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$\mathcal{N}$-IDEALS OF SUBTRACTION ALGEBRAS

Young Bae Jun, Jacob Kavikumar, and Keum Sook So

Abstract. Using $\mathcal{N}$-structures, the notion of an $\mathcal{N}$-ideal in a subtraction algebra is introduced. Characterizations of an $\mathcal{N}$-ideal are discussed. Conditions for an $\mathcal{N}$-structure to be an $\mathcal{N}$-ideal are provided. The description of a created $\mathcal{N}$-ideal is established.

1. Introduction

A (crisp) set A in a universe X can be defined in the form of its characteristic function $\mu_A : X \to \{0, 1\}$ yielding the value 1 for elements belonging to the set A and the value 0 for elements excluded from the set A. So far most of the generalization of the crisp set have been conducted on the unit interval $[0, 1]$ and they are consistent with the asymmetry observation. In other words, the generalization of the crisp set to fuzzy sets relied on spreading positive information that fit the crisp point $\{1\}$ into the interval $[0, 1]$. Because no negative meaning of information is suggested, we now feel a need to deal with negative information. To do so, we also feel a need to supply mathematical tool. To attain such object, Jun et al. [4] introduced a new function which is called negative-valued function, and constructed $\mathcal{N}$-structures. They discussed $\mathcal{N}$-subalgebras and $\mathcal{N}$-ideals in BCK/BCI-algebras. Schein [6] considered systems of the form $(\Phi; \circ, \setminus)$, where Φ is a set of functions closed under the composition "$\circ$" of functions (and hence $(\Phi; \circ)$ is a function semigroup) and the set theoretic subtraction "$\setminus$" (and hence $(\Phi; \setminus)$ is a subtraction algebra in the sense of [1]). He proved that every subtraction semigroup is isomorphic to a difference semigroup of invertible functions. Zelinka [7] discussed a problem proposed by Schein concerning the structure of multiplication in a subtraction semigroup. He solved the problem for subtraction algebras of a special type, called the atomic subtraction algebras. Jun et al. [2, 3] introduced the notion of ideals in subtraction algebras and discussed characterization of ideals. Jun et al. [5] provided conditions for an ideal to be irreducible. They introduced the notion of an order system in a subtraction algebra, and investigated related properties. They provided relations between ideals and order systems, and dealt with

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the concept of a fixed map in a subtraction algebra, and investigate related properties.

In this paper, we introduced the notion of a (created) $\mathcal{N}$-ideal of subtraction algebras, and investigate several characterizations of $\mathcal{N}$-ideals. We discuss how to make a created $\mathcal{N}$-ideal of an $\mathcal{N}$-structure (X, f).

2. Preliminaries

By a *subtraction algebra* we mean an algebra $(X; -)$ with a single binary operation "$-$" that satisfies the following identities: for any $x, y, z \in X$,

(S1) $x - (y - x) = x$;
(S2) $x - (x - y) = y - (y - x)$;
(S3) $(x - y) - z = (x - z) - y$.

The last identity permits us to omit parentheses in expressions of the form $(x - y) - z$. The subtraction determines an order relation on X: $a \leq b \Leftrightarrow a - b = 0$, where $0 = a - a$ is an element that does not depend on the choice of $a \in X$. The ordered set $(X; \leq)$ is a semi-Boolean algebra in the sense of [1], that is, it is a meet semilattice with zero 0 in which every interval $[0, a]$ is a Boolean algebra with respect to the induced order. Here $a \wedge b = a - (a - b)$; the complement of an element $b \in [0, a]$ is $a - b$; and if $b, c \in [0, a]$, then

$$\begin{aligned} b \vee c = (b' \wedge c')' &= a - ((a - b) \wedge (a - c)) \\ &= a - ((a - b) - ((a - b) - (a - c))). \end{aligned}$$

In a subtraction algebra, the following are true (see [3]):

(a1) $(x - y) - y = x - y$.
(a2) $x - 0 = x$ and $0 - x = 0$.
(a3) $(x - y) - x = 0$.
(a4) $x - (x - y) \leq y$.
(a5) $(x - y) - (y - x) = x - y$.
(a6) $x - (x - (x - y)) = x - y$.
(a7) $(x - y) - (z - y) \leq x - z$.
(a8) $x \leq y$ if and only if $x = y - w$ for some $w \in X$.
(a9) $x \leq y$ implies $x - z \leq y - z$ and $z - y \leq z - x$ for all $z \in X$.
(a10) $x, y \leq z$ implies $x - y = x \wedge (z - y)$.
(a11) $(x \wedge y) - (x \wedge z) \leq x \wedge (y - z)$.

Definition 2.1 ([3])**.** A nonempty subset A of a subtraction algebra X is called an *ideal* of X if it satisfies:

(b1) $a - x \in A$ for all $a \in A$ and $x \in X$.
(b2) for all $a, b \in A$, whenever $a \vee b$ exists in X then $a \vee b \in A$.

Proposition 2.2 ([3])**.** *A nonempty subset A of a subtraction algebra X is an ideal of X if and only if it satisfies*:

(b3) $0 \in A$,
(b4) $(\forall x \in X)(\forall y \in A)(x - y \in A \Rightarrow x \in A)$.

TABLE 1. Cayley table

$-$	0	a	b
0	0	0	0
a	a	0	a
b	b	b	0

Proposition 2.3 ([3]). *Let X be a subtraction algebra and let $x, y \in X$. If $w \in X$ is an upper bound for x and y, then the element*

$$x \vee y := w - ((w - y) - x)$$

is a least upper bound for x and y.

3. $\mathcal{N}$-ideals of subtraction algebras

Denote by $\mathcal{F}(X, [-1, 0])$ the collection of functions from a set X to $[-1, 0]$. We say that an element of $\mathcal{F}(X, [-1, 0])$ is a *negative-valued function* from X to $[-1, 0]$ (briefly, *$\mathcal{N}$-function* on X). By an *$\mathcal{N}$-structure* we mean an ordered pair (X, f) of X and an $\mathcal{N}$-function f on X. In what follows, let X denote a subtraction algebra and f an $\mathcal{N}$-function on X unless otherwise specified.

For any $\mathcal{N}$-function f on X and $t \in [-1, 0)$, the set

$$C(f; t) := \{x \in X \mid f(x) \leq t\}$$

is called a *closed (f, t)-cut* of (X, f).

Definition 3.1. By an *ideal* (resp. *subalgebra*) of X based on $\mathcal{N}$-function f (briefly, *$\mathcal{N}$-ideal* (resp. *$\mathcal{N}$-subalgebra*) of X), we mean an $\mathcal{N}$-structure (X, f) in which every nonempty closed (f, t)-cut of (X, f) is an ideal (resp. subalgebra) of X for all $t \in [-1, 0)$.

Example 3.2. Let $X = \{0, a, b\}$ be a subtraction algebra with the Cayley table which is given in Table 1 (see [5]). Let (X, f) be an $\mathcal{N}$-structure in which f is given by

$$f = \begin{pmatrix} 0 & a & b \\ -0.7 & -0.3 & -0.5 \end{pmatrix}.$$

It is easy to check that (X, f) is an $\mathcal{N}$-ideal of X.

Theorem 3.3. *An $\mathcal{N}$-structure (X, f) is an $\mathcal{N}$-ideal of X if and only if it satisfies the following assertions*:

(1) $(\forall x, y \in X)$ $(f(x - y) \leq f(x))$,
(2) $(\forall x, y \in X)$ $(\exists x \vee y \Rightarrow f(x \vee y) \leq \max\{f(x), f(y)\})$.

Proof. Assume that an $\mathcal{N}$-structure (X, f) satisfies two conditions (1) and (2). Let $t \in [-1, 0)$ be such that $C(f; t) \neq \emptyset$. Let $x \in X$ and $a \in C(f; t)$. Then

$f(a) \leq t$, and so $f(a-x) \leq f(a) \leq t$ by (1), i.e., $a - x \in C(f;t)$. Assume that $a \vee b$ exists in X for all $a, b \in C(f;t)$. Then

$$f(a \vee b) \leq \max\{f(a), f(b)\} \leq t$$

by (2), and so $a \vee b \in C(f;t)$. Therefore $C(f;t)$ is an ideal of X, that is, (X, f) is an $\mathcal{N}$-ideal of X.

Conversely, suppose that (X, f) is an $\mathcal{N}$-ideal of X, that is, every nonempty closed (f,t)-cut of (X, f) is an ideal of X for all $t \in [-1, 0)$. If there are $a, b \in X$ such that $f(a-b) > f(a)$, then $f(a-b) > t_a \geq f(a)$ for some $t_a \in [-1, 0)$. Thus $a \in C(f;t_a)$, but $a-b \notin C(f;t_a)$. This is a contradiction, and so $f(x-y) \leq f(x)$ for all $x, y \in X$. Assume that there exist $a, b \in X$ such that $a \vee b$ exists and $f(a \vee b) > \max\{f(a), f(b)\}$. Then $f(a \vee b) > t_0 \geq \max\{f(a), f(b)\}$ for some $t_0 \in [-1, 0)$. It follows that $a, b \in C(f;t_0)$ and $a \vee b \notin C(f;t_0)$ which is a contradiction. Therefore (2) is valid. □

Corollary 3.4. *Every $\mathcal{N}$-ideal (X, f) satisfies the following inequality:*

$$(\forall x \in X)\ (f(0) \leq f(x)). \tag{3.1}$$

Proof. Straightforward. □

Theorem 3.5. *For a fixed element $w \in X$, let (X, f_w) be an $\mathcal{N}$-structure in which f_w is give by*

$$f_w(x) = \begin{cases} t_1 & \text{if } x - w = 0, \\ t_2 & \text{otherwise} \end{cases}$$

for all $x \in X$ and $t_1, t_2 \in [-1, 0)$ with $t_1 < t_2$. Then (X, f_w) is an $\mathcal{N}$-ideal of X.

Proof. Let $x, y \in X$. If $x - w \neq 0$, then $f_w(x) = t_2 \geq f_w(x-y)$. If $x - w = 0$, then $x - y \leq x \leq w$, i.e., $(x-y) - w = 0$. Thus $f_w(x-y) = t_1 = f_w(x)$. Now if $x - w \neq 0$ or $y - w \neq 0$, then $f_w(x) = t_2$ or $f_w(y) = t_2$. Hence

$$f_w(x \vee y) \leq t_2 = \max\{f_w(x), f_w(y)\}$$

whenever $x \vee y$ exists in X. Assume that $x - w = 0$ and $y - w = 0$. Then w is an upper bound for x and y. It follows from Proposition 2.3 that $x \vee y$ exists and $x \vee y = w - ((w-y) - x) \leq w$, i.e., $x \vee y - w = 0$. Therefore

$$f_w(x \vee y) = t_1 = \max\{f_w(x), f_w(y)\}.$$

Using Theorem 3.3, we conclude that (X, f_w) is an $\mathcal{N}$-ideal of X. □

Theorem 3.6. *An $\mathcal{N}$-structure (X, f) is an $\mathcal{N}$-ideal of X if and only if it satisfies:*

$$(\forall x, a, b \in X)\ (f(x - ((x-a) - b)) \leq \max\{f(a), f(b)\}). \tag{3.2}$$

Proof. Let (X, f) be an $\mathcal{N}$-structure satisfying (3.2). Using (a2) and (S3), we have

$$x - y = (x - y) - (((x - y) - x) - x)$$

for all $x, y \in X$. It follows from (3.2) that

$$f(x - y) = f((x - y) - (((x - y) - x) - x)) \leq \max\{f(x), f(x)\} = f(x)$$

for all $x, y \in X$. Suppose $x \vee y$ exists for $x, y \in X$. Putting $a := x \vee y$, we have $x \vee y = a - ((a - y) - x) = a - ((a - x) - y)$ by Proposition 2.3 and (S3). Using (3.2) implies that

$$f(x \vee y) = f(a - ((a - x) - y)) \leq \max\{f(x), f(y)\}$$

for all $x, y \in X$. Therefore (X, f) is an $\mathcal{N}$-ideal of X by Theorem 3.3.

Conversely, suppose that (X, f) is an $\mathcal{N}$-ideal of X. Then the nonempty closed (f, t)-cut of (X, f) is an ideal of X for all $t \in [-1, 0)$. Let $\theta_{C(f;t)}$ be a relation on X defined by

$$(\forall x, y \in X)\ \big((x, y) \in \theta_{C(f;t)}\ \Leftrightarrow\ x - y \in C(f; t),\ y - x \in C(f; t)\big).$$

Then $\theta_{C(f;t)}$ is a congruence relation on X. For any $a, b \in C(f; t)$ and $x \in X$, we have $(x, x) \in \theta_{C(f;t)}$, $(a, 0) \in \theta_{C(f;t)}$ and $(b, 0) \in \theta_{C(f;t)}$. Hence

$$(x - ((x - a) - b), 0) = (x - ((x - a) - b), x - ((x - 0) - 0)) \in \theta_{C(f;t)},$$

and so $x - ((x - a) - b) \in C(f; t)$. It follows that

$$f(x - ((x - a) - b)) \leq \max\{f(a), f(b)\}$$

for all $a, b, x \in X$ because if there exist $a_0, b_0 \in X$ such that

$$f(x - ((x - a_0) - b_0)) > \max\{f(a_0), f(b_0)\},$$

then $f(x - ((x - a_0) - b_0)) > t_0 \geq \max\{f(a_0), f(b_0)\}$ for some $t_0 \in [-1, 0)$. Thus $a_0 \in C(f; t_0)$ and $b_0 \in C(f; t_0)$, but $x - ((x - a_0) - b_0) \notin C(f; t_0)$. This is a contradiction. □

Theorem 3.7. *An $\mathcal{N}$-structure (X, f) is an $\mathcal{N}$-ideal of X if and only if it satisfies the condition* (3.1) *and*

(3.3) $\qquad (\forall x, y, z \in X)\ (f(x - z) \leq \max\{f((x - y) - z), f(y)\}).$

Proof. Assume that (X, f) is an $\mathcal{N}$-ideal of X. Then the condition (3.1) is valid by Corollary 3.4. If we put $x = x - z$ in (a3), then $((x - z) - y) - (x - z) = 0$, i.e., $(x - z) - y \leq x - z$. If we put $x = y$ and $y = x - z$ in (a4), then $y - (y - (x - z)) \leq x - z$. Hence $x - z$ is an upper bound for $(x - z) - y$ and

$y-(y-(x-z))$. It follows from Proposition 2.3, (S2), (S3) and (a2) that

$$\begin{aligned}((x-y)-z)\vee y &= ((x-y)-z)\vee(y-0)\\ &= ((x-y)-z)\vee(y-(y-y))\\ &= ((x-y)-z)\vee(y-(y-(x-z)))\\ &= ((x-z)-y)\vee(y-(y-(x-z)))\\ &= (x-z)-(((x-z)-(y-(y-(x-z))))-((x-z)-y))\\ &= (x-z)-(((x-z)-((x-z)-y))-(y-(y-(x-z))))\\ &= (x-z)-((y-(y-(x-z)))-(y-(y-(x-z))))\\ &= (x-z)-0 \;=\; x-z\end{aligned}$$

so from Theorem 3.3(2) that

$$f(x-z)=f(((x-y)-z)\vee y)\le \max\{f((x-y)-z),f(y)\}$$

for all $x,y\in X$.

Conversely, let (X,f) be an $\mathcal{N}$-structure satisfying two conditions (3.1) and (3.3). Let $t\in[-1,0)$ be such that $C(f;t)\neq\emptyset$. Obviously, $0\in C(f;t)$ by the condition (3.1). Let $x\in X$ and $a\in C(f;t)$ be such that $x-a\in C(f;t)$. Then $f(a)\le t$ and $f(x-a)\le t$. It follows from (3.3) and (a2) that

$$\begin{aligned}f(x)=f(x-0)&\le \max\{f((x-a)-0),f(a)\}\\ &=\max\{f(x-a),f(a)\}\;\le\; t\end{aligned}$$

so that $x\in C(f;t)$. Hence $C(f;t)$ is an ideal of X for all $t\in[-1,0)$ by Proposition 2.2, and so (X,f) is an $\mathcal{N}$-ideal of X. □

Corollary 3.8. *Every $\mathcal{N}$-ideal (X,f) satisfies*:

$$(\forall x,y\in X)\ (x\le y\implies f(x)\le f(y)).$$

Proof. Let $x,y\in X$ be such that $x\le y$. Then $x-y=0$, and so

$$f(x)=f(x-0)\le\max\{f((x-y)-0),f(y)\}=\max\{f(0),f(y)\}=f(y)$$

by using (a2), (3.1) and (3.3). This completes the proof. □

Theorem 3.9. *An $\mathcal{N}$-structure (X,f) is an $\mathcal{N}$-ideal of X if and only if it satisfies the condition* (3.1) *and*

$$(\forall x,y\in X)\ (f(x)\le\max\{f(x-y),f(y)\}). \tag{3.4}$$

Proof. Assume that (X,f) is an $\mathcal{N}$-ideal of X. Then the condition (3.1) is valid by Corollary 3.4, and the condition (3.4) is by taking $z=0$ in (3.3) and using (a2).

Conversely, let (X,f) be an $\mathcal{N}$-structure satisfying two conditions (3.1) and (3.4). Since

$$(x-((x-a)-b))-b=(x-b)-((x-a)-b)\le x-(x-a)\le a,$$

that is, $((x-((x-a)-b))-b)-a=0$ for all $x,a,b\in X$, it follows from (3.1) and (3.4) that

$$\begin{aligned} & f(x-((x-a)-b)) \\ \le\ & \max\{f((x-((x-a)-b))-b), f(b)\} \\ \le\ & \max\{\max\{f(((x-((x-a)-b))-b)-a), f(a)\}, f(b)\} \\ =\ & \max\{\max\{f(0), f(a)\}, f(b)\} \\ =\ & \max\{f(a), f(b)\} \end{aligned}$$

for all $x,a,b\in X$. Therefore (X,f) is an $\mathcal{N}$-ideal of X by Theorem 3.6. □

Theorem 3.10. *For fixed elements $a,b\in X$, let (X,f_a^b) be an $\mathcal{N}$-structure in which f_a^b is give by*

$$f_a^b(x)=\begin{cases} t_1 & \text{if } (x-a)-b=0, \\ t_2 & \text{otherwise} \end{cases}$$

for all $x\in X$ and $t_1,t_2\in[-1,0)$ with $t_1<t_2$. Then (X,f_a^b) is an $\mathcal{N}$-ideal of X.

Proof. Since $(0-a)-b=0$, we have $f_a^b(0)=t_1\le f_a^b(x)$ for all $x\in X$. Let $x,y\in X$. If $(x-a)-b=0$, then $f_a^b(x)=t_1\le\max\{f_a^b(x-y),f_a^b(y)\}$. Suppose that $(x-a)-b\ne 0$. If $(y-a)-b=0$ and $((x-y)-a)-b=0$, then

$$\begin{aligned} (x-a)-b &= ((x-a)-b)-0 \\ &= ((x-a)-b)-((y-a)-b) \\ &= ((x-a)-(y-a))-b \\ &= ((x-y)-a)-b=0, \end{aligned}$$

a contradiction. Hence $(y-a)-b\ne 0$ or $((x-y)-a)-b\ne 0$, and thus $f_a^b(y)=t_2$ or $f_a^b(x-y)=t_2$. It follows that

$$f_a^b(x)=t_2=\max\{f_a^b(x-y),f_a^b(y)\}.$$

Hence, by Theorem 3.9, (X,f_a^b) is an $\mathcal{N}$-ideal of X. □

Theorem 3.11. *For an ideal A of X and a fixed element $w\in X$, let (X,f_A^w) be an $\mathcal{N}$-structure in which f_A^w is give by*

$$f_A^w(x)=\begin{cases} t_1 & \text{if } x-w\in A, \\ t_2 & \text{otherwise} \end{cases}$$

for all $x\in X$ and $t_1,t_2\in[-1,0)$ with $t_1<t_2$. Then (X,f_A^w) is an $\mathcal{N}$-ideal of X.

Proof. Since $0-w=0\in A$, we get $f_A^w(0)=t_1\le f_A^w(x)$ for all $x\in X$. Let $x,y\in X$. If $x-w\in A$, then $f_A^w(x)=t_1\le\max\{f_A^w(x-y),f_A^w(y)\}$. Suppose that $x-w\notin A$. If $y-w\in A$ and $(x-y)-w\in A$, then

$$(x-w)-(y-w)=(x-y)-w\in A.$$

Since A is an ideal and $y - w \in A$, it follows from (b4) that $x - w \in A$ which is a contradiction. Therefore $y - w \notin A$ or $(x - y) - w \notin A$, and so $f_A^w(y) = t_2$ or $f_A^w(x - y) = t_2$. Thus

$$f_A^w(x) = t_2 = \max\{f_A^w(x - y), f_A^w(y)\}.$$

Using Theorem 3.9, we know that (X, f_A^w) is an $\mathcal{N}$-ideal of X. □

Theorem 3.12. *An $\mathcal{N}$-structure (X, f) is an $\mathcal{N}$-ideal of X if and only if it satisfies*:

$$(3.5) \qquad (\forall a, b, x \in X)\ (x - a \leq b \implies f(x) \leq \max\{f(a), f(b)\}).$$

Proof. Assume that (X, f) is an $\mathcal{N}$-ideal of X. Let $a, b, x \in X$ be such that $x - a \leq b$. Then $(x - a) - b = 0$, and so

$$\begin{aligned} f(x) &\leq \max\{f(x - a), f(a)\} \\ &\leq \max\{\max\{f((x - a) - b), f(b)\}, f(a)\} \\ &= \max\{\max\{f(0), f(b)\}, f(a)\} \\ &= \max\{f(a), f(b)\} \end{aligned}$$

by (3.4) and (3.1).

Conversely, let (X, f) be an $\mathcal{N}$-structure satisfying the condition (3.5). Since $0 - x \leq x$ for all $x \in X$, it follows from (3.5) that

$$f(0) \leq \max\{f(x), f(x)\} = f(x)$$

for all $x \in X$. Note that $x - (x - y) \leq y$ for all $x, y \in X$. Using (3.5), we have $f(x) \leq \max\{f(x - y), f(y)\}$ for all $x, y \in X$. Hence (X, f) is an $\mathcal{N}$-ideal of X by Theorem 3.9. □

Theorem 3.13. *An $\mathcal{N}$-structure (X, f) is an $\mathcal{N}$-ideal (X, f) of X if and only if it satisfies*:

$$(3.6) \qquad f(x) \leq \max\{f(a_i) \mid i = 1, 2, \ldots, n\}$$

for all $x, a_1, a_2, \ldots, a_n \in X$ with $(\cdots((x - a_1) - a_2) - \cdots) - a_n = 0$.

Proof. Assume that (X, f) is an $\mathcal{N}$-ideal of X. If $x - a = 0$ for any $x, a \in X$, then $f(x) \leq f(a)$ by Corollary 3.8. Let $a, b, x \in X$ be such that $(x - a) - b = 0$. Then $f(x) \leq \max\{f(a), f(b)\}$ by Theorem 3.12. Now let $x, a_1, a_2, \ldots, a_n \in X$ be such that

$$(\cdots((x - a_1) - a_2) - \cdots) - a_n = 0.$$

By induction on n, we conclude that $f(x) \leq \max\{f(a_i) \mid i = 1, 2, \ldots, n\}$.

Conversely, let (X, f) be an $\mathcal{N}$-structure in which (3.6) is valid for all $x, a_1, a_2, \ldots, a_n \in X$ with $(\cdots((x - a_1) - a_2) - \cdots) - a_n = 0$. Then

$$(3.7) \qquad f(x) \leq \max\{f(y), f(z)\}$$

for all $x, y, z \in X$ with $(x - y) - z = 0$. Since $(0 - x) - x = 0$ for all $x \in X$, it follows from (3.7) that $f(0) \leq \max\{f(x), f(x)\} = f(x)$. Note that $(x - (x -$

$y)) - y = 0$ for all $x, y \in X$. Using (3.7), we have $f(x) \leq \max\{f(x-y), f(y)\}$ for all $x, y \in X$. Therefore (X, f) is an $\mathcal{N}$-ideal of X by Theorem 3.9. □

Proposition 3.14. *In an $\mathcal{N}$-ideal (X, f) of X, the following assertions are equivalent*:

(1) $(\forall x, y \in X)\ (f(x-y) \leq f((x-y)-y))$.
(2) $(\forall x, y, z \in X)\ (f((x-z)-(y-z)) \leq f((x-y)-z))$.

Proof. Assume that (1) is valid and let $x, y, z \in X$. Since

$$((x-(y-z))-z)-z = ((x-z)-(y-z))-z \leq (x-y)-z,$$

it follows from Corollary 3.8 that $f(((x-(y-z))-z)-z) \leq f((x-y)-z)$ so from (S3) and (1) that

$$\begin{aligned} f((x-z)-(y-z)) &= f((x-(y-z))-z) \\ &\leq f(((x-(y-z))-z)-z) \\ &\leq f((x-y)-z). \end{aligned}$$

Conversely, suppose that (2) is valid. If we use z instead of y in (2), then

$$f(x-z) = f((x-z)-0) = f((x-z)-(z-z)) \leq f((x-z)-z)$$

for all $x, z \in X$ by using (a2). This proves (1). □

For any element w of X, we consider the set

$$X_w := \{x \in X \mid f(x) \leq f(w)\}.$$

Obviously, $w \in X_w$, and so X_w is a non-empty subset of X.

Theorem 3.15. *Let w be an element of X. If (X, f) is an $\mathcal{N}$-ideal of X, then the set X_w is an ideal of X.*

Proof. Obviously, $0 \in X_w$ by (3.1). Let $x, y \in X$ be such that $x - y \in X_w$ and $y \in X_w$. Then $f(x-y) \leq f(w)$ and $f(y) \leq f(w)$. Since (X, f) is an $\mathcal{N}$-ideal of X, it follows from (3.4) that

$$f(x) \leq \max\{f(x-y), f(y)\} \leq f(w)$$

so that $x \in X_w$. Hence X_w is an ideal of X. □

Theorem 3.16. *Let w be an element of X and let (X, f) be an $\mathcal{N}$-structure of X and f. Then*

(1) *If X_w is an ideal of X, then (X, f) satisfies the following assertion*:

(3.8) $(\forall x, y, z \in X)(f(x) \geq \max\{f(y-z), f(z)\} \Rightarrow f(x) \geq f(y))$.

(2) *If (X, f) satisfies* (3.1) *and* (3.8)*, then X_w is an ideal of X.*

Proof. (1) Assume that X_w is an ideal of X for each $w \in X$. Let $x, y, z \in X$ be such that $f(x) \geq \max\{f(y-z), f(z)\}$. Then $y - z \in X_x$ and $z \in X_x$. Since X_x is an ideal of X, it follows that $y \in X_x$, that is, $f(y) \leq f(x)$.

(2) Suppose that (X, f) satisfies (3.1) and (3.8). For each $w \in X$, let $x, y \in X$ be such that $x - y \in X_w$ and $y \in X_w$. Then $f(x-y) \leq f(w)$ and $f(y) \leq f(w)$, which imply that $\max\{f(x-y), f(y)\} \leq f(w)$. Using (3.8), we have $f(w) \geq f(x)$ and so $x \in X_w$. Obviously $0 \in X_w$. Therefore X_w is an ideal of X. □

Let (X, f) and (X, g) be two $\mathcal{N}$-structures. We say that (X, f) is a *retrenchment* of (X, g) (see [4]) if $f(x) \leq g(x)$ for all $x \in X$.

Definition 3.17. Let (X, f) be an $\mathcal{N}$-structure. An $\mathcal{N}$-structure (X, g) is called a *created $\mathcal{N}$-ideal* of (X, f) if it satisfies:

(i) (X, g) is an $\mathcal{N}$-ideal of X.
(ii) (X, g) is a retrenchment of (X, f).
(iii) For any $\mathcal{N}$-ideal (X, h) of X, if (X, h) is a retrenchment of (X, f), then (X, h) is a retrenchment of (X, g).

The created $\mathcal{N}$-ideal of (X, f) will be denoted by $(X, [f])$. Note that the created $\mathcal{N}$-ideal of (X, f) is the greatest $\mathcal{N}$-ideal in X which is a retrenchment of (X, f). We discuss how to make a created $\mathcal{N}$-ideal of an $\mathcal{N}$-structure (X, f).

Theorem 3.18. *For any $\mathcal{N}$-structure (X, f), the created $\mathcal{N}$-ideal $(X, [f])$ of (X, f) is described as follows:*

$$[f](x) = \inf\left\{ \max\{f(a_i) \mid i = 1, 2, \ldots n\} \;\middle|\; \begin{array}{c} (\cdots((x - a_1) - a_2) - \\ \cdots) - a_n = 0 \end{array} \right\}.$$

Proof. Let (X, g) be an $\mathcal{N}$-structure in which g is defined by

$$g(x) = \inf\left\{ \max\{f(a_i) \mid i = 1, 2, \ldots n\} \;\middle|\; \begin{array}{c} (\cdots((x - a_1) - a_2) - \\ \cdots) - a_n = 0 \end{array} \right\}.$$

Let $x, a, b \in X$ be such that

$$(x - a) - b = 0. \tag{3.9}$$

For any $\varepsilon > 0$, there exist $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_m \in X$ such that

$$\begin{array}{c} (\cdots((a - a_1) - a_2) - \cdots) - a_n = 0, \\ (\cdots((b - b_1) - b_2) - \cdots) - b_m = 0 \end{array} \tag{3.10}$$

and

$$\begin{array}{c} g(a) > \max\{f(a_i) \mid i = 1, 2, \ldots, n\} - \varepsilon, \\ g(b) > \max\{f(b_j) \mid j = 1, 2, \ldots, m\} - \varepsilon. \end{array} \tag{3.11}$$

Using (3.9) and (3.10), we have

$$(\cdots((((\cdots((x - a_1) - a_2) - \cdots) - a_n) - b_1) - b_2) - \cdots) - b_m = 0.$$

Applying the definition of g and using (3.11), we have

$$\begin{aligned} g(x) &\leq \max\{f(a_1), f(a_2), \ldots, f(a_n), f(b_1), f(b_2), \ldots, f(b_m)\} \\ &< \max\{g(a) + \varepsilon, g(b) + \varepsilon\} \\ &= \max\{g(a), g(b)\} + \varepsilon. \end{aligned}$$

Since ε is arbitrary, it follows that $g(x) \leq \max\{g(a), g(b)\}$ so from Theorem 3.12 that (X, g) is an $\mathcal{N}$-ideal in X. Now since $x - x = 0$ for all $x \in X$, we obtain $g(x) \leq f(x)$ for all $x \in X$, and so (X, g) is a retrenchment of (X, f). Let (X, h) be an $\mathcal{N}$-ideal in X which is a retrenchment of (X, f). For any $x \in X$, we have

$$\begin{aligned} g(x) &= \inf\left\{ \max\{f(a_i) \mid i = 1, 2, \cdots n\} \;\middle|\; \begin{array}{c} (\cdots((x - a_1) - a_2) - \\ \cdots) - a_n = 0 \end{array} \right\} \\ &\geq \inf\left\{ \max\{h(a_i) \mid i = 1, 2, \cdots n\} \;\middle|\; \begin{array}{c} (\cdots((x - a_1) - a_2) - \\ \cdots) - a_n = 0 \end{array} \right\} \\ &\geq \inf\{h(x)\} = h(x), \end{aligned}$$

and so (X, h) is a retrenchment of (X, g). Therefore (X, g) is a created $\mathcal{N}$-ideal of (X, f). Since $(X, [f])$ is greatest, we have $g = [f]$. This completes the proof. □

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Young Bae Jun
Department of Mathematics Education (and RINS)
Gyeongsang National University
Chinju 660-701, Korea
E-mail address: skywine@gmail.com

Jacob Kavikumar
Centre for Science Studies
Universiti Tun Hussein Onn Malaysia
86400 Batu Pahat, Johor, Malaysia
E-mail address: kaviphd@gmail.com

Keum Sook So
Department of Mathematics
Hallym University
Chuncheon 200-702, Korea
E-mail address: ksso@hallym.ac.kr