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LOCAL COHOMOLOGY MODULES WHICH ARE SUPPORTED ONLY AT FINITELY MANY MAXIMAL IDEALS

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Abstract. Let $\mathfrak{a}$ be an ideal of a commutative Noetherian ring R, M a finitely generated R-module and N a weakly Laskerian R-module. We show that if N has finite dimension d, then $\mathrm{Ass}_R(H^d_{\mathfrak{a}}(N))$ consists of finitely many maximal ideals of R. Also, we find the least integer i, such that $H^i_{\mathfrak{a}}(M,N)$ is not consisting of finitely many maximal ideals of R.

1. Introduction

Throughout this paper, R is a commutative Noetherian ring and $\mathfrak{a}$ an ideal of R. This paper concerns the set of associated primes of local cohomology modules. For basic results in the theory of local cohomology modules, we refer the reader to the standard text book [5]. Huneke [11] has conjectured that when M is a finitely generated R-module, then each local cohomology module $H^i_{\mathfrak{a}}(M)$ has only finitely many associated primes. This conjecture has been shown to be true in many cases. But it dose not hold in general. See [15] for a counter-example in the general case and [12] for such an example in the local case.

Let M be an R-module. In this paper, we investigate the question when $\mathrm{Ass}_R(H^i_{\mathfrak{a}}(M))$ consists of finitely many maximal ideals. To have the most generality, we will present our results for generalized local cohomology modules, the notion which has been introduced by Herzog [10]. Recall that for two R-modules M and N, the ith generalized local cohomology module of M and N with respect to $\mathfrak{a}$ is defined by

$$H^i_{\mathfrak{a}}(M,N) := \varinjlim_n \mathrm{Ext}^i_R(M/\mathfrak{a}^n M, N).$$

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Let M and N be two finitely generated R-modules over a local ring $(R,\mathfrak{m})$. In [7, Theorem 3.1] and [6, Theorem 2.2], it is proved that

$$\inf\{i : H^i_{\mathfrak{a}}(M,N) \text{ is not Artinian}\} = f - \operatorname{depth}(\mathfrak{a} + \operatorname{Ann}_R M, N).$$

Here, over a not necessarily local ring R for a finitely generated R-module M and a weakly Laskerian R-module N, we prove that the infimum of the integers i such that $\operatorname{Ass}_R(H^i_{\mathfrak{a}}(M,N))$ does not consist of finitely many maximal ideals is equal to the infimum of the integers i such that $\operatorname{Ass}_R(H^i_{\mathfrak{a}+\operatorname{Ann}_R M}(N))$ does not consist of finitely many maximal ideals. Several other description for this infimum can also be deduced from Theorem 2.7 below, which is our first main result in Section 2. Also, as our second main result in Section 2, for a weakly Laskerian R-module N of finite dimension d , we show that the set $\operatorname{Ass}_R(H^d_{\mathfrak{a}}(N)$ consists of finitely many maximal ideals.

In Section 3, we introduce the notion of weakly filter-regular sequences over a local ring $(R,\mathfrak{m})$. For a finitely generated R-module M and a weakly Laskerian R-module N, we show that the length of any maximal weakly filter N-regular sequence in $\mathfrak{a} + \operatorname{Ann}_R M$ is equal to the infimum of the integers i such that $\operatorname{Ass}_R(H^i_{\mathfrak{a}}(M,N))$ does not consist of finitely many maximal ideals.

2. Weakly Artinian modules

We start this section by introducing the notion of weakly Artinian modules.

Definition 2.1. An R-module M is said to be *weakly Artinian* if $E_R(M)$, its injective envelope, can be written as $E_R(M) := \oplus_{i=1}^n \mu^0(\mathfrak{m}_i, M)E_R(R/m_i)$, where $\mathfrak{m}_1,\dots,\mathfrak{m}_n$ are maximal ideals of R.

Example 2.2. i) Any Artinian R-module is weakly Artinian.

ii) If R is semilocal, then any R-module which is supported in $\operatorname{Max} R$ is weakly Artinian.

iii) Let $\mathfrak{m} \in \operatorname{Max} R$. Then the R-module $M := \oplus_{i\in\mathbb{N}} R/m$ is weakly Artinian, but it is not Artinian.

In [9, Definition 2.1], an R-module M is defined to be weakly Laskerian if the set of associated prime ideals of any quotient of M is finite. In part b) of the following result, we determine the relation between the two notions weakly Laskerian and weakly Artinian modules.

Part a) of the following lemma indicates that weakly Artinian modules are very close to Artinian ones.

Lemma 2.3. *Let M be an R-module.*

a) *M is Artinian if and only if M is weakly Artinian and $\mu^0(\mathfrak{m}, M) < \infty$ for all $\mathfrak{m} \in \operatorname{Ass}_R M$.*

b) *The following are equivalent*:

i) *M is weakly Artinian.*

ii) *$\operatorname{Ass}_R M$ consists of finitely many maximal ideals.*

iii) *$\operatorname{Supp}_R M$ consists of finitely many maximal ideals.*

iv) $\operatorname{Ass}_R M = \operatorname{Supp}_R M$ *and it consists of finitely many maximal ideals.*

v) M *is weakly Laskerian and* $\operatorname{Ass}_R M \subseteq \operatorname{Max} R$.

c) *The classes of weakly Artinian R-modules and weakly Laskerian R-modules are Serre subcategories. In particular, let M be a finitely generated R-module and N a weakly Artinian (resp. weakly Laskerian) R-module. Then for any integer i, the modules* $\operatorname{Tor}_i^R(M,N)$ *and* $\operatorname{Ext}_R^i(M,N)$ *are weakly Artinian (resp. weakly Laskerian).*

Proof. The proof in the parts a) and b) are obvious and we leave it to the reader.

c) The claim for weakly Laskerian modules is proved in [9, Lemma 2.3]. The prove of weakly Artinian case is easy. One only need to note that for any exact sequence $0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0$, $\operatorname{Supp}_R Y = \operatorname{Supp}_R X \cup \operatorname{Supp}_R Z$ and that for each i, both modules $\operatorname{Tor}_i^R(M,N)$ and $\operatorname{Ext}_R^i(M,N)$ are supported in $\operatorname{Supp}_R N$. $\square$

Theorem 2.4. *Let $\mathfrak{a}$ be an ideal of R, M a finitely generated R-module and N a weakly Laskerian R-module. Let t be a non-negative integer such that $H^i_{\mathfrak{a}}(M,N)$ is weakly Laskerian for all $i < t$. Then* $\operatorname{Hom}_R(R/\mathfrak{a}, H^t_{\mathfrak{a}}(M,N))$ *is weakly Laskerian.*

Proof. We adopt the method of the proof in [2, Theorem 1.2]. So, we use induction on t. Since $H^0_{\mathfrak{a}}(M,N) \cong \operatorname{Hom}_R(M, H^0_{\mathfrak{a}}(N))$, the assertion holds for $t = 0$. Hence inductively, we assume that $t \geq 1$ and that the claim has been proved for $t-1$. From the exact sequence

$$0 \longrightarrow \Gamma_{\mathfrak{a}}(N) \longrightarrow N \longrightarrow N/\Gamma_{\mathfrak{a}}(N) \longrightarrow 0,$$

we deduce the exact sequence

$$\operatorname{Ext}^t_R(M, \Gamma_{\mathfrak{a}}(N)) \xrightarrow{f} H^t_{\mathfrak{a}}(M,N) \xrightarrow{g} H^t_{\mathfrak{a}}(M, N/\Gamma_{\mathfrak{a}}(N)).$$

Since $\operatorname{im} f$ is weakly Laskerian, by applying the functor $\operatorname{Hom}_R(R/\mathfrak{a}, \cdot)$ to the exact sequence

$$0 \longrightarrow \operatorname{im} f \longrightarrow H^t_{\mathfrak{a}}(M,N) \xrightarrow{g} H^t_{\mathfrak{a}}(M, N/\Gamma_{\mathfrak{a}}(N)),$$

it becomes clear that it is enough to show that $\operatorname{Hom}_R(R/\mathfrak{a}, H^t_{\mathfrak{a}}(M, N/\Gamma_{\mathfrak{a}}(N)))$ is weakly Laskerian. So, we can assume that N is $\mathfrak{a}$-torsion free. Then $\mathfrak{a}$ contains an N-regular element x. The exact sequence

$$0 \longrightarrow N \xrightarrow{x} N \longrightarrow N/xN \longrightarrow 0$$

induces the exact sequence

$$\cdots \to H^i_{\mathfrak{a}}(M,N) \longrightarrow H^i_{\mathfrak{a}}(M,N/xN) \longrightarrow H^{i+1}_{\mathfrak{a}}(M,N) \xrightarrow{x} H^{i+1}_{\mathfrak{a}}(M,N) \to \cdots,$$

which implies that $H^i_{\mathfrak{a}}(M, N/xN)$ is weakly Laskerian for all $i < t-1$. Thus by the induction hypothesis, $\operatorname{Hom}_R(R/\mathfrak{a}, H^{t-1}_{\mathfrak{a}}(M, N/xN))$ is weakly Laskerian.

Now, from the exact sequence

$$H^{t-1}_{\mathfrak{a}}(M,N) \xrightarrow{h} H^{t-1}_{\mathfrak{a}}(M,N/xN) \xrightarrow{k} H^{t}_{\mathfrak{a}}(M,N) \xrightarrow{x} H^{t}_{\mathfrak{a}}(M,N),$$

we get the exact sequences

$$(*) \qquad 0 \longrightarrow \operatorname{im} h \longrightarrow H^{t-1}_{\mathfrak{a}}(M,N/xN) \longrightarrow \operatorname{im} k \longrightarrow 0$$

and

$$(\dagger) \qquad 0 \longrightarrow \operatorname{im} k \longrightarrow H^{t}_{\mathfrak{a}}(M,N) \xrightarrow{x} H^{t}_{\mathfrak{a}}(M,N).$$

Applying the long Ext exact sequence on $(*)$ yields the exact sequence

$$\operatorname{Hom}_R(R/\mathfrak{a}, H^{t-1}_{\mathfrak{a}}(M, \frac{N}{xN})) \longrightarrow \operatorname{Hom}_R(R/\mathfrak{a}, \operatorname{im} k) \longrightarrow \operatorname{Ext}^1_R(R/\mathfrak{a}, \operatorname{im} h).$$

Since $x \in \mathfrak{a}$, by applying the functor $\operatorname{Hom}_R(R/\mathfrak{a}, \cdot)$ on $(\dagger)$, we get the isomorphism

$$\operatorname{Hom}_R(R/\mathfrak{a}, \operatorname{im} k) \cong \operatorname{Hom}_R(R/\mathfrak{a}, H^{t}_{\mathfrak{a}}(M,N)).$$

Now, since $\operatorname{Hom}_R(R/\mathfrak{a}, H^{t-1}_{\mathfrak{a}}(M, \frac{N}{xN}))$ and $\operatorname{Ext}^1_R(R/\mathfrak{a}, \operatorname{im} h)$ are weakly Laskerian, it turns out that $\operatorname{Hom}_R(R/\mathfrak{a}, H^{t}_{\mathfrak{a}}(M,N))$ is also weakly Laskerian. Hence the proof is complete by induction. □

Corollary 2.5. *Let $\mathfrak{a}$ be an ideal of R, M a finitely generated R-module and N a weakly Laskerian R-module. Let t be a non-negative integer such that $\operatorname{Supp}_R(H^{i}_{\mathfrak{a}}(M,N)) \subseteq \operatorname{Max} R$ for all $i < t$. Then $H^{i}_{\mathfrak{a}}(M,N)$ is weakly Artinian for all $i < t$ and $\operatorname{Hom}_R(R/\mathfrak{a}, H^{t}_{\mathfrak{a}}(M,N))$ is weakly Laskerian.*

Proof. We argue by induction on t. Obviously, the claim holds for $t = 0$. Now, inductively assume that $t > 0$ and that the assertion has been shown for $t-1$. By the induction hypothesis, $H^{i}_{\mathfrak{a}}(M,N)$ is weakly Artinian for all $i < t-1$ and $\operatorname{Hom}_R(R/\mathfrak{a}, H^{t-1}_{\mathfrak{a}}(M,N))$ is weakly Laskerian. Since

$$\operatorname{Ass}_R(H^{t-1}_{\mathfrak{a}}(M,N)) = \operatorname{Ass}_R(\operatorname{Hom}_R(R/\mathfrak{a}, H^{t-1}_{\mathfrak{a}}(M,N))),$$

it follows that $H^{t-1}_{\mathfrak{a}}(M,N)$ is supported in finitely many maximal ideals of R. Hence, Lemma 2.3 b) implies that $H^{t-1}_{\mathfrak{a}}(M,N)$ is weakly Artinian. Since any weakly Artinian R-module is weakly Laskerian, Theorem 2.4 implies that $\operatorname{Hom}_R(R/\mathfrak{a}, H^{t}_{\mathfrak{a}}(M,N))$ is weakly Laskerian. □

Lemma 2.6. *Let $\mathfrak{a}$ be an ideal of the local ring $(R, \mathfrak{m})$ and M a finitely generated R-module and N an arbitrary R-module. Let s be an integer such that $\operatorname{Supp}_R(H^{i}_{\mathfrak{a}}(M,N)) \subseteq \{\mathfrak{m}\}$ for all $i \leq s$. Then $H^{i}_{\mathfrak{a}}(M,N) \cong H^{i}_{\mathfrak{m}}(M,N)$ for all $i \leq s$.*

Proof. Consider the functors $F(\cdot) := H^{0}_{\mathfrak{m}}(\cdot)$ and $G(\cdot) := H^{0}_{\mathfrak{a}}(\operatorname{Hom}_R(M, \cdot))$. Then one has $F(G(\cdot)) = H^{0}_{\mathfrak{m}}(M, \cdot)$. For any injective R-module E, [3, Lemma 2.1] yields that $H^{i}_{\mathfrak{m}}(\operatorname{Hom}_R(M,E)) = 0$ for all $i > 0$. Let E be an injective R-module. Then it is known that $H^{0}_{\mathfrak{a}}(E)$ is also an injective R-module.

Hence $H^i_{\mathfrak{m}}(H^0_{\mathfrak{a}}(\operatorname{Hom}_R(M,E))) = 0$ for all $i > 0$. So, by [14, Theorem 11.38], one has the spectral sequence

$$E_2^{p,q} := H^p_{\mathfrak{m}}(H^q_{\mathfrak{a}}(M,N)) \underset{p}{\Longrightarrow} H^{p+q}_{\mathfrak{m}}(M,N).$$

Since the modules $H^0_{\mathfrak{a}}(M,N), H^1_{\mathfrak{a}}(M,N), \dots, H^s_{\mathfrak{a}}(M,N)$ are weakly Artinian, one has $\dim H^i_{\mathfrak{a}}(M,N) = 0$ for all $i \le s$, and so $E_2^{p,q} = 0$ for all pairs (p,q) such that $p > 0$ and $q \le s$. Now let $0 \le n \le s$ be an integer. There is a chain

$$0 = H^{-1} \subseteq H^0 \subseteq \cdots \subseteq H^{n-1} \subseteq H^n = H^n_{\mathfrak{m}}(M,N)$$

of submodules of $H^n_{\mathfrak{m}}(M,N)$ such that $H^i/H^{i-1} \cong E_\infty^{i,n-i}$ for all $i = 0, \dots, n$. Since $E_\infty^{p,q}$ is a subquotient of $E_2^{p,q}$, it turns out that $E_\infty^{i,n-i} = 0$ for all $i > 0$. Thus $H^n_{\mathfrak{m}}(M,N) \cong E_\infty^{0,n}$. Let $r \ge 2$. Consider the sequence

$$E_r^{-r,n+r-1} \overset{d^{-r,n+r-1}}{\longrightarrow} E_r^{0,n} \overset{d^{0,n}}{\longrightarrow} E_r^{r,n-r+1}.$$

Since both of $E_r^{-r,n+r-1}$ and $E_r^{r,n-r+1}$ are zero, it follows that $E_2^{0,n} \cong E_3^{0,n} \cong \cdots \cong E_\infty^{0,n} \cong H^n_{\mathfrak{m}}(M,N)$. Thus $H^n_{\mathfrak{m}}(M,N) \cong H^0_{\mathfrak{m}}(H^n_{\mathfrak{a}}(M,N)) = H^n_{\mathfrak{a}}(M,N)$. □

In [7, Theorem 3.1 and Corollary 3.4], it is shown that if $(R,\mathfrak{m})$ is a local ring and M,N two finitely generated R-modules, then $\inf\{i : H^i_{\mathfrak{a}}(M,N) \ncong H^i_{\mathfrak{m}}(M,N)\} = \inf\{i : H^i_{\mathfrak{a}}(M,N) \text{ is not Artinian}\}$. Next, we extend this result.

Theorem 2.7. *Let $\mathfrak{a}$ be an ideal of R, M a finitely generated R-module and N a weakly Laskerian R-module. Let s be an integer. The following are equivalent*:

i) $\operatorname{Supp}_R(H^i_{\mathfrak{a}}(M,N)) \subseteq \operatorname{Max} R$ *for all* $i \le s$.
ii) $H^i_{\mathfrak{a}}(M,N)$ *is weakly Artinian for all* $i \le s$.

In the case N is finitely generated, the conditions i) *and* ii) *are equivalent to*

iii) $H^i_{\mathfrak{a}}(M,N)$ *is Artinian for all* $i \le s$.

Also, in the case $(R,\mathfrak{m})$ is local, the conditions i) *and* ii) *are equivalent to*

iv) $H^i_{\mathfrak{a}}(M,N) \cong H^i_{\mathfrak{m}}(M,N)$ *for all* $i \le s$.

Proof. i) $\Leftrightarrow$ ii) follows by Corollary 2.5.

Let N be finitely generated. The implication iii) $\Rightarrow$ ii) is obvious. By induction on s, we show that if $H^i_{\mathfrak{a}}(M,N)$ is weakly Artinian for all $i \le s$, then $H^i_{\mathfrak{a}}(M,N)$ is Artinian for all $i \le s$. Let $s = 0$. Then $H^0_{\mathfrak{a}}(M,N) \cong \operatorname{Hom}_R(M, H^0_{\mathfrak{a}}(N))$ is a finitely generated weakly Artinian module. So $H^0_{\mathfrak{a}}(M,N)$ is Artinian by Lemma 2.3 a).

Now, suppose that $s > 0$ and that the claim has been proved for $s-1$. The exact sequence $0 \longrightarrow H^0_{\mathfrak{a}}(N) \longrightarrow N \longrightarrow N/H^0_{\mathfrak{a}}(N) \longrightarrow 0$ induces the long exact sequence

$$\cdots \to H^i_{\mathfrak{a}}(M, H^0_{\mathfrak{a}}(N)) \to H^i_{\mathfrak{a}}(M,N) \to H^i_{\mathfrak{a}}(M, N/H^0_{\mathfrak{a}}(N)) \to H^{i+1}_{\mathfrak{a}}(M, H^0_{\mathfrak{a}}(N)) \to \cdots.$$

Since $H^0_{\mathfrak{a}}(\operatorname{Hom}_R(M,\cdot)) = H^0_{\mathfrak{a}+\operatorname{Ann}_R M}(\operatorname{Hom}_R(M,\cdot))$, we may and do assume that $\mathfrak{a}$ contains $\operatorname{Ann}_R M$. One has $\operatorname{Ass}_R(H^0_{\mathfrak{a}}(N)) \subseteq \operatorname{V}(\mathfrak{a}) \subseteq \operatorname{Supp}_R M$, and so [4, page 267, Proposition 10] implies that

$$\begin{aligned}\operatorname{Ass}_R(H^0_{\mathfrak{a}}(M,N)) &= \operatorname{Ass}_R(\operatorname{Hom}_R(M, H^0_{\mathfrak{a}}(N)))\\ &= \operatorname{Supp}_R M \cap \operatorname{Ass}_R(H^0_{\mathfrak{a}}(N)) = \operatorname{Ass}_R(H^0_{\mathfrak{a}}(N)).\end{aligned}$$

Hence by Lemma 2.3 a), $H^0_{\mathfrak{a}}(N)$ is Artinian, and so $H^j_{\mathfrak{a}}(M, H^0_{\mathfrak{a}}(N))$ is Artinian for all j. Note that by [8, Corollary 2.8 i], one has $H^j_{\mathfrak{a}}(M, H^0_{\mathfrak{a}}(N)) \cong \operatorname{Ext}^j_R(M, H^0_{\mathfrak{a}}(N))$ for all j. So, we may and do assume that N is $\mathfrak{a}$-torsion free. Hence we can take an element $x \in \mathfrak{a}$ which is not zero a divisor on N. Then the exact sequence

$$0 \longrightarrow N \xrightarrow{x} N \longrightarrow N/xN \longrightarrow 0$$

implies the following exact sequences
$(*)$
$$\cdots \to H^{i-1}_{\mathfrak{a}}(M,N) \to H^{i-1}_{\mathfrak{a}}(M,N/xN) \longrightarrow H^i_{\mathfrak{a}}(M,N) \xrightarrow{x} H^i_{\mathfrak{a}}(M,N) \to \cdots .$$

It implies that $H^i_{\mathfrak{a}}(M,N/xN)$ is weakly Artinian for all $i \le s-1$. Thus by the induction hypothesis, we conclude that $H^i_{\mathfrak{a}}(M,N/xN)$ is Artinian for all $i \le s-1$. In particular, for each integer $i \le s$ from $(*)$, one deduce that $0 :_{H^i_{\mathfrak{a}}(M,N)} x$ is Artinian. Therefore by [13, Theorem 1.3], it turns out that $H^i_{\mathfrak{a}}(M,N)$ is Artinian for all $i \le s$. This completes the induction.

Now, assume that $(R,\mathfrak{m})$ is local. Then the equivalence ii) $\Leftrightarrow$ iv) follows by Lemma 2.6. □

Lemma 2.8. *Let $\mathfrak{a}$ be an ideal of R and M an $\mathfrak{a}$-torsion R-module. If $0 :_M \mathfrak{a}$ is weakly Artinian, then M is also weakly Artinian.*

Proof. Let $\mathcal{S}$ be the class of all weakly Artinian modules. It is obvious from the definition that if an R-module M belongs to $\mathcal{S}$, then its injective envelope belongs to $\mathcal{S}$ too. So, the conclusion is immediate by [1, Lemma 2.2]. □

Theorem 2.9. *Let $\mathfrak{a}$ be an ideal of R, M a finitely generated R-module and N a weakly Laskerian R-module. For any integer s, the following are equivalent*:

i) $H^i_{\mathfrak{a}}(M,N)$ *is weakly Artinian for all* $i \le s$.
ii) $H^i_{\mathfrak{a}+\operatorname{Ann}_R M}(N)$ *is weakly Artinian for all* $i \le s$.

Proof. Since $H^0_{\mathfrak{a}}(\operatorname{Hom}_R(M,\cdot)) = H^0_{\mathfrak{a}+\operatorname{Ann}_R M}(\operatorname{Hom}_R(M,\cdot))$, we may and do assume that $\mathfrak{a}$ contains $\operatorname{Ann}_R M$. So, it is enough to show that for any integer s, $H^0_{\mathfrak{a}}(M,N), H^1_{\mathfrak{a}}(M,N), \dots, H^s_{\mathfrak{a}}(M,N)$ are weakly Artinian if and only if $H^0_{\mathfrak{a}}(N), H^1_{\mathfrak{a}}(N), \dots, H^s_{\mathfrak{a}}(N)$ are weakly Artinian.

Let the integer $s \ge 0$ be such that $H^i_{\mathfrak{a}}(N)$ is weakly Artinian for all $i \le s$. Consider the spectral sequence $E^{p,q}_2 := \operatorname{Ext}^p_R(M, H^q_{\mathfrak{a}}(N)) \underset{p}{\Longrightarrow} H^{p+q}_{\mathfrak{a}}(M,N)$. Lemma 2.3 implies that $E^{p,q}_2$ is weakly Artinian for all $q \le s$. Let $n \le s$ be an integer. There is a finite chain of submodules of $H^n_{\mathfrak{a}}(M,N)$ such that any successive quotient in this chain is a subquotient of $E_2^{p,q}$ for some p and q with $p+q=n$. Hence $H_{\mathfrak{a}}^n(M,N)$ is supported in $\bigcup_{p+q=n} \operatorname{Supp}_R E_2^{p,q}$, as so applying Lemma 2.3 once more yields that $H_{\mathfrak{a}}^n(M,N)$ is weakly Artinian.

Conversely, let the integer $s \geq 0$ be such that $H_{\mathfrak{a}}^i(M,N)$ is weakly Artinian for all $i \leq s$. By induction on s, we show that $H_{\mathfrak{a}}^i(N)$ is weakly Artinian for all $i \leq s$. Let $s=0$. Then as the proof of Theorem 2.7, we have

$$\operatorname{Ass}_R(H_{\mathfrak{a}}^0(M,N)) = \operatorname{Ass}_R(H_{\mathfrak{a}}^0(N)).$$

Hence $H_{\mathfrak{a}}^0(N)$ is weakly Artinian. Now, let $s>0$ and assume that the assertion holds for $s-1$. Then by the induction hypothesis $H_{\mathfrak{a}}^i(N)$ is weakly Artinian for all $i \leq s-1$. So, it remains to show that $H_{\mathfrak{a}}^s(N)$ is weakly Artinian. The exact sequence

$$0 \longrightarrow H_{\mathfrak{a}}^0(N) \longrightarrow N \longrightarrow N/H_{\mathfrak{a}}^0(N) \longrightarrow 0$$

induces the long exact sequence

$$\cdots \longrightarrow H_{\mathfrak{a}}^i(M,N) \longrightarrow H_{\mathfrak{a}}^i(M,N/H_{\mathfrak{a}}^0(N)) \longrightarrow \operatorname{Ext}_R^{i+1}(M,H_{\mathfrak{a}}^0(N)) \longrightarrow \cdots.$$

Thus $H_{\mathfrak{a}}^i(M,N/H_{\mathfrak{a}}^0(N))$ is weakly Artinian for all $i \leq s$, and so we may and do assume that N is $\mathfrak{a}$-torsion free. (Note that $H_{\mathfrak{a}}^i(N/H_{\mathfrak{a}}^0(N)) \cong H_{\mathfrak{a}}^i(N)$ for all $i>0$.) So, we can take an element $x \in \mathfrak{a}$ which is not a zero-divisor on N. Then the exact sequence $0 \longrightarrow N \xrightarrow{x} N \longrightarrow N/xN \longrightarrow 0$ implies the following exact sequences

$$(*) \qquad \cdots \longrightarrow H_{\mathfrak{a}}^i(M,N) \longrightarrow H_{\mathfrak{a}}^i(M,N/xN) \longrightarrow H_{\mathfrak{a}}^{i+1}(M,N) \longrightarrow \cdots$$

and

$$(**) \qquad \cdots \longrightarrow H_{\mathfrak{a}}^{s-1}(N/xN) \longrightarrow H_{\mathfrak{a}}^s(N) \xrightarrow{x} H_{\mathfrak{a}}^s(N) \longrightarrow \cdots.$$

Next $(*)$ yields that $H_{\mathfrak{a}}^i(M,N/xN)$ is weakly Artinian for all $i \leq s-1$. Thus from induction hypothesis, we deduce that $H_{\mathfrak{a}}^{s-1}(N/xN)$ is weakly Artinian. Now, from $(**)$ it becomes clear that $0 :_{H_{\mathfrak{a}}^s(N)} x$ is weakly Artinian. So we conclude from Lemma 2.8 that $H_{\mathfrak{a}}^s(N)$ is weakly Artinian. □

Corollary 2.10. *Let $\mathfrak{a}$ be an ideal of R, M and N two finitely generated R-modules. Then the infimum of the integers i such that $H_{\mathfrak{a}}^i(M,N)$ is not Artinian is equal the infimum of the integers i such that $H_{\mathfrak{a}+\operatorname{Ann}_R M}^i(N)$ is not Artinian.*

Proof. It follows by Theorems 2.7 and 2.9. □

It is known that if M is a finitely generated R-module of finite dimension d, then the local cohomology module $H_{\mathfrak{a}}^d(M)$ is Artinian, see [5, Exercise 7.1.7]. The following might be considered as a generalization of this fact to weakly Laskerian modules.

Theorem 2.11. *Let $\mathfrak{a}$ be an ideal of R. Let N be a weakly Laskerian R-module of finite dimension d. Then $H_{\mathfrak{a}}^d(N)$ is weakly Artinian.*

Proof. We argue by induction on d. For $d = 0$, there is nothing to prove, because $H^0_{\mathfrak{a}}(N)$ is a submodule of N and zero dimensional modules are clearly weakly Artinian. Now suppose that $d > 0$, and that the claim holds for $d-1$. Since $d > 0$, we may assume that N is a $\mathfrak{a}$-torsion free. So, we can take an N-regular element x in $\mathfrak{a}$. Then N/xN is a weakly Laskerian R-module of dimension $\leq d-1$. From the exact sequence

$$0 \longrightarrow N \xrightarrow{x} N \longrightarrow N/xN \longrightarrow 0,$$

we deduce the long exact sequence

$$\cdots \longrightarrow H^{d-1}_{\mathfrak{a}}(N/xN) \longrightarrow H^d_{\mathfrak{a}}(N) \xrightarrow{x} H^d_{\mathfrak{a}}(N) \longrightarrow \cdots .$$

By the induction hypothesis, $H^{d-1}_{\mathfrak{a}}(N/xN)$ is weakly Artinian, and so $0 :_{H^d_{\mathfrak{a}}(N)} x$ is weakly Artinian. Hence, Lemma 2.8 implies that $H^d_{\mathfrak{a}}(N)$ is weakly Artinian. □

Remark 2.12. Let $\mathfrak{a}$ be an ideal of R, M a finitely generated R-module and N an arbitrary R-module.

i) Suppose that N is weakly Laskerian and $d = \dim N < \infty$. Then $H^d_{\mathfrak{a}}(N)$ is Artinian if and only if $\mu^0(\mathfrak{m}, H^d_{\mathfrak{a}}(N)) < \infty$ for all $\mathfrak{m} \in \operatorname{Max} R$.

ii) Suppose that $\dim R/\mathfrak{a} = 0$ and let i be a non-negative integer. Since $\operatorname{Supp}_R(H^i_{\mathfrak{a}}(M,N)) \subseteq \mathrm{V}(\mathfrak{a})$ always holds, it turns out that $\operatorname{Supp}_R(H^i_{\mathfrak{a}}(M,N))$ consists of finitely many maximal ideals of R. Hence, $H^i_{\mathfrak{a}}(M,N)$ is weakly Artinian. Therefore, the R-module $H^i_{\mathfrak{a}}(M,N)$ is Artinian if and only if $\mu^0(\mathfrak{m}, H^i_{\mathfrak{a}}(M,N)) < \infty$ for all $\mathfrak{m} \in \operatorname{Max} R$.

3. Weakly filter regular sequences

Throughout this section $\mathfrak{a}$ is an ideal of a local ring $(R, \mathfrak{m})$. Let M, N be two finitely generated R-modules. Using the notion of filter M-regular sequences, Chu and Tang [6, Theorem 2.2] have shown that

$$\inf\{i : H^i_{\mathfrak{a}}(M,N) \text{is not Artinian}\} = f-\operatorname{depth}(\mathfrak{a} + \operatorname{Ann}_R M, N).$$

Recall that $f-\operatorname{depth}(\mathfrak{a} + \operatorname{Ann}_R M, N)$ is the length of any maximal filter N-regular sequence in $\mathfrak{a} + \operatorname{Ann}_R M$. Here, we introduce the notion of weakly filter M-regular sequences. Let M be a finitely generated R-module and N a weakly Laskerian R-module. As our main result in this section, we determine the least integer i, such that $H^i_{\mathfrak{a}}(M,N)$ is not weakly Artinian.

Definition 3.1. Let M be an R-module and $a_1, \ldots, a_n$ a sequence of elements in $\mathfrak{m}$. We say that $a_1, \ldots, a_n$ is a weakly filter M-regular sequence if for each $i = 1, \ldots, n$, the R-module $0 :_{M/(a_1,\ldots,a_{i-1})M} a_i$ is weakly Artinian.

Note that a weakly filter M-regular sequence is a special case of an $\mathcal{S}$-sequence on M, which was introduced in [1]. $\mathcal{S}$ is here to be taken as the Serre subcategory of weakly Artinian modules.

Let the situation be as in the above definition. It is easy to see that $a_1, \dots, a_n$ is a weakly filter M-regular sequence if and only if for any $1 \leq i \leq n$, $a_1, \dots, a_{i-1}$ is a weakly filter M-regular sequence and $a_i, \dots, a_n$ is a weakly filter $M/(a_1, \dots, a_{i-1})M$-regular sequence.

Lemma 3.2. *Let M be an R-module $\mathfrak{p} \in \operatorname{Supp}_R M - \{\mathfrak{m}\}$. Let $a_1, \dots, a_n \in \mathfrak{m}$ be a weakly filter M-regular sequence. Then $a_1/1, \dots, a_n/1 \in R_{\mathfrak{p}}$ is a poor $M_{\mathfrak{p}}$-sequence.*

Proof. Using induction on n, it is enough to prove the claim for $n = 1$. Since $\operatorname{Supp}_R(0 :_M a_1) \subseteq \{\mathfrak{m}\}$, one has $(0 :_{M_{\mathfrak{p}}} a_1 R_{\mathfrak{p}}) \cong (0 :_M a_1)_{\mathfrak{p}} = 0$, and so the proof is complete. □

Theorem 3.3. *Let M be a weakly Laskerian R-module and t a non-negative integer. The following are equivalent:*

i) $\operatorname{Ext}^i_R(R/\mathfrak{a}, M)$ *is weakly Artinian for all $i < t$.*

ii) $\mathfrak{a}$ *contains a weakly filter M-regular sequence of length t.*

Proof. Let $\operatorname{Ext}^i_R(R/\mathfrak{a}, M)$ be weakly Artinian for all $i < t$. By induction on t, we show that $\mathfrak{a}$ contains a weakly filter M-regular sequence of length t. Let $t = 1$. Then $\operatorname{Hom}_R(R/\mathfrak{a}, M)$ is weakly Artinian. If for some $\mathfrak{p} \in \operatorname{Ass}_R(M) - \{\mathfrak{m}\}$, one has $\mathfrak{a} \subseteq \mathfrak{p}$, then $\mathfrak{p} \in \operatorname{Supp}_R(\operatorname{Hom}_R(R/\mathfrak{a}, M))$, which is a contradiction. So, there exists $a_1 \in \mathfrak{a}$ such that $\operatorname{Hom}_R(R/(a_1), M) \cong 0 :_M a_1$ is weakly Artinian. Now, let $t > 1$ and assume that the claim is proved for $t - 1$. Then $\mathfrak{a}$ contains a weakly filter M-regular sequence a_1, say. The short exact sequences

$$0 \longrightarrow 0 :_M a_1 \longrightarrow M \xrightarrow{a_1} a_1 M \longrightarrow 0$$

and

$$0 \longrightarrow a_1 M \longrightarrow M \longrightarrow M/a_1 M \longrightarrow 0,$$

respectively induce the long exact sequences,

$$\begin{aligned} 0 &\longrightarrow \operatorname{Hom}_R(R/\mathfrak{a}, 0 :_M a_1) \longrightarrow \operatorname{Hom}_R(R/\mathfrak{a}, M) \xrightarrow{a_1} \operatorname{Hom}_R(R/\mathfrak{a}, a_1 M) \\ &\longrightarrow \operatorname{Ext}^1_R(R/\mathfrak{a}, 0 :_M a_1) \longrightarrow \operatorname{Ext}^1_R(R/\mathfrak{a}, M) \xrightarrow{a_1} \operatorname{Ext}^1_R(R/\mathfrak{a}, a_1 M) \\ &\longrightarrow \operatorname{Ext}^2_R(R/\mathfrak{a}, 0 :_M a_1) \longrightarrow \cdots \end{aligned}$$

and

$$\begin{aligned} 0 &\longrightarrow \operatorname{Hom}_R(R/\mathfrak{a}, a_1 M) \longrightarrow \operatorname{Hom}_R(R/\mathfrak{a}, M) \longrightarrow \operatorname{Hom}_R(R/\mathfrak{a}, M/a_1 M) \\ &\longrightarrow \operatorname{Ext}^1_R(R/\mathfrak{a}, a_1 M) \longrightarrow \operatorname{Ext}^1_R(R/\mathfrak{a}, M) \longrightarrow \operatorname{Ext}^1_R(R/\mathfrak{a}, M/a_1 M) \\ &\longrightarrow \operatorname{Ext}^2_R(R/\mathfrak{a}, a_1 M) \longrightarrow \cdots . \end{aligned}$$

From the first long exact sequence, it follows that $\operatorname{Ext}^i_R(R/\mathfrak{a}, a_1 M)$ is weakly Artinian for all $i < t$. Hence, from the second long exact sequence, we deduce that $\operatorname{Ext}^i_R(R/\mathfrak{a}, M/a_1 M)$ is weakly Artinian for all $i < t - 1$. By the induction hypothesis, $\mathfrak{a}$ contains a weakly filter $M/a_1 M$-regular sequence $a_2, \dots, a_t$. Hence $a_1, \dots, a_t \in \mathfrak{a}$ is a weakly filter M-regular sequence.

Conversely, let $a_1, \ldots, a_t \in \mathfrak{a}$ be a weakly filter M-regular sequence. Let $\mathfrak{p} \in \operatorname{Supp}_R M - \{\mathfrak{m}\}$. Then by Lemma 3.2, $a_1/1, \ldots, a_t/1 \in \mathfrak{a}R_{\mathfrak{p}}$ is a poor $M_{\mathfrak{p}}$-regular sequence, and so $\operatorname{Ext}^i_R(R/\mathfrak{a}, M)_{\mathfrak{p}} = 0$ for all $i < t$. This yields that $\operatorname{Ext}^i_R(R/\mathfrak{a}, M)$ is weakly Artinian for all $i < t$. $\square$

Corollary 3.4. *Let M be a weakly Laskerian R-module. Then any two maximal weakly filter M-regular sequences in $\mathfrak{a}$ have the same length.*

Definition 3.5. Let $\mathfrak{a}$ be an ideal of the local ring $(R, \mathfrak{m})$. The $w - f - \operatorname{depth}(\mathfrak{a}, M)$ is defined to be the length of any maximal weakly filter M-regular sequence in $\mathfrak{a}$.

The next corollary generalizes [6, Theorem 2.2] from the finitely generated case to the weakly Laskerian case.

Corollary 3.6. *Let M be a finitely generated R-module and N a weakly Laskerian R-module. Then*

$$\inf\{i : H^i_{\mathfrak{a}}(M, N) \text{ is not weakly Artinian}\} = w - f - \operatorname{depth}(\mathfrak{a} + \operatorname{Ann}_R M, N).$$

Proof. By Theorem 3.3 and [1, Theorem 2.9],

$$\begin{aligned} & w - f - \operatorname{depth}(\mathfrak{a} + \operatorname{Ann}_R M, N) \\ = \ & \inf\{i : \operatorname{Ext}^i_R(R/\mathfrak{a} + \operatorname{Ann}_R M, N) \text{ is not weakly Artinian}\} \\ = \ & \inf\{i : H^i_{(\mathfrak{a} + \operatorname{Ann}_R M)}(N) \text{ is not weakly Artinian}\}. \end{aligned}$$

Thus, the conclusion follows by Theorem 2.9. $\square$

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