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FULL QUADRATURE SUMS FOR GENERALIZED POLYNOMIALS WITH FREUD WEIGHTS

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Abstract. Generalized nonnegative polynomials are defined as products of nonnegative polynomials raised to positive real powers. The generalized degree can be defined in a natural way. In this paper we extend quadrature sums involving pth powers of polynomials to those for generalized polynomials.

1. Introduction

In 1969, Askey [1] proposed the following problem: Let $P \in \mathbb{P}_n$, where $\mathbb{P}_n$ denotes the set of all polynomials of degree at most n. Let $\{x_{n,j}\}$ be the zeros of the orthogonal polynomials with respect to $d\alpha$, a positive measure on $[-1, 1]$, and $\{\lambda_{n,j}\}$ be the Cotes numbers for $d\alpha$. When is it true that

$$\sum_{j=1}^{n} \lambda_{n,j} |P(x_{n,j})|^p \leq C \int_{-1}^{1} |P(x)|^p d\alpha(x), \tag{1.1}$$

where C is independent of P and n? Such inequalities are essential in various problems in approximation theory, and in particular, in investigating mean convergence of Lagrange interpolation and orthogonal expansions.

Of course when $p = 2$, the Gauss quadrature formula asserts equality in (1.1) with $C = 1$. Askey proved (1.1) for certain Jacobi weights for $p \geq 1$. Nevai [15] proved (1.1) for generalized Jacobi weights and $P \in \mathbb{P}_{ln}$ with $l \geq 2$ fixed. A further generalization, valid for $0 < p < \infty$, and $P \in \mathbb{P}_{ln}$ with $l > 1$ fixed, was proved in [10]. A converse inequality has been proven in [16].

For the Freud weights, Lubinsky, Máté, and Nevai [10, Corollary 9, p. 536] proved (1.1) with the range of summation suitably restricted, and subsequently, Lubinsky and Matjila [13] provided a solution for the Freud weights as follows: Let $r > 0$, and $b \in (-\infty, 2]$. Then we have for $1 \leq p < \infty$

$$\sum_{j=1}^{n} \lambda_{n,j} |PW|^p (x_{n,j}) W^{-b}(x_{n,j}) \leq C \int_{-\infty}^{\infty} |PW|(t) W^{2-b}(t) dt \tag{1.2}$$

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for every polynomial P of degree at most $n+rn^{1/3}$, where $W^2(x) = \exp(-|x|^\alpha)$, $(\alpha > 1)$, $\{\lambda_{n,j}\}$ are the Cotes numbers, and $\{x_{n,j}\}$ are the zeros of the orthonormal polynomials for W^2.

The aim of this paper is to extend inequalities such as (1.2) to generalized polynomials.

A generalized nonnegative algebraic polynomial is a function of the type

$$f(z) = |\omega| \prod_{j=1}^{m} |z - z_j|^{r_j} \quad (0 \neq \omega \in \mathbb{C})$$

with $r_j \in \mathbb{R}^+$, $z_j \in \mathbb{C}$, and the number

$$n \stackrel{\text{def}}{=} \sum_{j=1}^{m} r_j$$

is called the generalized degree of f. Note that $n > 0$ is not necessarily an integer. Thus throughout this paper we assume that $n \in \mathbb{R}^+$ unless stated otherwise.

We denote by GANP_n the set of all generalized nonnegative algebraic polynomials of degree at most $n \in \mathbb{R}^+$.

Using

$$|z - z_j|^{r_j} = ((z - z_j)(z - \bar{z}_j))^{r_j/2}, \quad z \in \mathbb{R},$$

we can easily check that when $f \in \mathrm{GANP}_n$ is restricted to the real line, then it can be written as

$$f = \prod_{j=1}^{m} P_j^{r_j/2}, \quad 0 \leq P_j \in \mathbb{P}_2, \quad r_j \in \mathbb{R}^+, \quad \sum_{j=1}^{m} r_j \leq n,$$

which is the product of nonnegative polynomials raised to positive real powers. This explains the name *generalized nonnegative polynomials.* Many properties of generalized nonnegative polynomials were investigated in a series of papers ([2, 3, 4, 5]).

Associated with the Freud weight $W_\alpha(x) = \exp(-|x|^\alpha)$, $\alpha > 0$, there are Mhaskar-Rahmanov-Saff numbers $a_n = a_n(\alpha)$, which is the positive solution of the equation

$$n = \frac{2}{\pi} \int_0^1 a_n t Q'(a_n t)(1 - t^2)^{-\frac{1}{2}} dt, \quad n \in \mathbb{R}^+,$$

where $Q(x) = |x|^\alpha$, $\alpha > 0$. Explicitly,

$$a_n = a_n(\alpha) = \left(\frac{n}{\lambda_\alpha}\right)^{1/\alpha}, \quad n \in \mathbb{R}^+,$$

where

$$\lambda_\alpha = \frac{2^{2-\alpha}\Gamma(\alpha)}{\{\Gamma(\alpha/2)\}^2}.$$

Its importance lies partly in the identity

$$\|PW_\alpha\|_{L^\infty(\mathbb{R})} = \|PW_\alpha\|_{L^\infty([-a_n,a_n])}, \quad P \in \mathbb{P}_n. \tag{1.3}$$

Now we state our result.

Theorem 1.1. *Let $W_\alpha(x) = \exp(-|x|^\alpha)$, $\alpha > 1$, and $0 < p < \infty$. Let $K > 0$, $\ell \geq 1$, and let*

$$-Ka_n \leq y_M < y_{M-1} < \cdots < y_1 \leq Ka_n$$

and

$$\delta = \min\{y_{j-1} - y_j \ : j = 2, 3, \ldots, M\} > 0.$$

Let Ψ be convex, nonnegative, and nondecreasing in $[0,\infty)$. Then for all $f \in \mathrm{GANP}_{\ell n}$, $\frac{2p^2}{\ell p+4} \leq n \in \mathbb{R}^+$,

$$\sum_{j=1}^{M} \Psi(f(y_j)W_\alpha^p(y_j)) \leq C_1 \left(\frac{n}{a_n} + \frac{1}{\delta}\right) \int_{-\infty}^{\infty} \Psi(C_2 f(u) W_\alpha^p(u)) du.$$

The constants C_1 and C_2 are independent of M, δ, $\{y_j\}$, n, and f.

Theorem 1.2. *Let $W_\alpha(x) = \exp(-|x|^\alpha)$, $\alpha > 1$. Let $0 < p < \infty$, $\ell \geq 1$, and $\frac{2p^2}{\ell p+4} \leq n \in \mathbb{N}$. Let $\{x_{n,j}\}$ be the zeros of orthogonal polynomial $P_n(W_\alpha^2; x)$ and $\{\lambda_{n,j}\}$ be the Cotes numbers for W_α^2. Then, there exists a positive constant C such that*

$$\begin{aligned} &\sum_{j=1}^{n} \lambda_{n,j} f(x_{n,j}) W_\alpha^{p-2}(x_{n,j}) \left(\max\left\{n^{-\frac{2}{3}}, 1 - \frac{|x_{n,j}|}{a_n}\right\}\right)^{\frac{1}{2}} \\ &\leq C \int_{-\infty}^{\infty} f(u) W_\alpha^p(u) du \end{aligned} \tag{1.4}$$

for $f \in \mathrm{GANP}_{\ell n}$.

In proving (1.2), refined Markov type inequalities [13] were used. We have to insert the square root factor on the left hand side of (1.4) because we do not have refined Markov type inequalities for generalized polynomials.

Throughout this paper we write $g_n(x) \sim h_n(x)$ if for every n and for every x in consideration

$$0 < c_1 \leq \frac{g_n(x)}{h_n(x)} \leq c_2 < \infty,$$

and $g(x) \sim h(x)$, $n \sim N$ have similar meanings.

2. Proof of theorems

In order to prove Theorems 1.1 and 1.2, we need lemmas on Infinite-Finite range inequalities and estimates of Christoffel functions for generalized polynomials.

In the analysis of extremal polynomials on $\mathbb{R}$, the estimation of the norm of a weighted polynomial $\|PW\|_{L^p(\mathbb{R})}$ in terms of the norm $\|PW\|_{L^p(-c_n,c_n)}$ over an finite interval $(-c_n, c_n)$ is important because such estimations or inequalities reduce problems over an infinite interval to problems on a finite interval. Mhaskar and Saff [14] established sharper inequalities that led to nth root asymptotics for extremal polynomials, for $p = \infty$ they showed that

$$\|PW_\alpha\|_{L^\infty(\mathbb{R})} = \|PW_\alpha\|_{L^\infty([-a_n, a_n])}, \quad P \in \mathbb{P}_n.$$

For generalized nonnegative polynomials we have the following lemma, which is the restatement of Theorem 2.1 in [7. p. 124].

Lemma 2.1. *Let $W_\alpha(x) = \exp(-|x|^\alpha)$, $\alpha > 0$. Then*

$$\|fW_\alpha\|_{L^\infty(\mathbb{R})} = \|fW_\alpha\|_{L^\infty([-a_n, a_n])} \tag{2.1}$$

for all $f \in \mathrm{GANP}_n$, $n \in \mathbb{R}^+$.

If $0 < p < \infty$, then there exist positive constants C_1 and C_2 so that, whenever

$$\frac{n}{(\log n)^2} \geq K_n \geq C_1, \quad 2 \leq n \in \mathbb{R}^+ \tag{2.2}$$

and

$$\delta_n = \left(\frac{K_n \log n}{n}\right)^{2/3}, \quad 2 \leq n \in \mathbb{R}^+, \tag{2.3}$$

then

$$\|fW_\alpha\|_{L^p(\mathbb{R})} \leq (1 + n^{-C_2 K_n}) \|fW_\alpha\|_{L^p([-a_n(1+\delta_n), a_n(1+\delta_n)])} \tag{2.4}$$

for all $f \in \mathrm{GANP}_n$, $n \geq 2$.

Proof. See the proof of Theorem 2.1 in [7. p. 124]. □

Next we define generalized Christoffel functions. Let $0 < p < \infty$. Then the generalized Christoffel function for ordinary polynomials is defined by

$$\lambda_{n,p}(W_\alpha; x) = \min_{P \in \mathbb{P}_{n-1}} \int_{-\infty}^{\infty} \frac{|P(t)W_\alpha(t)|^p}{|P(x)|^p} dt, \quad x \in \mathbb{R}, \quad n \in \mathbb{N}.$$

The generalized Christoffel function for generalized nonnegative polynomials is defined by

$$\omega_{n,p}(W_\alpha; x) = \inf_{f \in \mathrm{GANP}_n} \int_{-\infty}^{\infty} \frac{(f(t)W_\alpha(t))^p}{f^p(x)} dt, \quad x \in \mathbb{R}, \quad n \in \mathbb{R}^+.$$

For the estimates of $\omega_{n,p}(W_\alpha; x)$, we need the following lemma, which is the restatement of Theorem 2.3 in [7, p. 125].

Lemma 2.2. *Let $W_\alpha(x) = \exp(-|x|^\alpha)$, $\alpha > 1$. Let $0 < p < \infty$. Then*

$$\omega_{n,p}(W_\alpha; x) \geq C \frac{a_n}{n} W_\alpha^p(x), \quad x \in \mathbb{R}, \quad n \in \mathbb{R}^+,$$

and

$$\omega_{n,p}(W_\alpha; x) \leq \lambda_{[n]+1,p}(W_\alpha; x), \quad x \in \mathbb{R}, \quad n \in \mathbb{R}^+,$$

where $[n]$ denotes the integer part of n.

Proof. See the proof of Theorem 2.3 in [7, p. 125]. □

Remark. It is well known (see, for example, [8]) that if $\alpha > 1$, then there exist positive constants C_1 and C_2 depending on p and α, such that

$$\lambda_{[n]+1,p}(W_\alpha; x) \le C_1 \frac{a_n}{n} W_\alpha^p(x), \quad |x| \le C_2 a_n.$$

Consequently

$$\omega_{n,p}(W_\alpha; x) \sim \frac{a_n}{n} W_\alpha^p(x), \quad |x| \le C_2 a_n.$$

Now we prove our results.

Proof of Theorem 1.1. Let $W_\alpha(x) = \exp(-|x|^\alpha)$, $\alpha > 1$, $0 < p < \infty$. Fix $K > 0$ and $\ell \ge 1$. Let Ψ be convex, nonnegative, and nondecreasing in $[0, \infty)$. By Lemma 2.1, there exists a positive constant B^* such that

$$\|fW_\alpha\|_{L^p(\mathbb{R})} \le 2\|fW_\alpha\|_{L^p([-B^*a_n, B^*a_n])}, \quad (0 < p < \infty) \tag{2.5}$$

for $n \ge 2$, $f \in \mathrm{GANP}_n$. Let $k = 4/p$ and $B \ge B^*(\ell + k)$. Then

$$B^* a_{(\ell+k)n} \le B a_n. \tag{2.6}$$

Now let $p_j(v, x)$, $j = 0, 1, 2, \dots$, be the orthonormal Chebyshev polynomials associated with the Chebyshev weight

$$v(t) = \begin{cases} (1 - t^2)^{-1/2}, & t \in (-1, 1), \\ 0, & t \notin (-1, 1). \end{cases}$$

and let

$$K_m(v, x, t) = \sum_{j=0}^{m-1} p_j(v, x) p_j(v, t), \quad m \in \mathbb{N}.$$

Let $f \in \mathrm{GANP}_{\ell n}$, $(\ell + k)n \ge 2$, and let $N = [n] + 1$. Then for each fixed x,

$$f(t) \left| K_N \left(v, \frac{x}{Ba_n}, \frac{t}{Ba_n} \right) \right|^k$$

is a generalized polynomial in t of degree less than $(\ell + k)n$. By Lemma 2.2, (2.5), and (2.6), we have for all $t \in \mathbb{R}$,

$$\begin{aligned} &f^p(t) W_\alpha^p(t) |K_N^k(v, x/(Ba_n), t/(Ba_n))|^p \\ &\le c_1 \frac{n}{a_n} \int_{-\infty}^{\infty} f^p(u) W_\alpha^p(u) K_N^4(v, x/(Ba_n), u/(Ba_n)) du \\ &\le c_2 \frac{n}{a_n} \int_{-Ba_n}^{Ba_n} f^p(u) W_\alpha^p(u) K_N^4(v, x/(Ba_n), u/(Ba_n)) du. \end{aligned}$$

Set $t = x$. Then for all $x \in \mathbb{R}$,

$$\begin{aligned} &f^p(x) W_\alpha^p(x) K_N^4(v, x/(Ba_n), x/(Ba_n)) \\ &\le c_2 \frac{n}{a_n} \int_{-Ba_n}^{Ba_n} f^p(u) W_\alpha^p(u) K_N^4(v, x/(Ba_n), u/(Ba_n)) du. \end{aligned}$$

Since

$$K_N^4(v, x/(Ba_n), x/(Ba_n)) \sim N^4 \sim n^4 \quad \text{for} \quad |x| \le Ba_n,$$

(see, [15, p. 108]), we have

$$f^p(x)W_\alpha^p(x) \le c_3 \frac{1}{n^3 a_n} \int_{-Ba_n}^{Ba_n} f^p(u)W_\alpha^p(u)K_N^4(v, x/(Ba_n), u/(Ba_n))du \tag{2.7}$$

for $|x| \le Ba_n$. By Theorem 2.2 in [10, p. 537], we have for $|x| \le \frac{Ba_n}{2}$,

$$\begin{aligned} &\int_{-ba_n}^{Ba_n} K_N^4(v, x/(Ba_n), u/(Ba_n))du \\ &= Ba_n \int_{-1}^{1} K_N^4(v, x/(Ba_n), u)du \sim a_n N^3 \sim a_n n^3. \end{aligned}$$

Using Jensen's inequality and (2.7), we obtain for $|x| \le \frac{Ba_n}{2}$,

$$\begin{aligned} &\Psi(f^p(x)W_\alpha^p(x)) \\ &\le \Psi\left(\frac{\int_{-Ba_n}^{Ba_n} c_4 f^p(u)W_\alpha^p(u)K_N^4(v, x/(Ba_n), u/(Ba_n))du}{\int_{-Ba_n}^{Ba_n} K_N^4(v, x/(Ba_n), u/(Ba_n))du}\right) \\ &\le \frac{\int_{-Ba_n}^{Ba_n} \Psi(c_4 f^p(u)W_\alpha^p(u))K_N^4(v, x/(Ba_n), u/(Ba_n))du}{\int_{-Ba_n}^{Ba_n} K_N^4(v, x/(Ba_n), u/(Ba_n))du} \\ &\le c_5 \frac{1}{a_n n^3} \int_{-Ba_n}^{Ba_n} \Psi(c_4 f^p(u)W_\alpha^p(u))K_N^4(v, x/(Ba_n), u/(Ba_n))du. \end{aligned}$$

Since

$$K_N^4\left(v, \frac{x}{Ba_n}, \frac{u}{Ba_n}\right) \le c_6 n^2 K_N^2\left(v, \frac{x}{Ba_n}, \frac{u}{Ba_n}\right), \quad |x| \le Ba_n, \quad |u| \le Ba_n,$$

we have

$$\begin{aligned} &\Phi(f^p(x)W_\alpha^p(x)) \\ &\le c_7 \frac{1}{a_n n} \int_{-Ba_n}^{Ba_n} \Psi(c_4 f^p(u)W_\alpha^p(u))K_N^2(v, x/(Ba_n), u/(Ba_n))du \end{aligned} \tag{2.8}$$

for $|x| \le \frac{Ba_n}{2}$.

Now, let

$$-Ka_n \le y_M < y_{M-1} < \cdots < y_1 \le Ka_n,$$

and

$$\delta = \min\{y_{j-1} - y_i : j = 2, 3, \ldots, M\} > 0.$$

We can assume that $K \le B/2$ so that $|y_j/(Ba_n)| \le 1/2$ for $j = 1, 2, \ldots, M$. As

$$\frac{d}{dx} \arccos(x) \sim -1 \quad \text{for} \quad |x| \le 1/2,$$

we have

$$\arccos(y_j/(Ba_n)) - \arccos(y_{j-1}/(Ba_n)) \geq c_8 \frac{y_{j-1} - y_j}{a_n} \geq c_8 \frac{\delta}{a_n}.$$

Then by Lemma 2.3 in [10, p. 539], we obtain

$$\sum_{j=1}^{M} K_N^2 \left(v, \frac{y_j}{Ba_n}, \frac{u}{Ba_n}\right) \leq \frac{8}{\pi^2} N \left(N + \frac{c_9 a_n}{\delta}\right) \quad \text{for} \quad |u| \leq Ba_n.$$

Using (2.8) and the above inequality, we have for all $f \in \mathrm{GANP}_{\ell n}$, $n \geq \frac{2}{\ell+k} = \frac{2p}{\ell p+4}$,

$$\sum_{j=1}^{M} \Psi(f^p(y_j) W_\alpha^p(y_j)) \leq c_{10} \left(\frac{n}{a_n} + \frac{1}{\delta}\right) \int_{-\infty}^{\infty} \Psi(c_4 f^p(u) W_\alpha^p(u)) du,$$

which yields Theorem 1.1. □

Lemma 2.3. *Let $x_{n,n} < x_{n,n-1} < \cdots < x_{n,1}$ be the zeros of orthogonal polynomial $P_n(W_\alpha^2; x)$, $\alpha > 1$. Then there exists a positive constant C such that*

$$x_{n,j-1} - x_{n,j} \geq C \frac{a_n}{n} \quad \textit{for} \quad j = 2, \ldots, n.$$

Proof. By Theorem 5.1 in [6, p. 36], there exists $\eta \in (0,1)$ such that

$$x_{n,j-1} - x_{n,j} \sim \frac{a_n}{n} \quad \text{for} \quad |x_{n,j}| \leq \eta a_n. \tag{2.9}$$

By Theorem 7.6 in [8, p. 168]

$$\left(\int_{-\infty}^{\infty} |P(t) W_\alpha(t)|^2 dt\right)^{\frac{1}{2}} \geq c_1 \left(\frac{a_n}{n}\right)^{\frac{1}{2}} \|PW_\alpha\|_{L^\infty(\mathbb{R})}, \quad P \in \mathbb{P}_{n-1}. \tag{2.10}$$

Let

$$g_n(x) = \left(\max\left\{n^{-\frac{2}{3}}, 1 - \frac{|x|}{a_n}\right\}\right)^{\frac{1}{2}}.$$

By Theorem 1.9 in [9, p. 470],

$$\|PW_\alpha\|_{L^\infty(\mathbb{R})} \geq c_2 \frac{a_n}{n} g_n^{-1}(x) |(PW_\alpha)'(x)| \tag{2.11}$$

for $|x| \geq \eta a_n$, $P \in \mathbb{P}_{n-1}$. By (2.10) and (2.11), we have

$$\frac{\int_{-\infty}^{\infty} |P(t) W_\alpha(t)|^2 dt}{|(PW_\alpha)'^2} \geq c_3 \left(\frac{a_n}{n}\right)^3 g_n^{-2}(x)$$

for $|x| \geq \eta a_n$, and for all $P \in \mathbb{P}_{n-1}$, hence

$$\inf_{P \in \mathbb{P}_{n-1}} \frac{\int_{-\infty}^{\infty} |P(t) W_\alpha(t)|^2 dt}{|(PW_\alpha)'^2} \geq c_3 \left(\frac{a_n}{n}\right)^3 g_n^{-2}(x) \quad \text{for} \quad |x| \geq \eta a_n.$$

Since

$$\inf_{P\in\mathbb{P}_{n-1}} \frac{\int_{-\infty}^{\infty} |P(t)W_\alpha(t)|^2 dt}{|(PW_\alpha)'^2} = \frac{1}{\sum_{i=0}^{n-1} \left(\frac{d}{dx} P_i(W_\alpha^2;x)W_\alpha(x)\right)^2},$$

(see, [6, Lemma 2.1, p. 24]), we have

$$\sum_{i=0}^{n-1} \left(\frac{d}{dx} P_i(W_\alpha^2;x)W_\alpha(x)\right)^2 \le c_4 \left(\frac{n}{a_n}\right)^3 g_n^2(x), \quad |x| \ge \eta a_n. \tag{2.12}$$

Now let

$$K_n(W_\alpha^2;t,x) = \sum_{i=0}^{n-1} P_i(W_\alpha^2;t)P_i(W_\alpha^2;x).$$

Then by the Christofell-Darboux formula, we have

$$K_n(W_\alpha^2;x_{n,j},x) = \frac{\gamma_{n-1}(W_\alpha^2)}{\gamma_n(W_\alpha^2)} \frac{P_{n-1}(W_\alpha^2;x_{n,j})P_n(W_\alpha^2;x)}{x - x_{n,j}},$$

which implies that

$$K_n(W_\alpha^2;x_{n,j},x_{n,j+1}) = 0. \tag{2.13}$$

By Theorem 1.1 in [9, p. 465] and Corollary 1.2 in [9, p. 466], we have

$$\sum_{i=0}^{n-1} P_i^2(W_\alpha^2;x_{n,j}) \sim \frac{n}{a_n} W_\alpha^{-2}(x_{n,j})g_n(x_{n,j}) \tag{2.14}$$

for all $n = 1, 2, \ldots,$ and for $j = 1, 2, \ldots, n$.

Now suppose that $x_{n,j} \ge x \ge x_{n,j+1} \ge \eta a_n$. Then

$$g_n(x_{n,j}) \le g_n(x) \le g_n(x_{n,j+1}). \tag{2.15}$$

By (2.12), (2.14), and (2.15), we have

$$\begin{aligned}
&\left|\frac{d}{dx} K_n(W_\alpha^2;x_{n,j},x)W_\alpha(x)\right| \\
&= \left|\sum_{i=0}^{n-1} P_i(W_\alpha^2;x_{n,j}) \frac{d}{dx}(P_i(W_\alpha^2;x)W_\alpha(x))\right| \\
&\le \left(\sum_{i=0}^{n-1} P_i^2(W_\alpha^2;x_{n,j}) \sum_{i=0}^{n-1} \left(\frac{d}{dx}(P_i(W_\alpha^2;x)W_\alpha(x))\right)^2\right)^{1/2} \qquad (2.16)\\
&\le c_5 \left(\left(\frac{n}{a_n}\right)^4 W_\alpha^{-2}(x_{n,j})g_n(x_{n,j})g_n^2(x)\right)^{1/2} \\
&\le c_5 \left(\frac{n}{a_n}\right)^2 W_\alpha^{-1}(x_{n,j})g_n^{\frac{3}{2}}(x_{n,j+1}).
\end{aligned}$$

From (2.13), (2.14), and (2.16), we have

$$\begin{aligned}
&c_6 \frac{n}{a_n} g_n(x_{n,j}) W_\alpha^{-1}(x_{n,j}) \\
&\le K_n(W_\alpha^2; x_{n,j}, x_{n,j}) W_\alpha(x_{n,j}) \\
&= K_n(W_\alpha^2; x_{n,j}, x_{n,j}) W_\alpha(x_{n,j}) - K_n(W_\alpha^2; x_{n,j}, x_{n,j+1}) W_\alpha(x_{n,j+1}) \\
&= \int_{x_{n,j+1}}^{x_{n,j}} \frac{d}{dx}(K_n(W_\alpha^2; x_{n,j}, x) W_\alpha(x)) dx \\
&\le c_7 \left(\frac{n}{a_n}\right)^2 W_\alpha^{-1}(x_{n,j}) g_n^{\frac{3}{2}}(x_{n,j+1})(x_{n,j} - x_{n,j+1}).
\end{aligned} \tag{2.17}$$

Since $g_n(x_{n,j}) \sim g_n(x_{n,j+1})$, (see, [9, (11.10), p. 521]), from (2.17), we have for $x_{n,j} > x_{n,j+1} \ge \eta a_n$,

$$x_{n,j} - x_{n,j+1} \ge c_8 \frac{a_n}{n} g_n^{-\frac{1}{2}}(x_{n,j+1}) \ge c_9 \frac{a_n}{n}. \tag{2.18}$$

The proof of (2.18) for $x_{n,j+1} < x_{n,j} \le -\eta a_n$, is similar, hence by (2.8) and (2.18), Lemma 2.3 follows. □

Proof of Theorem 1.2. Let $0 < p < \infty$ and let $n \in \mathbb{N}$. Let $\{x_{n,j}\}$ be the zeros of orthogonal polynomial $P_n(W_\alpha^2; x)$ and $\{\lambda_{n,j}\}$ be the Cotes numbers for W_α^2. By Theorem 1.1 in [9, p. 465] and Corollary 1.2 in [9, p. 466], we have for all $n \in \mathbb{N}$, and $j = 1, 2, \ldots, n$,

$$\lambda_{n,j} \le c \frac{a_n}{n} \left(\max\left\{ n^{-\frac{2}{3}}, 1 - \frac{|x_{n,j}|}{a_n} \right\} \right)^{-1/2} W_\alpha^2(x_{n,j}). \tag{2.19}$$

Let $f \in \mathrm{GANP}_{\ell n}$. Then by (2.19), we have for all $j = 1, 2, \ldots, n$,

$$\begin{aligned}
&\lambda_{n,j} f(x_{n,j}) W_\alpha^{p-2}(x_{n,j}) \quad \left(\max\left\{ n^{-\frac{2}{3}}, 1 - \frac{|x_{n,j}|}{a_n} \right\} \right)^{1/2} \\
&\le c \frac{a_n}{n} f(x_{n,j}) W_\alpha^p(x_{n,j}),
\end{aligned}$$

hence

$$\begin{aligned}
&\sum_{j=1}^n \lambda_{n,j} f(x_{n,j}) W_\alpha^{p-2}(x_{n,j}) \left(\max\left\{ n^{-\frac{2}{3}}, 1 - \frac{|x_{n,j}|}{a_n} \right\} \right)^{1/2} \\
&\le c \frac{a_n}{n} \sum_{j=1}^n f(x_{n,j}) W_\alpha^p(x_{n,j}).
\end{aligned} \tag{2.20}$$

Using Theorem 1.1 with $\Psi(x) = x$, we have

$$\sum_{j=1}^n f(x_{n,j}) W_\alpha^p(x_{n,j}) \le c_1 \left(\frac{n}{a_n} + \frac{1}{\delta} \right) \int_{-\infty}^{\infty} f(u) W_\alpha^p(u) du,$$

where

$$\delta = \min\{x_{n,j} - x_{n,j-1} \, : j = 2, 3, \ldots, n\} > 0.$$

By Lemma 2.3,

$$\frac{1}{\delta} \le c_2 \frac{n}{a_n},$$

thus

$$\sum_{j=1}^{n} f(x_{n,j}) W_\alpha^p(x_{n,j}) \le c_3 \frac{n}{a_n} \int_{-\infty}^{\infty} f(u) W_\alpha^p(u) du. \tag{2.21}$$

Then Theorem 1.2 follows from (2.20) and (2.21). □

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