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SCREEN CONFORMAL EINSTEIN LIGHTLIKE HYPERSURFACES OF A LORENTZIAN SPACE FORM

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Abstract. In this paper, we study the geometry of lightlike hypersurfaces of a semi-Riemannian manifold. We prove a classification theorem for Einstein lightlike hypersurfaces M of a Lorentzian space form subject such that the second fundamental forms of M and its screen distribution $S(TM)$ are conformally related by some non-vanishing smooth function.

1. Introduction

It is well known that the normal bundle $TM^\perp$ of the lightlike hypersurfaces (M, g) of a semi-Riemannian manifold $(\bar{M}, \bar{g})$ is a vector subbundle of TM, of rank 1. A complementary vector bundle $S(TM)$ of $TM^\perp$ in TM is non-degenerate distribution on M, called a *screen distribution* on M, such that

$$TM = TM^\perp \oplus_{\text{orth}} S(TM), \tag{1.1}$$

where $\oplus_{\text{orth}}$ denotes the orthogonal direct sum. We denote such a lightlike hypersurface by $(M, g, S(TM))$. Denote by $F(M)$ the algebra of smooth functions on M and by $\Gamma(E)$ the $F(M)$ module of smooth sections of a vector bundle E over M. We use the same notation for any other vector bundle. For any null section ξ of $TM^\perp$ on a coordinate neighborhood $\mathcal{U} \subset M$, there exists a null section N of a vector bundle $\text{tr}(TM)$ in $S(TM)^\perp$ [3] satisfying

$$\bar{g}(\xi, N) = 1,\ \bar{g}(N, N) = \bar{g}(N, X) = 0,\ \forall\, X \in \Gamma\left(S(TM)|_{\mathcal{U}}\right). \tag{1.2}$$

Then the tangent bundle $T\bar{M}$ of $\bar{M}$ is decomposed as follows:

$$T\bar{M} = TM \oplus \text{tr}(TM) = \{TM^\perp \oplus \text{tr}(TM)\} \oplus_{\text{orth}} S(TM). \tag{1.3}$$

We call $\text{tr}(TM)$ and N the *transversal vector bundle* and the *null transversal vector field* of M with respect to the screen distribution $S(TM)$ respectively.

Recently, Atindogbe-Ezin-Tossa have proved the following theorem for Einstein lightlike hypersurfaces of a Lorentzian space form in their paper [2]:

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Theorem A ([2]). *Let $(M, g, S(TM))$ be a screen homothetic lightlike hypersurface of a Lorentzian space form $(\bar{M}^{m+2}(c), \bar{g})$, $c \geq 0$. If M is Einstein, that is, $Ric = \gamma g$ (γ constant), then $\gamma \geq mc$ and*

(1) *If $\gamma = mc$, then M is locally a product manifold $L \times M^*$, where the integral submanifold M^* of $S(TM)$ is a Riemannian m-space form with the same curvature c as $\bar{M}$ and L is an open subset of a lightlike geodesic ray in $\bar{M}$.*
(2) *If $\gamma > mc$, then M is locally a product $L \times M^*$, where M^* is a Riemannian m-space form of positive constant curvature $c + 2(\gamma - mc)$ which is isometric to a sphere.*

The purpose of this paper is to prove a characterization theorem for screen conformal Einstein lightlike hypersurfaces M of a Lorentzian space form $(\bar{M}(c), \bar{g})$.

Theorem 1.1. *Let $(M, g, S(TM))$ be a screen conformal Einstein lightlike hypersurface of a Lorentzian space form $(\bar{M}^{m+2}(c), \bar{g})$; $m > 2$. Then $c = 0$ and M is locally a product manifold $L \times M_\alpha \times M_\beta$, where L is an open subset of a lightlike geodesic ray in $\bar{M}$ and M_α and M_β are leaves of some integerable distributions of M such that*

(1) *If $\gamma \neq 0$, either M_α or M_β is an m-dimensional totally umbilical Einstein Riemannian space form which is isometric to a sphere or a hyperbolic space according to the sign of γ and the other is a point.*
(2) *If $\gamma = 0$, M_α is an $(m-1)$ or an m-dimensional Euclidean space and M_β is a non-null curve or a point.*

Comparing our Theorem 1.1 with above result Theorem A, we observe that Theorem 1.1 has the following new features of geometric significance:

(1) Since the key player of lightlike hypersurfaces is the integral submanifold $M^* = M_\alpha \times M_\beta$ of the screen distribution $S(TM)$, Theorem 1.1 provides more deeper geometry of M^* than Theorem A.

(2) We prove $c = 0$ if M is screen conformal and $m > 2$. This is a significant result. The screen conformal is more weak condition than the screen homothetic. We can also find $c = 0$ for arbitrary m (without the condition $m > 2$ due to Note 2) if M is screen homothetic (as Theorem A). Contrary to this, there is no discussion on such a relationship in Atindogbe-Ezin-Tossa's above result. Recall the following structure equations:

Let $\bar{\nabla}$ be the Levi-Civita connection of $\bar{M}$ and P the projection morphism of $\Gamma(TM)$ on $\Gamma(S(TM))$ with respect to the decomposition (1.1). Then the local Gauss and Weingartan formulas are given by

$$\bar{\nabla}_X Y = \nabla_X Y + B(X, Y)N, \tag{1.4}$$

$$\bar{\nabla}_X N = -A_N X + \tau(X)N, \tag{1.5}$$

$$\nabla_X PY = \nabla^*_X PY + C(X, PY)\xi, \tag{1.6}$$

$$\nabla_X \xi = -A^*_\xi X - \tau(X)\xi \tag{1.7}$$

for any $X, Y \in \Gamma(TM)$, where the symbols ∇ and ∇^* are the induced linear connections on TM and $S(TM)$ respectively, B and C are the local second fundamental forms on TM and $S(TM)$ respectively, A_N and A^*_ξ are the shape operators on TM and $S(TM)$ respectively and τ is a 1-form on TM.

Since $\bar{\nabla}$ is torsion-free, ∇ is also torsion-free and B is symmetric. From the fact that $B(X,Y) = \bar{g}(\bar{\nabla}_X Y, \xi)$ for all $X, Y \in \Gamma(TM)$, we know that B is independent of the choice of a screen distribution and satisfies

$$B(X,\, \xi) = 0,\ \forall\ X \in \Gamma(TM). \tag{1.8}$$

The induced connection ∇ of M is not metric and satisfies

$$(\nabla_X g)(Y,\, Z) = B(X,\, Y)\, \eta(Z) + B(X,\, Z)\, \eta(Y) \tag{1.9}$$

for any $X, Y, Z \in \Gamma(TM)$, where η is a 1-form such that

$$\eta(X) = \bar{g}(X,\, N),\ \forall\ X \in \Gamma(TM). \tag{1.10}$$

But ∇^* is a metric connection. The above local second fundamental forms B and C of M and on $S(TM)$ are related to their shape operators by

$$B(X,\, Y) = g(A^*_\xi X,\, Y), \qquad \bar{g}(A^*_\xi X,\, N) = 0, \tag{1.11}$$

$$C(X,\, PY) = g(A_N X,\, PY), \quad \bar{g}(A_N X,\, N) = 0. \tag{1.12}$$

From (1.11), A^*_ξ is $S(TM)$-valued and self-adjoint on TM such that

$$A^*_\xi \xi = 0, \tag{1.13}$$

that is, ξ is an eigenvector field of A^*_ξ corresponding to the eigenvalue 0.

We denote by $\bar{R}$, R and R^* the curvature tensors of $\bar{\nabla}$, ∇ and ∇^* respectively. Using the Gauss-Weingarten equations for M and $S(TM)$, we obtain the Gauss-Codazzi equations for M and $S(TM)$ such that, for any $X, Y, Z, W \in \Gamma(TM)$,

$$\begin{aligned} \bar{g}(\bar{R}(X,\, Y)Z,\, PW) = g(R(X,\, Y)Z,\, PW)& \\ + B(X,\, Z)C(Y,\, PW) - B(Y,\, Z)C(X,\, PW)&, \end{aligned} \tag{1.14}$$

$$\begin{aligned} \bar{g}(\bar{R}(X,\, Y)Z,\, \xi) &= g(R(X,\, Y)Z,\, \xi) \\ &= (\nabla_X B)(Y,\, Z) - (\nabla_Y B)(X,\, Z) \\ &\quad + B(Y,\, Z)\tau(X) - B(X,\, Z)\tau(Y), \end{aligned} \tag{1.15}$$

$$\bar{g}(\bar{R}(X,\, Y)Z,\, N) = g(R(X,\, Y)Z,\, N), \tag{1.16}$$

$$\begin{aligned} g(R(X,\, Y)PZ,\, PW) &= g(R^*(X,\, Y)PZ,\, PW) \\ &\quad + C(X,PZ)B(Y,PW) \\ &\quad - C(Y,PZ)B(X,PW), \end{aligned} \tag{1.17}$$

$$\begin{aligned} g(R(X,\, Y)PZ,\, N) &= (\nabla_X C)(Y,\, PZ) - (\nabla_Y C)(X,\, PZ) \\ &\quad + C(X,\, PZ)\tau(Y) - C(Y,\, PZ)\tau(X). \end{aligned} \tag{1.18}$$

2. Screen conformal hypersurfaces

A lightlike hypersurface $(M, g, S(TM))$ of a semi-Riemannian manifold $(\bar{M}, \bar{g})$ is *screen conformal* [1] if the shape operators A_N and A^*_ξ of M and $S(TM)$ respectively are related by $A_N = \varphi\, A^*_\xi$, or equivalently,

$$C(X, PY) = \varphi\, B(X, Y),\ \forall\, X,\, Y \in \Gamma(TM), \tag{2.1}$$

where φ is a non-vanishing smooth function on a neighborhood $\mathcal{U}$ in M. In particular, if φ is a non-zero constant, M is called *screen homothetic.*

Note 1. For a screen conformal M, C is symmetric on $S(TM)$. Thus, by [3], $S(TM)$ is integrable and M is locally a product manifold $L \times M^*$, where L is an open subset of a lightlike geodesic ray in $\bar{M}$ and M^* is a leaf of $S(TM)$.

Let $\bar{M}$ be a semi-Riemannian space form $\bar{M}(c)$, by (1.15), we have

$$(\nabla_X B)(Y, Z) - (\nabla_Y B)(X, Z) = B(X, Z)\tau(Y) - B(Y, Z)\tau(X) \tag{2.2}$$

for all X, Y, $Z \in \Gamma(TM)$. Using this, (1.16), (1.18) and (2.1), we obtain

$$\begin{aligned} &\{X[\varphi] - 2\varphi\tau(X)\}B(Y, PZ) - \{Y[\varphi] - 2\varphi\tau(Y)\}B(X, PZ) \\ &= c\{g(Y, PZ)\eta(X) - g(X, PZ)\eta(Y)\}. \end{aligned} \tag{2.3}$$

Replacing Y by ξ in (2.3), we obtain

$$\{\xi[\varphi] - 2\varphi\tau(\xi)\}B(X, PZ) = cg(X, PZ). \tag{2.4}$$

Using this equation, we have the following result.

Theorem 2.1 ([6])**.** *Let $(M, g, S(TM))$ be a screen conformal lightlike hypersurface of a semi-Riemannian space form $(\bar{M}^{m+2}(c), \bar{g})$; $m > 2$. Then $c = 0$.*

Proof. Assume that $c \neq 0$. Then $\xi[\varphi] - 2\varphi\tau(\xi) \neq 0$ and $B \neq 0$, that is, M is not a totally geodesic. From (2.1) and (2.4), we have

$$B(X, Y) = \rho\, g(X, Y),\ C(X, Y) = \varphi\rho\, g(X, Y),\ \forall\ X,\, Y \in \Gamma(TM), \tag{2.5}$$

where $\rho = c(\xi[\varphi] - 2\varphi\tau(\xi))^{-1} \neq 0$. From (2.1) and (2.5), we get $\varphi\rho \neq 0$. Thus M and $S(TM)$ are not totally geodesic but totally umbilical. Since M is screen conformal, by Note 1, M is locally a product manifold $L \times M^*$, where L is an open subset of a lightlike geodesic ray in $\bar{M}$ and M^* is a leaf of $S(TM)$. Since $\bar{M}$ is a space of constant curvature, from (1.14), (1.17) and (2.5), we have

$$R^*(X, Y)Z = (c + 2\varphi\rho^2)\{g(Y, Z)X - g(X, Z)Y\} \tag{2.6}$$

for all X, Y, $Z \in \Gamma(S(TM))$. Thus the leaf M^* of $S(TM)$ is a semi-Riemannian manifold of curvature $(c + 2\varphi\rho^2)$. Let Ric^* be the induced symmetric Ricci tensor of M^*. From (2.6), we have

$$Ric^*(X, Y) = (c + 2\varphi\rho^2)(m - 1)\, g(X, Y),\ \forall\ X,\, Y \in \Gamma(S(TM)). \tag{2.7}$$

Thus M^* is an Einstein manifold. Since M^* is a semi-Riemannian manifold and $m > 2$, we show that $(c + 2\varphi\rho^2)$ is a constant and M^* has constant curvature $(c + 2\varphi\rho^2)$. Using (1.9), (2.2) and (2.5), we have

$$\{X[\rho] + \rho\tau(X) - \rho^2\eta(X)\}PY = \{Y[\rho] + \rho\tau(Y) - \rho^2\eta(Y)\}PX. \tag{2.8}$$

Suppose there exists a vector field $X_o \in \Gamma(TM)$ such that $X_o[\rho] + \rho\tau(X_o) - \rho^2\eta(X_o) \neq 0$ at each point $x \in M$. Then $PY = fPX_o$ for any $Y \in \Gamma(TM)$, where f is a smooth function. It follows that all vectors from the fiber $S(TM)_x$ are co-linear with $(PX_o)_x$. It is a contradiction as $\dim(S(TM)_x) > 2$. Thus

$$X[\rho] + \rho\tau(X) - \rho^2\eta(X) = 0, \ \forall\, X \in \Gamma(TM).$$

This implies $\xi[\rho] = \rho^2 - \rho\tau(\xi)$. Therefore, $0 = \xi[\varphi\rho^2] = \rho(c + 2\varphi\rho^2)$. Since $(c + 2\varphi\rho^2)$ is a constant and $\rho \neq 0$, we have $c + 2\varphi\rho^2 = 0$. Thus M^* is a semi-Euclidean space and $C = 0$. Thus, from (2.4), we have $\varphi\rho = 0$. This means $c = 0$. It is contradiction to $c \neq 0$. Thus we have $c = 0$. □

3. Einstein lightlike hypersurfaces

The Ricci tensor $\bar{Ric}$ of $\bar{M}$ and the induced Ricci type tensor $R^{(0,\,2)}$ of M are defined by

$$\bar{Ric}(X,\, Y) \ = \ \mathrm{trace}\{Z \to \bar{R}(X,\, Z)Y\},\ \forall\ X,\, Y \in \Gamma(T\bar{M}), \tag{3.1}$$

$$R^{(0,\,2)}(X, Y) = \mathrm{trace}\{Z \to R(Z, X)Y\},\ \forall\ X,\, Y \in \Gamma(TM). \tag{3.2}$$

Substituting the Gauss-Codazzi equations (1.14) and (1.16) in (3.1) and using the relations (1.11) and (1.12), for all $X,\, Y \in \Gamma(TM)$, we obtain

$$R^{(0,\,2)}(X, Y) = \bar{Ric}(X, Y) + B(X, Y)\mathrm{tr}A_N - g(A_N X, A^*_\xi Y) - \bar{g}(R(\xi, Y)X,\, N).$$

A tensor field $R^{(0,\,2)}$ of M is called its *induced Ricci tensor*, denoted by Ric, if it is symmetric. If $\bar{M}$ is a semi-Riemannian space form $(\bar{M}(c),\, \bar{g})$, then we have $\bar{R}(\xi,\, Y)X = c\bar{g}(X,\, Y)\xi$ and $\bar{Ric}(X,\, Y) = (m+1)c\,\bar{g}(X,\, Y)$. Thus

$$R^{(0,\,2)}(X, Y) = mc\, g(X, Y) + B(X, Y)\mathrm{tr}A_N - g(A_N X, A^*_\xi Y). \tag{3.3}$$

For the rest of this section, by $(M, g, S(TM))$ we shall mean a screen conformal lightlike hypersurfaces of a Lorentzian space form $(\bar{M}^{m+2}(c),\, \bar{g})$; $m > 2$ unless otherwise specified. In this case, $S(TM)$ is Riemannian and integrable distribution and the sectional curvature c of $\bar{M}(c)$ satisfies $c = 0$. For this class of lightlike hypersurfaces, $R^{(0,\,2)}$ is a symmetric Ricci tensor Ric.

Note 2. It is well known that $R^{(0,\,2)}$ is symmetric if and only if each 1-form τ is closed, i.e., $d\tau = 0$, on any $\mathcal{U} \subset M$ [5]. Therefore, suppose $R^{(0,\,2)}$ is symmetric, there exists a smooth function f on $\mathcal{U}$ such that $\tau = df$. Consequently we get $\tau(X) = X(f)$. If we take $\bar{\xi} = \alpha\xi$, it follows that $\tau(X) = \bar{\tau}(X) + X(\mathrm{In}\,\alpha)$. Setting $\alpha = \exp(f)$ in this equation, we get $\bar{\tau}(X) = 0$ for any $X \in \Gamma(TM_{|\,\mathcal{U}})$. We call the pair $\{\xi,\, N\}$ on $\mathcal{U}$ such that the corresponding 1-form τ vanishes the *distinguished null pair* of M. Although $S(TM)$ is not unique, it is canonically isomorphic to the factor vector bundle $TM^\sharp = TM/\mathrm{Rad(TM)}$ considered by Kupeli [7]. Thus all $S(TM)$ are mutually isomorphic. For this reason, let $(M, g, S(TM))$ be a screen conformal Einstein lightlike hypersurface equipped with the distinguished null pair $\{\xi, N\}$ of a Lorentzian space form $(\bar{M}^{m+2}(c), \bar{g})$; $m > 2$. Under this hypothesis, we show that $\xi[\varphi]B(X, Y) = cg(X, Y)$ due to (2.4). Thus if M is screen homothetic, then we have $c = 0$.

Let M be an Einstein manifold, that is, $R^{(0,\,2)} = Ric = \gamma g$, where γ is a constant if $m > 2$. Since ξ is an eigenvector field of A^*_ξ corresponding to the eigenvalue 0 due to (1.13) and A^*_ξ is $\Gamma(S(TM))$-valued real symmetric, A^*_ξ have m real orthonormal eigenvector fields in $S(TM)$ and is diagonalizable. Consider a frame field of eigenvectors $\{\xi, E_1, \ldots, E_m\}$ of A^*_ξ such that $\{E_1, \ldots, E_m\}$ is an orthonormal frame field of $S(TM)$. Then

$$A^*_\xi E_i = \lambda_i E_i,\ 1 \le i \le m.$$

Since M is screen conformal and $Ric = \gamma g$, the equation (3.3) reduces to

$$g(A^*_\xi X,\, A^*_\xi Y) - sg(A^*_\xi X,\, Y) + \varphi^{-1}\gamma\, g(X,\, Y) = 0, \tag{3.4}$$

where $s = \mathrm{tr}A^*_\xi$. Put $X = Y = E_i$ in (3.4), λ_i is a solution of equation

$$x^2 - sx + \varphi^{-1}\gamma = 0. \tag{3.5}$$

The equation (3.5) has at most two distinct solutions which are smooth real valued function on $\mathcal{U}$. Assume that there exists $p \in \{0, 1, \ldots, m\}$ such that $\lambda_1 = \cdots = \lambda_p = \alpha$ and $\lambda_{p+1} = \cdots = \lambda_m = \beta$, by renumbering if necessary. From (3.5), we have

$$s = \alpha + \beta = p\alpha + (m - p)\beta\,;\ \alpha\beta = \varphi^{-1}\gamma. \tag{3.6}$$

Theorem 3.1. *Let $(M, g, S(TM))$ be a screen conformal Einstein lightlike hypersurface of a Lorentzian space form $(\bar{M}^{m+2}(c), \bar{g})$; $m > 2$. Then M is locally a product manifold $L \times M_\alpha \times M_\beta$, where L is an open subset of a lightlike geodesic ray in $\bar{M}$ and M_α and M_β are totally umbilical leaves of some integerable distributions of M.*

Proof. If the equation (3.5) has only one solution α, then, by Note 1, we have $M = L \times M^* \cong L \times M^* \times \{x\}$ for any $x \in M$, where $M^* = M_\alpha$. Since $B(X, Y) = g(A^*_\xi X, Y) = \alpha g(X, Y)$ for all $X, Y \in \Gamma(TM)$, M is totally umbilical. By (2.1), we get $C(X, Y) = \varphi\alpha g(X, Y)$ for all $X, Y \in \Gamma(TM)$. Thus M^* is also totally umbilical. In this case, our assertion is true.

Assume the equation (3.5) has exactly two distinct solutions α and β. If $p = 0$ or $p = m$, then we also show that $M = L \times M^* \cong L \times M^* \times \{x\}$ for any $x \in M$ and $M^* = M_\alpha$ or M_β. In these cases, M and M^* are also totally umbilical. Let $0 < p < m$. Consider the following four distributions

D_α, D_β, D^s_α and D^s_β on M:

$$\Gamma(D_\alpha) = \{X \in \Gamma(TM) \mid A^*_\xi X = \alpha PX\}, \; D^s_\alpha = PD_\alpha;$$
$$\Gamma(D_\beta) = \{U \in \Gamma(TM) \mid A^*_\xi U = \beta PU\}, \; D^s_\beta = PD_\beta.$$

Then $D_\alpha \cap D_\beta = TM^\perp$ and $D^s_\alpha \cap D^s_\beta = \{0\}$. As $A^*_\xi PX = A^*_\xi X = \alpha PX$ for all $X \in \Gamma(D_\alpha)$ and $A^*_\xi PU = A^*_\xi U = \beta PU$ for all $U \in \Gamma(D_\beta)$, PX and PU are eigenvector fields of the real symmetric operator A^*_ξ corresponding to the different eigenvalues α and β respectively. Thus $PX \perp_g PU$ and $g(X,U) = g(PX,PU) = 0$, that is, $D_\alpha \perp_g D_\beta$. Also, since $B(X,U) = g(A^*_\xi X, U) = \alpha g(PX, PU) = 0$, we show that $D_\alpha \perp_B D_\beta$.

Since $\{E_i\}_{1\le i\le p}$ and $\{E_a\}_{p+1\le a\le m}$ are vector fields of D^s_α and D^s_β respectively and D^s_α and D^s_β are mutually orthogonal vector subbundle of $S(TM)$, D^s_α and D^s_β are non-degenerate distributions of rank p and rank $(m-p)$ respectively. Thus we have $S(TM) = D^s_\alpha \oplus_{\rm orth} D^s_\beta$.

From (3.4), we show that $(A^*_\xi)^2 - (\alpha+\beta)A^*_\xi + \alpha\beta P = 0$. Let $Y \in {\rm Im}(A^*_\xi - \alpha P)$, then there exists $X \in \Gamma(TM)$ such that $Y = (A^*_\xi - \alpha P)X$. Then $(A^*_\xi - \beta P)Y = 0$ and $Y \in \Gamma(D_\beta)$. Thus ${\rm Im}(A^*_\xi - \alpha P) \subset \Gamma(D_\beta)$. Since the morphism $A^*_\xi - \alpha P$ maps $\Gamma(TM)$ onto $\Gamma(S(TM))$, we have ${\rm Im}(A^*_\xi - \alpha P) \subset \Gamma(D^s_\beta)$. By duality, we also have ${\rm Im}(A^*_\xi - \beta P) \subset \Gamma(D^s_\alpha)$.

For X, $Y \in \Gamma(D_\alpha)$ and $U \in \Gamma(D_\beta)$, we have

$$(\nabla_X B)(Y,U) = -g((A^*_\xi - \alpha P)\nabla_X Y, U) + \alpha B(X,Y)\eta(U)$$

and $(\nabla_X B)(Y,U) = (\nabla_Y B)(X,U)$ due to (1.15). Thus $g((A^*_\xi - \alpha P)[X,Y], U) = 0$. Since the distribution D^s_β is non-degenerate and ${\rm Im}(A^*_\xi - \alpha P) \subset \Gamma(D^s_\beta)$, we have $(A^*_\xi - \alpha P)[X,Y] = 0$. Thus $[X,Y] \in \Gamma(D_\alpha)$ and D_α is integrable. By duality, D_β is also integrable. Since $S(TM)$ is integrable, for any X, $Y \in \Gamma(D^s_\alpha)$, we have $[X,Y] \in \Gamma(D_\alpha)$ and $[X,Y] \in \Gamma(S(TM))$. Thus $[X,Y] \in \Gamma(D^s_\alpha)$ and D^s_α is integrable. So is D^s_β.

For X, $Y \in \Gamma(D_\alpha)$, we have

$$\begin{aligned}(\nabla_X B)(Y,Z) = & -g((A^*_\xi - \alpha P)\nabla_X Y, Z) + \alpha B(X,Y)\eta(Z) \\ & + (X\alpha)\, g(Y,Z) + \alpha^2 \eta(Y)\, g(X,Z).\end{aligned}$$

Using this and the fact that $(\nabla_X B)(Y,Z) = (\nabla_Y B)(X,Z)$, we obtain

$$\{X\alpha - \alpha^2\eta(X)\}g(Y,Z) = \{Y\alpha - \alpha^2\eta(Y)\}g(X,Z), \tag{3.7}$$

due to $(A^*_\xi - \alpha P)[X, Y] = 0$. Therefore, for X, $Y \in \Gamma(D^s_\alpha)$ and $Z \in \Gamma(S(TM))$, we obtain $(X\alpha)g(Y,Z) = (Y\alpha)g(X,Z)$. Since $S(TM)$ is non-degenerate, we have $d\alpha(X)Y = d\alpha(Y)X$. Suppose there exists a vector field $X_o \in \Gamma(D^s_\alpha)$ such that $d\alpha(X_o)_x \ne 0$ at each point $x \in M$. Then $Y = fX_o$ for any $Y \in \Gamma(D^s_\alpha)$, where f is a smooth function. It follows that all vectors from the fiber $(D^s_\alpha)_x$ are colinear with $(X_o)_x$. It is a contradiction as $\dim (D^s_\alpha)_x = p > 1$. Thus we have $d\alpha|_{D^s_\alpha} = 0$. By duality, we also have $d\beta|_{D^s_\beta} = 0$. Thus α is a constant along D^s_α and β is a constant along D^s_β. Since $(p-1)\alpha = -(m-p-1)\beta$, α and β are constants along $S(TM)$.

From (2.3) with $c = 0$, we have

$$(X\varphi)B(Y,Z) = (Y\varphi)B(X,Z), \ \forall\, X,\, Y,\, Z \in \Gamma(TM). \tag{3.8}$$

Take $X,\, Y,\, Z \in \Gamma(D^s_\alpha)$, the equation (3.8) reduces to

$$(X\varphi)\alpha\, g(Y,Z) = (Y\varphi)\alpha\, g(X,Z), \ \text{i.e., } d(X\varphi)\alpha\, Y = (Y\varphi)\alpha\, X.$$

Since $\dim (D^s_\alpha)_x > 1$, we have $(X\varphi)\alpha = 0$ for all $X \in \Gamma(D^s_\alpha)$. While, take $X \in \Gamma(D^s_\beta)$ and $Y,\, Z \in \Gamma(D^s_\alpha)$ in (3.8), we have $(X\varphi)\alpha = 0$ for all $X \in \Gamma(D^s_\beta)$. Consequently, we obtain $(X\varphi)\alpha = 0$ for all $X \in \Gamma(S(TM))$. By duality, we get $(X\varphi)\beta = 0$ for all $X \in \Gamma(S(TM))$. Since $(\alpha,\beta) \neq (0,0)$, we have $X\varphi = 0$ for all $X \in \Gamma(S(TM))$, that is, φ is a constant along $S(TM)$. For all $X,\, Y \in \Gamma(D^s_\alpha)$, we have $\xi[\varphi]\alpha = 0$ due to (2.3). Also, for all $X,\, Y \in \Gamma(D^s_\beta)$, we have $\xi[\varphi]\beta = 0$. Thus we have $\xi[\varphi] = 0$. Consequently we have $X[\varphi] = 0$ for all $X \in \Gamma(TM)$, i.e., φ is a constant on M. For all $X \in \Gamma(D^s_\alpha)$ and $U \in \Gamma(D^s_\beta)$, since $(\nabla_X B)(U,Z) = (\nabla_U B)(X,Z)$, we get

$$g(\{(A^*_\xi - \beta P)\nabla_X U - (A^*_\xi - \alpha P)\nabla_U X\},\, Z) \ = \ 0, \ \forall\, Z \in \Gamma(S(TM)).$$

Since $S(TM)$ is non-degenerate, we have $(A^*_\xi - \beta P)\nabla_X U = (A^*_\xi - \alpha P)\nabla_U X$. Since the left term of the last equation is in $\Gamma(D^s_\alpha)$ and the right term is in $\Gamma(D^s_\beta)$ and $D^s_\alpha \cap D^s_\beta = \{0\}$, we have $(A^*_\xi - \beta P)\nabla_X U = 0$ and $(A^*_\xi - \alpha P)\nabla_U X = 0$. This imply that $\nabla_X U \in \Gamma(D_\beta)$ and $\nabla_U X \in \Gamma(D_\alpha)$. On the other hand, $\nabla_X U = \nabla^*_X U$ and $\nabla_U X = \nabla^*_U X$ due to $D_\alpha \perp_B D_\beta$, we have

$$\nabla_X U \in \Gamma(D^s_\beta), \ \nabla_U X \in \Gamma(D^s_\alpha), \ \forall\, X \in \Gamma(D^s_\alpha); \ \forall\, U \in \Gamma(D^s_\beta). \tag{3.9}$$

For $X,\, Y \in \Gamma(D^s_\alpha)$ and $U,\, V \in \Gamma(D^s_\beta)$, since $g(X,U) = 0$, we have

$$g(\nabla_Y X, U) + g(X, \nabla_Y U) = 0, \ g(\nabla_V U, X) + g(U, \nabla_V X) = 0.$$

Using (3.9), we have $g(X, \nabla_Y U) = g(U, \nabla_V X) = 0$. Thus we get

$$g(\nabla_Y X,\, U) = 0\,; \ g(X,\, \nabla_V U) = 0. \tag{3.10}$$

Since the leaf M^* of $S(TM)$ is a Riemannian manifold and $S(TM) = D^s_\alpha \oplus_{\rm orth} D^s_\beta$, where D^s_α and D^s_β are parallel and integrable distributions with respect to the induced connection ∇^* on M^* due to (3.10), by the decomposition theorem of de Rham [8], we have $M^* = M_\alpha \times M_\beta$, where M_α and M_β are some leaves of D^s_α and D^s_β respectively. Thus we have our theorem. □

Proof of Theorem 1.1. First, we prove that $\gamma = 0$ and $\alpha\beta = 0$ for $0 < p < m$. From the facts that $(p-1)\alpha = -(m-p-1)\beta$ and $m > 2$, if $p = 1$, then $\beta = 0$ and if $p = m-1$, then $\alpha = 0$. Thus we have $\gamma = 0$. Let $1 < p < m-1$. Then, for $X \in \Gamma(D^s_\alpha)$ and $U \in \Gamma(D^s_\beta)$, using (3.9) and (3.10), we have

$$g(R(X,\, U)U,\, X) = g(\nabla_X \nabla_U U,\, X).$$

From the second equation of (3.10), we know that $\nabla_U U$ has no component of D_α. Since P maps $\Gamma(D_\beta)$ onto $\Gamma(D^s_\beta)$ and $S(TM) = D^s_\alpha \oplus_{\text{orth}} D^s_\beta$, we have

$$\nabla_U U = P(\nabla_U U) + \eta(\nabla_U U)\xi\,;\ P(\nabla_U U) \in \Gamma(D^s_\beta).$$

It follows that

$$\begin{aligned} g(\nabla_X \nabla_U U, X) &= g(\nabla_X P(\nabla_U U), X) + (\nabla_X \eta)(\nabla_U U))\, g(\xi, X) \\ &\quad + \eta(\nabla_X \nabla_U U) g(\xi, X) + \eta(\nabla_U U) g(\nabla_X \xi, X) \\ &= -\ \alpha\, \eta(\nabla_U U) g(X, X). \end{aligned}$$

Since $\eta(\nabla_U U) = g(U, A_N U) = \varphi g(U, A^*_\xi U) = \varphi\beta\, g(U, U)$, we have

$$g(R(X,\, U)U,\, X) = -\,\varphi\alpha\beta g(X,\, X) g(U,\, U).$$

While, from the Gauss equation (1.14), we have

$$g(R(X,\, U)U,\, X) = \varphi\alpha\beta g(X,\, X) g(U,\, U).$$

From the last two equations, we get $\gamma = \varphi\alpha\beta = 0$ for $1 < p < m-1$. Consequently we show that if $0 < p < m$, then $\gamma = 0$ and $\alpha\beta = 0$. □

(1) Let $\gamma \neq 0$: In case $(\mathrm{tr}A^*_\xi)^2 \neq 4\varphi^{-1}\gamma$. The equation (3.5) has two non-vanishing distinct solutions α and β. If $0 < p < m$, then we have $\gamma = 0$. Thus $p = 0$ or $p = m$. If $p = 0$, then $D^s_\alpha = \{0\}$ and $D^s_\beta = S(TM)$. If $p = m$, then $D^s_\alpha = S(TM)$ and $D^s_\beta = \{0\}$. From (1.14) and (1.18), we have

$$R^*(X, Y)Z = 2\varphi\alpha^2\{g(Y,\, Z)X - g(X,\, Z)Y\},\ \forall\ X,\, Y,\, Z \in \Gamma(D_\alpha);$$

$$R^*(U, V)W = 2\varphi\beta^2\{g(V,\, W)U - g(U,\, W)V\},\ \forall\ U,\, V,\, W \in \Gamma(D_\beta).$$

Thus either M_α or M_β, which are leafs of D_α or D_β respectively, is a Riemannian manifold M^* of constant curvature $2\varphi\alpha^2$ or $2\varphi\beta^2$ respectively and the other leaf is a point $\{x\}$. If $p = m$, that is, $M^* = M_\alpha$, since $B(X, Y) = \alpha g(X, Y)$ for all $X,\, Y \in \Gamma(S(TM))$, we have $C(X, Y) = \varphi\alpha g(X, Y)$ for all $X,\, Y \in \Gamma(S(TM))$. If $p = 0$, that is, $M^* = M_\beta$, since $B(U, V) = \beta g(U, V)$ for all $U,\, V \in \Gamma(S(TM))$, we have $C(U, V) = \varphi\beta g(U, V)$ for all $U,\, V \in \Gamma(S(TM))$. Thus the leaf M^* is a totally umbilical which is not a totally geodesics. Consequently M is locally a product manifold $L \times M^* \times \{x\}$ or $L \times \{x\} \times M^*$, where M^* is an m-dimensional totally umbilical Riemannian manifold of constant curvature $2\varphi\beta^2$ or $2\varphi\alpha^2$ which is isometric to a sphere or a hyperbolic space, $\{x\}$ is a point.

In case $(\mathrm{tr}A^*_\xi)^2 = 4\varphi^{-1}\gamma$. The equation (3.5) has only one non-zero constant solution, named by α and α is only one eigenvalue of A^*_ξ. In this case, the equations (3.6) reduce to $s = 2\alpha = m\alpha\,;\ \alpha^2 = \varphi^{-1}\gamma$. Thus we have $m = 2$. Thus this case is not appear.

(2) Let $\gamma = 0$. The equation (3.6) reduces to $x(x - s) = 0$. In case $\mathrm{tr}A^*_\xi \neq 0$. Let $\alpha = 0$ and $\beta = s$. Then we have $s = \beta = (m-p)\beta$, i.e., $(m-p-1)\beta = 0$. So $p = m-1$. Thus the leaf M_α of D^s_α is totally geodesic $(m-1)$-dimensional Riemannian manifold and the leaf M_β of D^s_β is a spacelike curve. In the sequel,

let $X, Y, Z \in \Gamma(D^s_\alpha)$ and $U \in \Gamma(D^s_\beta)$. From (1.14), (1.18) and $c = 0$, we have $R^*(X, Y)Z = R(X, Y)Z = \bar{R}(X, Y)Z = 0$. Using (3.10) and the fact that the connection ∇^* is metric, we have

$$g(\nabla^*_X Y, U) = -g(Y, \nabla^*_X U) = -g(Y, \nabla_X U) = 0.$$

Thus $\nabla^*_X Y \in \Gamma(D^s_\alpha)$. From this result, (1.6), (3.9) and the integrable property of D^s_α, we have $g(R^*(X, Y)Z, U) = 0$. This implies $\pi_\alpha R^*(X, Y)Z = R^*(X, Y)Z = 0$, where π_α is the projection morphism of $\Gamma(S(TM))$ on $\Gamma(D^s_\alpha)$ and $\pi_\alpha R^*$ is the curvature tensor of D^s_α. Thus M_α is a Euclidean manifold. Thus M is locally a product $L \times M_\alpha \times M_\beta$, where M_α is an $(m-1)$-dimensional Euclidean space and M_β is a spacelike curve in $\bar{M}$.

In case $\mathrm{tr}A^*_\xi = 0$. Then we have $\alpha = \beta = 0$ and $A^*_\xi = 0$ or equivalently $B = 0$ and $D^s_\alpha = D^s_\beta = S(TM)$. Thus M is totally geodesic in $\bar{M}$. Since M is screen conformal, we also have $C = A_N = 0$. Thus the leaf M^* of $S(TM)$ is also totally geodesic. Thus we have $\bar{\nabla}_X Y = \nabla^*_X Y$ for any tangent vector fields X and Y to the leaf M^*. This implies that M^* is a Euclidean m-space. Thus M is locally a product $L \times M^* \times \{x\}$, where L is a null curve and $\{x\}$ is a point.

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