

ON ALMOST r -PARACONTACT RIEMANNIAN MANIFOLD WITH A CERTAIN CONNECTION

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ABSTRACT. In a Riemannian manifold, the existence of a new connection is proved. In particular cases, this connection reduces to several symmetric, semi-symmetric and quarter symmetric connections, even some of them are not introduced so far. So, in this paper, we define a quarter symmetric semi-metric connection in an almost r -paracontact Riemannian manifold and consider invariant, non-invariant and anti-invariant hypersurfaces of an almost r -paracontact Riemannian manifold with that connection.

1. Introduction

The elementary idea of the considering manifold was originated from the almost contact manifold with Riemannian connection. The almost contact manifold can be naturally generalized as almost paracontact manifold and further almost r -paracontact one. On the other hand, K. Yano and T. Imai [18] gave the most general form of quarter symmetric metric connection and studied some special cases as their examples by using the torsion tensor.

Hypersurfaces of almost paracontact manifold were studied by T. Adati in [1]. A. Bucki also considered hypersurfaces of almost r -paracontact Riemannian manifold in [4]. Some properties of invariant hypersurfaces of r -paracontact Riemannian manifold were investigated by A. Bucki and A. Miernowski in [5]. In [2], M. Ahmad, J. B. Jun, and A. Haseeb studied hypersurfaces of almost r -paracontact Riemannian manifold with quarter symmetric metric connection. In [11], I. Mihai and K. Matsumoto studied submanifolds of an almost r -paracontact Riemannian manifold of P -Sasakian type.

The linear connection ∇ in an n -dimensional differentiable manifold M is called *symmetric* if its torsion tensor vanishes, otherwise it is non-symmetric. The connection ∇ is *metric* if there is a Riemannian metric g in M such that $\nabla g = 0$, otherwise it is non-metric. It is well known that a linear connection is symmetric and metric if and only if it is the Levi-Civita connection.

Received May 9, 2009; Revised February 24, 2010.

2000 *Mathematics Subject Classification.* 53C05, 53D12.

Key words and phrases. almost r -paracontact Riemannian manifold, quarter-symmetric semi-metric connection.

In 1973, B. G. Schmidt [14] proved that if the holonomy group of ∇ is a subgroup of the orthogonal group $\mathcal{O}(n)$, then ∇ is the Levi-Civita connection of a Riemannian metric. In 1924, A. Friedmann and J. A. Schouten [8] introduced the idea of a semi-symmetric linear connection in a differentiable manifold. A linear connection is said to be a semi-symmetric connection if its torsion tensor T is of the form

$$T(X, Y) = u(Y)X - u(X)Y,$$

where u is a 1-form. A Hayden connection with the torsion tensor of the above form is a semi-symmetric metric connection. In 1970, K. Yano [17] considered a semi-symmetric metric connection and studied some of its properties. He proved that a Riemannian manifold is conformally flat if and only if it admits a semi-symmetric metric connection whose curvature tensor vanishes identically. He also proved that a Riemannian manifold is of constant curvature if and only if it admits a semi-symmetric connection for which the manifold is a group manifold, where a group manifold [7] is a differentiable manifold admitting a linear connection ∇ such that its curvature tensor R vanishes and its torsion tensor is covariantly constant with respect to ∇ . In [16], L. Tamassy and T. Q. Binh proved that if in a Riemannian manifold of dimension ≥ 4 , ∇ is a metric linear connection of non-vanishing constant curvature for which

$$R(X, Y)Z + R(Y, Z)X + R(Z, X)Y = 0,$$

then ∇ is the Levi-Civita connection. Some different kind of semi-symmetric non-metric connections are studied in ([3], [10], [15]).

On the other hand, S. Golab [9] introduced the idea of a quarter symmetric linear connection if its torsion tensor T is of the form

$$T(X, Y) = u(Y)\phi X - u(X)\phi Y,$$

where u is a 1-form and ϕ is a tensor field of the type (1,1). In [2], [12] and [13], some kinds of quarter symmetric metric connection were studied.

The purpose of the paper is to define and study quarter symmetric semi-metric connection in an almost r -paracontact Riemannian manifold and consider its invariant, non-invariant and anti-invariant hypersurfaces.

In Section 2, we give a brief introduction about an almost r -paracontact Riemannian manifold. In Section 3, we show that the induced connection on a hypersurface of an almost r -paracontact Riemannian manifold with quarter symmetric semi-metric connection with respect to the normal vector field is also a quarter symmetric semi-metric connection. We find the characteristic properties of invariant, non-invariant and anti-invariant hypersurfaces of almost r -paracontact Riemannian manifold with a quarter symmetric semi-metric connection.

2. Definitions and identities

Let M be an n -dimensional Riemannian manifold with a positive definite metric g . If on M there exist a tensor field ϕ of type (1,1), r vector fields

$\xi_1, \xi_2, \dots, \xi_r (n > r)$ and r 1-forms $\eta^1, \eta^2, \dots, \eta^r$ such that

$$\begin{aligned} \eta^\alpha(\xi_\beta) &= \delta_\beta^\alpha, \quad \alpha, \beta \in (r) = \{1, 2, 3, \dots, r\}, \\ \phi^2(X) &= X - \eta^\alpha(X)\xi_\alpha, \quad \eta^\alpha(X) = g(X, \xi_\alpha), \quad \alpha \in (r), \\ g(\phi X, \phi Y) &= g(X, Y) - \sum_\alpha \eta^\alpha(X)\eta^\alpha(Y), \end{aligned}$$

where X and Y are vector fields on M and $a^\alpha b_\alpha \stackrel{\text{def}}{=} \sum_\alpha a^\alpha b_\alpha$, then the structure $\sum = (\phi, \xi_\alpha, \eta^\alpha, g)_{\alpha \in (r)}$ is said to be an *almost r -paracontact Riemannian structure* and M is an *almost r -paracontact Riemannian manifold* [1]. From the definitions of almost r -paracontact Riemannian structure, we have for $\alpha \in (r)$

$$(2.1) \quad \phi(\xi_\alpha) = 0, \quad \eta^\alpha \circ \phi = 0.$$

An almost r -paracontact Riemannian manifold M equipped with the Riemannian connection ∇^* with structure \sum is said to be of *S -paracontact type* if [2]

$$(2.2) \quad \Phi(X, Y) = (\nabla_Y^* \eta^\alpha)(X)$$

for all $\alpha \in (r)$, where $\Phi(X, Y) \stackrel{\text{def}}{=} g(\phi X, Y) = g(X, \phi Y)$. Furthermore M is said to be of *P -Sasakian type* if it satisfies (2.2) and

$$\begin{aligned} (\nabla_Z^* \Phi)(X, Y) &= - \sum_\alpha \eta^\alpha(X)[g(Y, Z) - \sum_\beta \eta^\beta(Y)\eta^\beta(Z)] \\ &\quad - \sum_\alpha \eta^\alpha(Y)[g(X, Z) - \sum_\beta \eta^\beta(X)\eta^\beta(Z)] \end{aligned}$$

for all vector fields X, Y and Z on M . The above two conditions are equivalent respectively to

$$(2.3) \quad \phi X = \nabla_X^* \xi_\alpha,$$

$$(2.4) \quad \begin{aligned} (\nabla_Y^* \phi)(X) &= - \sum_\alpha \eta^\alpha(X)[Y - \eta^\alpha(Y)\xi_\alpha] \\ &\quad - [g(X, Y) - \sum_\alpha \eta^\alpha(X)\eta^\alpha(Y)] \sum_\beta \xi_\beta. \end{aligned}$$

On the other hand, a *quarter symmetric semi-metric connection* ∇ on M is defined as for all $\alpha \in (r)$

$$(2.5) \quad \nabla_X Y = \nabla_X^* Y - \eta^\alpha(X)\phi Y + g(\phi X, Y)\xi_\alpha.$$

Using (2.1) and (2.5) in (2.3) and (2.4), we get respectively

$$(2.6) \quad \begin{aligned} (\nabla_Y \phi)(X) &= - \sum_\alpha \eta^\alpha(X)Y - [g(X, Y) - \sum_\alpha \eta^\alpha(X)\eta^\alpha(Y)] \sum_\beta \xi_\beta \\ &\quad + g(X, Y)\xi_\alpha, \end{aligned}$$

$$(2.7) \quad \nabla_X \xi_\alpha = \phi X.$$

3. Certain hypersurfaces of almost r -paracontact Riemannian manifold

Let M^n be a hypersurface of almost r -paracontact Riemannian manifold M^{n+1} with a Riemannian metric g . If τ_* denotes the differential of the immersion τ and \bar{X} is a vector field on M^n , then we shall identify \bar{X} and $\tau_*\bar{X}$. We denote the objects belonging to M^n by the mark of hyphen placed over them.

Let N be the unit normal vector field to M^n . Then the induced metric \bar{g} on M^n is defined by $\bar{g}(\bar{X}, \bar{Y}) = g(\bar{X}, \bar{Y})$. Then we have [6] $g(\bar{X}, N) = 0$, $g(N, N) = 1$. If $\bar{\nabla}^*$ is the induced connection on the hypersurface from the Riemannian connection ∇^* with respect to the unit normal vector field N , then the Gauss formula is given by

$$(3.1) \quad \nabla_{\bar{X}}^* \bar{Y} = \bar{\nabla}_{\bar{X}}^* \bar{Y} + h(\bar{X}, \bar{Y})N,$$

where h is the second fundamental tensor. Also if $\bar{\nabla}$ is the induced connection on the hypersurface from the quarter symmetric semi-metric connection ∇ with respect to the unit normal N , then we have

$$(3.2) \quad \nabla_{\bar{X}} \bar{Y} = \bar{\nabla}_{\bar{X}} \bar{Y} + m(\bar{X}, \bar{Y})N,$$

where m is a tensor field of type $(0, 2)$ of the hypersurface.

Now, suppose that \sum is an almost r -paracontact Riemannian structure on M^{n+1} . Then every vector field X on M^{n+1} is decomposed as

$$X = \bar{X} + \lambda(X)N,$$

where λ is a 1-form on M^{n+1} and \bar{X} is a vector field and N is a normal vector field on M^n . Then we have

$$(3.3) \quad \phi \bar{X} = \bar{\phi} \bar{X} + b(\bar{X})N,$$

$$(3.4) \quad \phi N = \bar{N} + KN,$$

where $\bar{\phi}$ is a tensor field of type $(1,1)$, b is a 1-form and K is a scalar function on M^n .

From now on, we have put for each $\alpha \in (r)$,

$$(3.5) \quad \xi_\alpha = \bar{\xi}_\alpha + a_\alpha N.$$

From equations (2.5), (3.1)-(3.3), we get

$$\begin{aligned} \bar{\nabla}_{\bar{X}} \bar{Y} + m(\bar{X}, \bar{Y})N &= \bar{\nabla}_{\bar{X}}^* \bar{Y} + h(\bar{X}, \bar{Y})N - \eta^\alpha(\bar{X})\{b(\bar{Y})N - \bar{\phi}\bar{Y}\} \\ &\quad + g(\bar{\phi}\bar{X}, \bar{Y})(\bar{\xi}_\alpha + a_\alpha N). \end{aligned}$$

By taking the tangential and normal parts from the both sides of the above equation, we get

$$(3.6) \quad \bar{\nabla}_{\bar{X}} \bar{Y} = \bar{\nabla}_{\bar{X}}^* \bar{Y} - \eta^\alpha(\bar{X})\bar{\phi}\bar{Y} + g(\bar{\phi}\bar{X}, \bar{Y})\bar{\xi}_\alpha,$$

which implies that the connection induced on a hypersurface of an almost r -paracontact Riemannian manifold with quarter symmetric semi-metric connection with respect to the unit normal is also a quarter symmetric semi-metric one.

From (3.2) and (3.6), we have

$$(3.7) \quad \nabla_{\bar{X}}\bar{Y} = \bar{\nabla}_{\bar{X}}\bar{Y} + \{h(\bar{X}, \bar{Y}) - \eta^\alpha(\bar{X})b(\bar{Y}) + a_\alpha g(\bar{\phi}\bar{X}, \bar{Y})\}N,$$

which is the Gauss formula for a quarter symmetric semi-metric connection. The Weingarten formula with respect to the Riemannian connection ∇^* is given by

$$(3.8) \quad \nabla_{\bar{X}}^*N = -H\bar{X}$$

for every \bar{X} in M^n , where H is a tensor field of type (1,1) of M^n given by $\bar{g}(H\bar{X}, \bar{Y}) = h(\bar{X}, \bar{Y}) = h(\bar{Y}, \bar{X})$.

From (2.5), we have

$$\nabla_{\bar{X}}N = \nabla_{\bar{X}}^*N - \eta^\alpha(\bar{X})\phi N + b(\bar{X})\xi_\alpha$$

by virtue of $b(\bar{X}) = g(\phi\bar{X}, N)$.

We have from (3.8) and the above equation

$$(3.9) \quad \nabla_{\bar{X}}N = -H\bar{X} - \eta^\alpha(\bar{X})\phi N + b(\bar{X})\xi_\alpha,$$

which is the Weingarten formula with respect to the quarter symmetric semi-metric connection.

Now, we define $\bar{\eta}^\alpha$ as

$$(3.10) \quad \bar{\eta}^\alpha(\bar{X}) = \eta^\alpha(\bar{X}), \quad \alpha \in (r).$$

Making use of (3.3)-(3.6), we obtain from the structure equations of the almost r -paracontact Riemannian manifold

$$(3.11) \quad b(\bar{N}) + K^2 = 1 - \sum_{\alpha} (a_\alpha)^2,$$

$$(3.12) \quad Ka_\alpha + b(\bar{\xi}_\alpha) = 0,$$

$$(3.13) \quad \Phi(\bar{X}, \bar{Y}) = \bar{g}(\bar{\phi}\bar{X}, \bar{Y}) = \bar{g}(\bar{X}, \bar{\phi}\bar{Y}) = \bar{\Phi}(\bar{X}, \bar{Y})$$

for all $\alpha \in (r)$. Making use of (2.1), (3.3) and (3.4), we have $g(\bar{\phi}\bar{X}, N) = g(\phi\bar{X}, N) - b(\bar{X}) = g(\bar{X}, \phi N) - b(\bar{X}) = 0$. Hence we get

$$(3.14) \quad \bar{g}(\bar{X}, \bar{N}) = b(\bar{X}).$$

Differentiating covariantly (3.3) and (3.4) along the hypersurface and making use of (3.7) and (3.9), we get

$$(3.15) \quad \begin{aligned} (\nabla_{\bar{Y}}\phi)\bar{X} &= (\bar{\nabla}_{\bar{Y}}\bar{\phi})\bar{X} - b(\bar{X})H(\bar{Y}) + [h(\bar{Y}, \bar{\phi}\bar{X}) - \bar{\eta}^\alpha(\bar{Y})b(\bar{\phi}\bar{X}) \\ &\quad + a_\alpha\bar{g}(\bar{\phi}\bar{Y}, \bar{\phi}\bar{X}) + (\bar{\nabla}_{\bar{Y}}b)(\bar{X})]N \\ &\quad - [h(\bar{X}, \bar{Y}) + a_\alpha\bar{g}(\bar{\phi}\bar{Y}, \bar{X})](\bar{N} + KN) + b(\bar{X})b(\bar{Y})(\bar{\xi}_\alpha + a_\alpha N). \end{aligned}$$

We choosing as $\bar{\eta}^\alpha(N) = -a_\alpha$, $\alpha \in (r)$. From identity $(\nabla_Z\Phi)(X, Y) = g((\nabla_Z\phi)(X), Y)$, making use of (3.13)-(3.15), we have

$$(3.16) \quad \begin{aligned} (\nabla_{\bar{Z}}\Phi)(\bar{X}, \bar{Y}) &= (\bar{\nabla}_{\bar{Z}}\bar{\Phi})(\bar{X}, \bar{Y}) - b(\bar{X})h(\bar{Z}, \bar{Y}) \\ &\quad - b(\bar{Y})[h(\bar{Z}, \bar{X}) + a_\alpha\bar{g}(\bar{\phi}\bar{Z}, \bar{X})] + b(\bar{X})b(\bar{Z})\bar{\eta}^\alpha(\bar{Y}). \end{aligned}$$

From (3.14) we know $b = 0$ is equivalent to the fact that $\bar{N} = 0$.

If we assume that hypersurface is invariant, that is, for $p \in M$, $\phi T_p M \subset T_p M$, then $b = 0$ and $\bar{N} = 0$. Thus from (3.12) we know $Ka_\alpha = 0$, $\alpha \in (r)$. Hence we have the two possibilities in this case as $K = 0$ or $K \neq 0$ as follows.

At first, if $K \neq 0$, then $a_\alpha = 0$ and $\xi_\alpha = \bar{\xi}_\alpha$. Thus, all ξ_α are tangent to M^n , which implies the structure $(\bar{\phi}, \bar{\xi}_\alpha, \bar{\eta}^\alpha, \bar{g})_{\alpha \in (r)}$ is an almost r -paracontact Riemannian structure on M^n .

Secondly, if $K = 0$, then $\phi(N) = 0$. So let $N = \xi_r$. Then $\bar{\xi}_r = 0$, $a_r = 1$, $\bar{\eta}^r = 0$. From (3.11), $\sum_\alpha (a_\alpha)^2 = 1$ and since $a_r = 1$, $\sum_i (a_i)^2 = 0$, $i \in (r-1)$. Thus $a_i = 0$ for all $i \in (r-1)$. Thus, $\xi_i = \bar{\xi}_i$, $\xi_r = N$ (all ξ_α but one tangent to M^n). Hence the structure $(\bar{\phi}, \bar{\xi}_i, \bar{\eta}^i, \bar{g})_{i \in (r-1)}$ is an almost $(r-1)$ -paracontact structure on M^n . Thus we have:

Theorem 3.1. *Let M^n be an invariant hypersurface immersed in an almost r -paracontact Riemannian manifold M^{n+1} with quarter symmetric semi-metric connection with structure \sum . Then either*

- (i) *All ξ_α are tangent to M^n and M^n admits an almost r -paracontact Riemannian structure $\sum_1 = (\bar{\phi}, \bar{\xi}_\alpha, \bar{\eta}^\alpha, \bar{g})_{\alpha \in (r)}$ ($n - r > 2$) or*
- (ii) *One of ξ_α (say, ξ_r) is normal to M^n and remaining ξ_α are tangent to M^n and M^n admits an almost $(r-1)$ -paracontact Riemannian structure $\sum_2 = (\bar{\phi}, \bar{\xi}_i, \bar{\eta}^i, \bar{g})_{i \in (r)}$ ($n - r > 1$).*

On the other hand, we have from (3.5),

$$(3.17) \quad (\nabla_{\bar{Y}}\eta^\alpha)(\bar{X}) = (\bar{\nabla}_{\bar{Y}}\bar{\eta}^\alpha)(\bar{X}) - a_\alpha h(\bar{Y}, \bar{X}).$$

From the equations (3.10), (3.13), (3.16) and (3.17), we can derive that conditions (2.7) and (2.16) are satisfied for two structures \sum_1 and \sum_2 . Thus we can conclude the following.

Theorem 3.2. *Let M^n be an invariant hypersurface immersed in an almost r -paracontact Riemannian manifold of P -Sasakian type with quarter symmetric semi-metric connection. Then the induced almost r -paracontact Riemannian structure \sum_1 or $(r-1)$ -paracontact Riemannian structure \sum_2 are also of P -Sasakian type.*

On the other hand, we obtain from (3.15)

$$(3.18) \quad \begin{aligned} (\bar{\nabla}_{\bar{Y}}\bar{\phi})\bar{X} - b(\bar{X})H(\bar{Y}) - h(\bar{X}, \bar{Y})\bar{N} - a_\alpha\bar{g}(\bar{\phi}\bar{Y}, \bar{X})\bar{N} + b(\bar{X})b(\bar{Y})\bar{\xi}_\alpha &= 0, \\ h(\bar{Y}, \bar{\phi}\bar{X}) - \bar{\eta}^\alpha(\bar{Y})b(\bar{\phi}\bar{X}) + a_\alpha\bar{g}(\bar{\phi}\bar{Y}, \bar{\phi}\bar{X}) + (\bar{\nabla}_{\bar{Y}}b)(\bar{X}) - h(\bar{X}, \bar{Y})K \\ - a_\alpha\bar{g}(\bar{\phi}\bar{Y}, \bar{X})K + a_\alpha b(\bar{X})b(\bar{Y}) &= 0. \end{aligned}$$

If the hypersurface M^n is totally geodesic, then $h = 0$ and $H = 0$. Thus from (3.18), we can deduce

$$(\bar{\nabla}_{\bar{Y}}\bar{\phi})\bar{X} - a_\alpha\bar{g}(\bar{\phi}\bar{Y}, \bar{X})\bar{N} + b(\bar{X})b(\bar{Y})\bar{\xi}_\alpha = 0.$$

Conversely, if $(\bar{\nabla}_{\bar{Y}}\bar{\phi})(\bar{X}) - a_\alpha\bar{g}(\bar{\phi}\bar{Y}, \bar{X})\bar{N} + b(\bar{X})b(\bar{Y})\bar{\xi}_\alpha = 0$, then we have

$$(3.19) \quad h(\bar{Y}, \bar{X})\bar{N} + b(\bar{X})H(\bar{Y}) = 0.$$

Then we have by virtue of (3.8) and (3.14)

$$(3.20) \quad h(\bar{X}, \bar{Y})b(\bar{Z}) + h(\bar{Y}, \bar{Z})b(\bar{X}) = 0.$$

Using (3.19), we get from (3.8)

$$(3.21) \quad h(\bar{Y}, \bar{Z})b(\bar{X}) = h(\bar{X}, \bar{Z})b(\bar{Y}).$$

We get from (3.20) and (3.21), $b(\bar{Z})h(\bar{X}, \bar{Y}) = 0$, which implies that $h = 0$ as $b \neq 0$. Making use of $h = 0$ in (3.19), we get $H = 0$. Thus $h = 0$ and $H = 0$. It just means that the hypersurface M^n is totally geodesic. Thus we have:

Theorem 3.3. *Let M^n be a non-invariant hypersurface of an almost r -paracontact Riemannian manifold M^{n+1} with quarter symmetric semi-metric connection with a structure \sum satisfying $\nabla\phi = 0$ along M^n . Then the hypersurface is totally geodesic if and only if*

$$(\bar{\nabla}_{\bar{Y}}\bar{\phi})(\bar{X}) - a_\alpha\bar{g}(\bar{\phi}\bar{Y}, \bar{X})\bar{N} + b(\bar{X})b(\bar{Y})\bar{\xi}_\alpha = 0.$$

We have from (3.5)

$$(3.22) \quad \begin{aligned} \bar{\nabla}_{\bar{Y}}\bar{\xi}_\alpha &= \bar{\nabla}_{\bar{Y}}\bar{\xi}_\alpha + h(\bar{Y}, \bar{\xi}_\alpha)N - \bar{\eta}^\alpha(\bar{Y})b(\bar{\xi}_\alpha)N + a_\alpha b(\bar{\phi}\bar{Y})N - a_\alpha H(\bar{Y}) \\ &\quad - a_\alpha\bar{\eta}^\alpha(\bar{Y})\bar{N} - a_\alpha\bar{\eta}^\alpha(\bar{Y})K\bar{N} + b(\bar{Y})\bar{\xi}_\alpha + a_\alpha b(\bar{Y})N + N\bar{Y}a_\alpha. \end{aligned}$$

In the sequel, let M^{n+1} be an almost r -paracontact Riemannian manifold of S -paracontact type. Then we get from (2.3), (3.3) and (3.22)

$$(3.23) \quad \bar{\phi}\bar{X} = \bar{\nabla}_{\bar{X}}\bar{\xi}_\alpha + b(\bar{X})\bar{\xi}_\alpha - a_\alpha\{H(\bar{X}) + \bar{\eta}^\alpha(\bar{X})\bar{N}\},$$

$$(3.24) \quad \begin{aligned} b(\bar{X}) &= h(\bar{X}, \bar{\xi}_\alpha) - \bar{\eta}^\alpha(\bar{X})b(\bar{\xi}_\alpha) + a_\alpha\bar{g}(\bar{\phi}\bar{X}, \bar{\xi}_\alpha) \\ &\quad - a_\alpha\bar{\eta}^\alpha(\bar{X}) + \bar{X}(a_\alpha) + b(\bar{X})a_\alpha \end{aligned}$$

for all $\alpha \in (r)$. We can know by using (3.24) that if M^n is totally geodesic, then $a_\alpha = 0$ and $h = 0$. Hence $b = 0$, which implies that the hypersurface is invariant. Thus we have the following.

Theorem 3.4. *If M^n is totally geodesic hypersurface of an almost r -paracontact Riemannian manifold M^{n+1} with quarter symmetric semi-metric connection of S -paracontact type with a structure \sum and all ξ_α are tangent to M^n , then the hypersurface is invariant.*

Finally if the hypersurface is anti-invariant, then we know $\bar{\phi} = 0$ and $a_\alpha = 0$. Hence we get from (3.23) that $\bar{\nabla}_{\bar{X}}\bar{\xi}_\alpha = -b(\bar{X})\bar{\xi}_\alpha$ holds. Thus we have:

Theorem 3.5. *If M^n is an anti-invariant hypersurface of an almost r -paracontact Riemannian manifold M^{n+1} with quarter symmetric semi-metric connection of S -paracontact type with a structure $\bar{\Sigma}$, then $\bar{\nabla}_{\bar{X}}\bar{\xi}_\alpha = -b(\bar{X})\bar{\xi}_\alpha$.*

References

- [1] T. Adati, *Hypersurfaces of almost paracontact Riemannian manifolds*, TRU Math. **17** (1981), no. 2, 189–198.
- [2] M. Ahmad, J. B. Jun, and A. Haseeb, *Hypersurfaces of almost r -paracontact Riemannian manifold endowed with a quarter symmetric metric connection*, Bull. Korean Math. Soc. **46** (2009), no. 3, 477–487.
- [3] O. C. Andonie and D. Smaranda, *Certaines connexions semi-symétriques*, Tensor (N.S.) **31** (1977), no. 1, 8–12.
- [4] A. Bucki, *Hypersurfaces of almost r -paracontact Riemannian manifolds*, Tensor (N.S.) **48** (1989), no. 3, 245–251.
- [5] A. Bucki and A. Miernowski, *Invariant hypersurfaces of an almost r -paracontact manifold*, Demonstratio Math. **19** (1986), no. 1, 113–121.
- [6] B. Y. Chen, *Geometry of Submanifolds*, Marcel Dekker, New York, 1973.
- [7] L. P. Eisenhart, *Continuous Groups of Transformations*, Dover Publications, Inc., New York, 1961.
- [8] A. Friedmann and J. A. Schouten, *über die Geometrie der halbsymmetrischen Übertragung*, Math. Z. **21** (1924), no. 1, 211–223.
- [9] S. Golab, *On semi-symmetric and quarter-symmetric linear connections*, Tensor (N.S.) **29** (1975), no. 3, 249–254.
- [10] Y. Liang, *On semi-symmetric recurrent-metric connection*, Tensor (N.S.) **55** (1994), no. 2, 107–112.
- [11] I. Mihai and K. Matsumoto, *Submanifolds of an almost r -paracontact Riemannian manifold of P -Sasakian type*, Tensor (N.S.) **48** (1989), no. 2, 136–142.
- [12] R. S. Mishra and S. N. Pandey, *On quarter symmetric metric F -connections*, Tensor (N.S.) **34** (1980), no. 1, 1–7.
- [13] S. C. Rastogi, *On quarter-symmetric metric connection*, C. R. Acad. Bulgare Sci. **31** (1978), no. 7, 811–814.
- [14] B. G. Schmidt, *Conditions on a connection to be a metric connection*, Comm. Math. Phys. **29** (1973), 55–59.
- [15] J. Sengupta, U. C. De, and T. Q. Binh, *On a type of semi-symmetric non-metric connection on a Riemannian manifold*, Indian J. Pure Appl. Math. **31** (2000), no. 12, 1659–1670.
- [16] L. Tamássy and T. Q. Binh, *On the nonexistence of certain Riemannian connections with torsion and of constant curvature*, Publ. Math. Debrecen **36** (1989), no. 1-4, 283–288.
- [17] K. Yano, *On semi-symmetric metric connection*, Rev. Roumaine Math. Pures Appl. **15** (1970), 1579–1586.
- [18] K. Yano and T. Imai, *Quarter-symmetric metric connections and their curvature tensors*, Tensor (N.S.) **38** (1982), 13–18.

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