n}}), \\
\Gamma^{2n}_{2n(2n)} &= \frac{2}{2n-2}(\frac{1}{x_{2n}}), \Gamma^{2n}_{11} = \Gamma^{2n}_{22} = \cdots \\
&= \Gamma^{2n}_{(2n-1)(2n-1)} = -\frac{2}{2n-2}(\frac{1}{x_{2n}})
\end{aligned}$$

and are otherwise zero. And so we have

$$\begin{aligned}
R^2_{121} &= R^3_{131} = \cdots = R^{2n-1}_{1(2n-1)1} = R^1_{212} = R^3_{232} = \cdots = R^{2n-1}_{2(2n-1)2} \\
&= \cdots = R^1_{(2n-2)1(2n-2)} = R^2_{(2n-2)2(2n-2)} = \cdots \\
&= R^{2n-1}_{(2n-2)(2n-1)(2n-2)} = R^1_{(2n-1)1(2n-1)} = R^2_{(2n-1)2(2n-1)} \\
&= \cdots = R^{2n-2}_{(2n-1)(2n-2)(2n-1)} = -\frac{4}{(2n-2)^2}(\frac{1}{x^2_{2n}}),
\end{aligned}$$

$$R^1_{(2n)1(2n)} = R^{2n}_{1(2n)1} = R^2_{(2n)2(2n)} = R^{2n}_{2(2n)2} = R^3_{(2n)3(2n)}$$
$$= R^{2n}_{3(2n)3} = \cdots = R^{2n}_{(2n-1)(2n)(2n-1)} = \frac{2}{2n-2}(\frac{1}{x_{2n}^2})$$

and are otherwise zero. Here $R^p_{jlk} = g^{pi}R_{ijlk}$ and $R_{ijlk} = g(R(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_k}), \frac{\partial}{\partial x_l})$. Hence we get

$$s = s^* = 0,$$

which yields from the equation (3.2)

$$s = k.$$

On the other hand, we have

$$d\Omega = \frac{4}{2n-2}(\frac{1}{x_{2n}})dx_{2n} \wedge \Omega,$$

which implies that the Lee form $\theta = \frac{4}{2n-2}(\frac{1}{x_{2n}})dx_{2n}$ of M^{2n} is closed. Summing up the above arguments, we obtain that R^{2n}_+ allows a lck structure (J, g) with $k = s$, which is not Kähler. This completes the proof of Theorem 1.3.

References

[1] V. Apostolov and P. Gauduchon, *The Riemannian Goldberg-Sachs theorem*, Internat. J. Math. **8** (1997), no. 4, 421–439.

[2] A. L. Besse, *Einstein Manifolds*, Springer-Verlag, Berlin, 1987.

[3] S. Dragomir and L. Ornea, *Locally Conformal Kähler Geometry*, Progress in Mathematics, 155. Birkhäuser Boston, Inc., Boston, MA, 1998.

[4] P. Gauduchon and S. Ivanov, *Einstein-Hermitian surfaces and Hermitian Einstein-Weyl structures in dimension* 4, Math. Z. **226** (1997), no. 2, 317–326.

[5] P. Gauduchon, *La* 1*-forme de torsion d'une variété hermitienne compacte*, Math. Ann. **267** (1984), no. 4, 495–518.

[6] Z. Hu, H. Li, and U. Simon, *Schouten curvature functions on locally conformally flat Riemannian manifolds*, J. Geom. **88** (2008), no. 1-2, 75–100.

[7] J. Kim, *Rigidity theorems for Einstein-Thorpe metrics*, Geom. Dedicata **80** (2000), no. 1-3, 281–287.

[8] ———, *On Einstein Hermitian manifolds*, Monatsh. Math. **152** (2007), no. 3, 251–254.

[9] I. Vaisman, *Some curvature properties of complex surfaces*, Ann. Mat. Pura Appl. (4) **132** (1982), 1–18.

[10] ———, *Some curvature properties of locally conformal Kähler manifolds*, Trans. Amer. Math. Soc. **259** (1980), no. 2, 439–447.

[11] ———, *A theorem on compact locally conformal Kähler manifolds*, Proc. Amer. Math. Soc. **75** (1979), no. 2, 279–283.

Department of Mathematics Education
Kangwon National University
Chunchon 200-701, Korea
E-mail address: jaeman64@kangwon.ac.kr