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ANTI-PERIODIC SOLUTIONS FOR HIGHER-ORDER NONLINEAR ORDINARY DIFFERENTIAL EQUATIONS

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Abstract. In this paper, the existence of anti-periodic solutions for higher-order nonlinear ordinary differential equations is studied by using degree theory and some known results are improved to some extent.

1. Introduction

Anti-periodic problems arise naturally from the mathematical models of a variety of physical processes and have important applications in auto-control, partial differential equations and engineering. Recently, there has been a great deal of research on anti-periodic boundary value problem (see [1], [2], [9], [10], [11], [12] and references therein). In mechanics, the simplest model of oscillation equation is single pendulum equation

$$x'' + \omega^2 \sin x = p(t) = p(t + 2\pi), \tag{1}$$

whose anti-periodic solutions satisfy

$$x\,(t + \pi) = -x(t), \qquad \forall t \in \mathbb{R}.$$

In particular, many authors have discussed the existence of anti-periodic solutions for the first order or second order nonlinear ordinary differential equation. Mawhin ([9]) generalized equation (1) to the general Duffing equation

$$x'' + g(x) = p(t),$$

and obtained the existence results for anti-periodic solutions by using critical point theory. In [1], the author proved the existence of anti-periodic solutions for the following abstract nonlinear second order evolution equation

$$-x''(t) + ax'(t) + A(t)x(t) = f(t)$$

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associated with maximal monotone operators in Hilbert spaces. In [2], some results of anti-periodic solutions for Liénard equation

$$x'' + f(x)x' + g(t, x) = p(t)$$

were established by Leray-Schauder principle.

In recent years, many results relative to the existence of periodic solutions for higher-order ordinary differential equations have been obtained (see [3], [6], [7], [8] and references therein). In this paper, we consider the existence of anti-periodic solutions for the following higher-order nonlinear ordinary differential equations

$$x^{(2m)} + \sum_{i=2}^{2m-1} a_i x^{(i)} + f(x)x' + g(t, x) = p(t), \tag{2}$$

$$x^{(2m+1)} + \sum_{i=2}^{2m} a_i x^{(i)} + f(x)x' + g(t, x) = p(t), \tag{3}$$

where $a_i \in \mathbb{R}(i = 2, 3, \ldots, 2m)$, $f(x) \in C(\mathbb{R}, \mathbb{R})$, $g(t, x) \in C(\mathbb{R}^2, \mathbb{R})$ and $g(t + 2\pi, x) = g(t, x)$, $p(t) \in C(\mathbb{R}, \mathbb{R})$ and $p(t + 2\pi) = p(t)$. We obtain several useful results by using Leray-Schauder principle.

The plan of this paper is as follows. Section 2 contains the necessary preliminaries. In section 3, we obtain the existence theorem of anti-periodic solutions for equation (2) (Theorem 3.1). Section 4 consists of two parts. In the first part, we establish two existence theorems of anti-periodic solutions for equation (3) (Theorem 4.1, 4.2). The second part is devoted to handling with the equation (3) when $g(t, x) = g(x)$ and we obtain two existence results of anti-periodic solutions (Theorem 4.3, 4.4). Our results improve and generalize some known results to some extent.

2. Preliminaries

Throughout the paper, we shall use the following notations

$$C^{k,\pi} = \left\{x \in C^k(\mathbb{R}, \mathbb{R}) : x(t + \pi) = -x(t),\ \forall t \in \mathbb{R}\right\},$$

$$\|x\|_2 = \left\{\int_0^{2\pi} |x(t)|^2 dt\right\}^{1/2}, \qquad \|x\|_\infty = \max_{t\in[0,2\pi]} |x(t)|,$$

$$\|x\|_{C^k} = \max_{i=0,1,\ldots,k} \left\{\left\|x^{(i)}\right\|_\infty\right\}.$$

For $x(t) \in C^{0,\pi}$, there exists the following Fourier expansion

$$x(t) = \sum_{i=0}^{\infty} \left[a_{2i+1} \cos(2i + 1)t + b_{2i+1} \sin(2i + 1)t\right].$$

Let us define $J : C^{0,\pi} \longrightarrow C^{1,\pi}$

$$\begin{aligned} (Jx)(t) &= \int_0^t x(s)ds - \sum_{i=0}^{\infty} \frac{b_{2i+1}}{2i+1} \\ &= \sum_{i=0}^{\infty} \left[\frac{a_{2i+1}}{2i+1} \sin(2i+1)t - \frac{b_{2i+1}}{2i+1} \cos(2i+1)t \right]. \end{aligned}$$

Obviously

$$\frac{d}{dt}[Jx(t)] = x(t).$$

By the definition of J, we have

$$\begin{aligned} |(Jx)(t)| &\le \int_0^{2\pi} |x(s)|ds + \sum_{i=0}^{\infty} \frac{|b_{2i+1}|}{2i+1} \\ &\le 2\pi\|x\|_\infty + \left(\sum_{i=0}^{\infty} b_{2i+1}^2 \right)^{1/2} \left[\sum_{i=0}^{\infty} \frac{1}{(2i+1)^2} \right]^{1/2}. \end{aligned}$$

Noting

$$\left[\sum_{i=0}^{\infty} \frac{1}{(2i+1)^2} \right]^{1/2} = \frac{\pi}{2\sqrt{2}}$$

and using the Parseval equality

$$\int_0^{2\pi} |x(s)|^2 ds = \pi \sum_{i=0}^{\infty} \left(a_{2i+1}^2 + b_{2i+1}^2 \right),$$

we obtain

$$\begin{aligned} |(Jx)(t)| &\le 2\pi\|x\|_\infty + \frac{\pi}{2\sqrt{2}} \left[\sum_{i=0}^{\infty} \left(a_{2i+1}^2 + b_{2i+1}^2 \right) \right]^{1/2} \\ &= 2\pi\|x\|_\infty + \frac{\pi}{2\sqrt{2}} \left(\frac{1}{\pi} \int_0^{2\pi} |x(s)|^2 ds \right)^{1/2} \\ &\le 2\pi\|x\|_\infty + \frac{\pi}{2}\|x\|_\infty = \frac{5\pi}{2}\|x\|_\infty, \quad \forall t \in [0,1]. \end{aligned}$$

Immediately

$$\|Jx\|_\infty \le \frac{5\pi}{2}\|x\|_\infty.$$

Therefore, J is continuous. It is easy to prove that J is a completely continuous operator by Arzela-Ascoli theorem.

Moreover, we will need the following lemmas.

Lemma 2.1 ([5])**.** *Assume that* $x(t) \in C^1(\mathbb{R}, \mathbb{R})$ *and* $x(0) = x(2\pi)$, $\int_0^{2\pi} x(t)dt = 0$*, then*

$$\int_0^{2\pi} |x(t)|^2 dt \le \int_0^{2\pi} \left|x'(t)\right|^2 dt.$$

Lemma 2.2 ([4]). *Suppose Ω is a bounded open set of normal space X, f is compact in $\overline{\Omega}$ and $p \in X \setminus f(\partial\Omega)$. Then the equation $f(x) = p$ has a solution in Ω, provided with $deg(f, \Omega, p) \neq 0$.*

3. The existence of anti-periodic solutions for equation (2)

In this section, we will prove the existence of anti-periodic solutions for equation (2).

Theorem 3.1. *Assume that*

(H_1) *for* $t \in \mathbb{R}, x \in \mathbb{R}$

$$f(-x) = f(x), \quad g(t+\pi, -x) = -g(t, x), \quad p(t+\pi) = -p(t);$$

(H_2) *there is* $\alpha \geq 0$ *such that*

$$\limsup_{|x|\to+\infty} \frac{|g(t,x)|}{|x|} = \alpha, \quad \forall t \in \mathbb{R};$$

(H_3) $1 - \sum_{i=1}^{m-1} |a_{2i}| - \alpha > 0$.

Then there exists at least one anti-periodic solution of equation (2).

Proof. We consider the auxiliary equation of (2)

$$\begin{aligned} x^{(2m)} &= -\lambda \sum_{i=2}^{2m-1} a_i x^{(i)} - \lambda f(x)x' - \lambda g(t,x) + \lambda p(t) \\ &:= \lambda q_1 \left(x^{(2m-1)}, \ldots, x', x, t \right), \end{aligned} \tag{4}$$

where $\lambda \in [0, 1]$. Obviously, $q_1 \left(x^{(2m-1)}, \ldots, x', x, t \right)$ is continuous.

Firstly, we can claim that there exists a prior bound in $C^{2m,\pi}$, for the possible solution $x(t)$ of equation (4).

Multiplying equation (4) with $x(t)$ and integrating it over $[0, 2\pi]$, we get

$$\begin{aligned} &\int_0^{2\pi} x^{(2m)}(t)x(t)dt \\ = &-\lambda \sum_{i=2}^{2m-1} a_i \int_0^{2\pi} x^{(i)}(t)x(t)dt - \lambda \int_0^{2\pi} f(x(t))x'(t)x(t)dt \\ &-\lambda \int_0^{2\pi} g(t, x(t))x(t)dt + \lambda \int_0^{2\pi} p(t)x(t)dt. \end{aligned}$$

Noting $\int_0^{2\pi} x^{(2i+1)}(t)x(t)dt = 0$ and

$$\int_0^{2\pi} f(x(t))x'(t)x(t)dt = \int_0^{2\pi} f(x(t))x(t)d(x(t)) = \int_{x(0)}^{x(2\pi)} f(\tau)\tau d\tau = 0,$$

combining with $\int_0^{2\pi} x^{(2m)}(t)x(t)dt = (-1)^m \int_0^{2\pi} \left|x^{(m)}(t)\right|^2 dt$, we have

$$\begin{aligned} & \int_0^{2\pi} \left|x^{(m)}(t)\right|^2 dt \\ = \quad & (-1)^{m+1}\lambda \sum_{i=1}^{m-1} a_{2i} \int_0^{2\pi} x^{(2i)}(t)x(t)dt + (-1)^{m+1}\lambda \int_0^{2\pi} g(t,x(t))x(t)dt \\ & +(-1)^m\lambda \int_0^{2\pi} p(t)x(t)dt \\ \leq \quad & \sum_{i=1}^{m-1} |a_{2i}| \left|\int_0^{2\pi} x^{(2i)}(t)x(t)dt\right| + \int_0^{2\pi} |g(t,x(t))x(t)|\,dt \\ & + \int_0^{2\pi} |p(t)x(t)|\,dt. \end{aligned}$$

By hypothesis (H_2), we can find some constant $\beta \geq 0$ such that

$$|g(t,x)| \leq \beta + \alpha|x|, \quad \forall t, x \in \mathbb{R}.$$

Thus

$$\begin{aligned} & \int_0^{2\pi} \left|x^{(m)}(t)\right|^2 dt \\ \leq \quad & \sum_{i=1}^{m-1} |a_{2i}| \int_0^{2\pi} \left|x^{(i)}(t)\right|^2 dt + \int_0^{2\pi} (\beta + \alpha|x(t)|)\,|x(t)|\,dt \\ & + \int_0^{2\pi} |p(t)x(t)|\,dt \\ \leq \quad & \sum_{i=1}^{m-1} |a_{2i}| \left\|x^{(i)}\right\|_2^2 + \alpha\|x\|_2^2 + \left(\sqrt{2\pi}\beta + \|p\|_2\right)\|x\|_2 \,. \end{aligned} \tag{5}$$

For $x(t) \in C^{2m,\pi}$, we get $\int_0^{2\pi} x^{(i)}(t)dt = 0$ $(i = 0, 1, \ldots, 2m-1)$. By Lemma 2.1, it can be shown that

$$\|x\|_2 \leq \|x'\|_2 \leq \cdots \leq \left\|x^{(2m)}\right\|_2 .$$

So, from (5)

$$\left\|x^{(m)}\right\|_2^2 \leq \sum_{i=1}^{m-1} |a_{2i}| \left\|x^{(m)}\right\|_2^2 + \alpha \left\|x^{(m)}\right\|_2^2 + \left(\sqrt{2\pi}\beta + \|p\|_2\right) \left\|x^{(m)}\right\|_2 .$$

By assumption (H_3), there exists $M_1 > 0$ (independent of λ) such that

$$\|x\|_2 \leq \|x'\|_2 \leq \cdots \leq \left\|x^{(m)}\right\|_2 \leq M_1.$$

Because $\int_0^{2\pi} x(t)dt = 0$, there exists $t_0 \in [0, 2\pi]$ such that $x(t_0) = 0$. Hence

$$\|x\|_\infty \leq \int_0^{2\pi} |x'(t)|\, dt \leq \sqrt{2\pi}\, \|x'\|_2 \leq \sqrt{2\pi} M_1. \tag{6}$$

A similar argument, we can prove

$$\left\| x^{(i)} \right\|_\infty \leq \sqrt{2\pi} M_1, \quad i = 1, 2, \ldots, m-1. \tag{7}$$

Multiplying equation (4) with $x^{(2m)}(t)$ and integrating it over $[0, 2\pi]$, we get

$$\begin{aligned}
&\int_0^{2\pi} \left| x^{(2m)}(t) \right|^2 dt \\
= \;& -\lambda \sum_{i=2}^{2m-1} a_i \int_0^{2\pi} x^{(i)}(t) x^{(2m)}(t) dt - \lambda \int_0^{2\pi} f(x(t)) x'(t) x^{(2m)}(t) dt \\
& -\lambda \int_0^{2\pi} g(t, x(t)) x^{(2m)}(t) dt + \lambda \int_0^{2\pi} p(t) x^{(2m)}(t) dt \\
\leq \;& \sum_{i=2}^{2m-1} |a_i| \left| \int_0^{2\pi} x^{(i)}(t) x^{(2m)}(t) dt \right| + \int_0^{2\pi} \left| f(x(t)) x'(t) x^{(2m)}(t) \right| dt \\
& + \int_0^{2\pi} \left| g(t, x(t)) x^{(2m)}(t) \right| dt + \int_0^{2\pi} \left| p(t) x^{(2m)}(t) \right| dt.
\end{aligned}$$

For $\int_0^{2\pi} x^{(2i+1)}(t) x^{(2m)}(t) dt = 0$, we can see

$$\begin{aligned}
&\int_0^{2\pi} \left| x^{(2m)}(t) \right|^2 dt \\
\leq \;& \sum_{i=1}^{m-1} |a_{2i}| \left| \int_0^{2\pi} x^{(2i)}(t) x^{(2m)}(t) dt \right| + \int_0^{2\pi} \left| f(x(t)) x'(t) x^{(2m)}(t) \right| dt \\
& + \int_0^{2\pi} \left| g(t, x(t)) x^{(2m)}(t) \right| dt + \int_0^{2\pi} \left| p(t) x^{(2m)}(t) \right| dt.
\end{aligned} \tag{8}$$

By (6), there exist $\gamma_1, \gamma_2 \geq 0$ such that $|f(x)| \leq \gamma_1, |g(t, x)| \leq \gamma_2,\ \forall t, x \in \mathbb{R}$. Hence, from (7) and (8), we can get

$$\begin{aligned}
&\int_0^{2\pi} \left| x^{(2m)}(t) \right|^2 dt \\
\leq \;& \sum_{i=1}^{m-1} |a_{2i}| \int_0^{2\pi} \left| x^{(i+m)}(t) \right|^2 dt + \left(\sqrt{2\pi} M_1 \gamma_1 + \gamma_2 \right) \int_0^{2\pi} \left| x^{(2m)}(t) \right| dt \\
& + \int_0^{2\pi} \left| p(t) x^{(2m)}(t) \right| dt \\
\leq \;& \sum_{i=1}^{m-1} |a_{2i}| \left\| x^{(i+m)} \right\|_2^2 + \left(2\pi M_1 \gamma_1 + \sqrt{2\pi} \gamma_2 + \|p\|_2 \right) \left\| x^{(2m)} \right\|_2.
\end{aligned}$$

By using Lemma 2.1, we know

$$\left\|x^{(2m)}\right\|_2^2 \leq \sum_{i=1}^{m-1} |a_{2i}| \left\|x^{(2m)}\right\|_2^2 + \left(2\pi M_1\gamma_1 + \sqrt{2\pi}\gamma_2 + \|p\|_2\right) \left\|x^{(2m)}\right\|_2 .$$

For (H_3), there exists $M_2 > 0$ (independent of λ) such that

$$\|x\|_2 \leq \|x'\|_2 \leq \cdots \leq \left\|x^{(2m)}\right\|_2 \leq M_2.$$

Similar with the proof of (6), we can prove

$$\left\|x^{(i)}\right\|_\infty \leq \sqrt{2\pi} M_2, \quad i = 0, 1, \ldots, 2m-1.$$

By the equation (4), there exists $M_3 > 0$ (independent of λ) such that

$$\left\|x^{(2m)}\right\|_\infty \leq M_3.$$

Set $T_1 = \max\left\{\sqrt{2\pi} M_2, M_3\right\} + 1$. Then

$$\|x\|_{C^{2m}} < T_1. \tag{9}$$

Secondly, we can prove the existence of anti-periodic solutions for equation (2). Set

$$\Omega = \left\{x(t) \in C^{2m,\pi} : \|x\|_{C^{2m}} < T_1\right\}.$$

Then Ω is a bounded open set in $C^{2m,\pi}$. By hypothesis (H_1), it is easy to see that

$$\begin{aligned} & q_1\left(x^{(2m-1)}(t+\pi), \ldots, x'(t+\pi), x(t+\pi), t+\pi\right) \\ = & -q_1\left(x^{(2m-1)}(t), \ldots, x'(t), x(t), t\right), \quad \forall x(t) \in C^{2m,\pi}. \end{aligned}$$

Hence $q_1 : C^{2m-1,\pi} \longrightarrow C^{0,\pi}$. Define $F_\lambda : \overline{\Omega} \longrightarrow C^{2m,\pi}$

$$F_\lambda x = \lambda J^{2m} q_1 x, \quad \lambda \in [0,1].$$

Obviously, F_λ is compact. Hence, the fixed points of F_1 in $\overline{\Omega}$ are the anti-periodic solutions of equation (2).

Let $h_\lambda(x) : \overline{\Omega} \times [0,1] \longrightarrow C^{2m,\pi}$

$$h_\lambda(x) = x - F_\lambda x.$$

By (9), we get $\theta \notin h_\lambda(\partial\Omega)$. Hence

$$\begin{aligned} \deg(id - F_1, \Omega, \theta) &= \deg(h_1, \Omega, \theta) = \deg(h_0, \Omega, \theta) \\ &= \deg(id, \Omega, \theta) = 1. \end{aligned}$$

Consequently, F_1 has at least one fixed point in Ω by Lemma 2.2. Namely, the equation (2) has at least one anti-periodic solution. □

4. The existence of anti-periodic solutions for equation (3)

In this section, we will prove the existence of anti-periodic solutions for equation (3).

Theorem 4.1. *Assume that*

(H_4) *for* $x \in \mathbb{R}$

$$(-1)^{m+1} f(x) \leq 0;$$

(H_5) $1 - \sum_{i=1}^{m-1} |a_{2i+1}| - \alpha > 0$

and the assumptions $(H_1), (H_2)$ *are true. Then there exists at least one anti-periodic solution of equation* (3).

Proof. We consider the auxiliary equation of (3)

$$\begin{aligned} x^{(2m+1)} &= -\lambda \sum_{i=2}^{2m} a_i x^{(i)} - \lambda f(x) x' - \lambda g(t,x) + \lambda p(t) \\ &:= \lambda q_2 \left(x^{(2m)}, \ldots, x', x, t \right), \end{aligned} \tag{10}$$

where $\lambda \in [0,1]$. Obviously, $q_2 \left(x^{(2m)}, \ldots, x', x, t \right)$ is continuous.

Similar with the proof of Theorem 3.1, we need only prove that there exists a prior bound in $C^{2m+1,\pi}$, for the possible solution $x(t)$ of equation (10).

Multiplying equation (10) with $x'(t)$ and integrating it over $[0, 2\pi]$, we get

$$\begin{aligned} & \int_0^{2\pi} x^{(2m+1)}(t) x'(t) dt \\ = & -\lambda \sum_{i=2}^{2m} a_i \int_0^{2\pi} x^{(i)}(t) x'(t) dt - \lambda \int_0^{2\pi} f(x(t)) \left(x'(t) \right)^2 dt \\ & -\lambda \int_0^{2\pi} g(t, x(t)) x'(t) dt + \lambda \int_0^{2\pi} p(t) x'(t) dt. \end{aligned}$$

Noting $\int_0^{2\pi} x^{(2i)}(t) x'(t) dt = 0$ and

$$\int_0^{2\pi} x^{(2m+1)}(t) x'(t) dt = (-1)^m \int_0^{2\pi} \left| x^{(m+1)}(t) \right|^2 dt,$$

we have

$$\begin{aligned} & \int_0^{2\pi} \left| x^{(m+1)}(t) \right|^2 dt \\ = & (-1)^{m+1} \lambda \sum_{i=1}^{m-1} a_{2i+1} \int_0^{2\pi} x^{(2i+1)}(t) x'(t) dt \\ & +(-1)^{m+1} \lambda \int_0^{2\pi} f(x(t)) \left(x'(t) \right)^2 dt \\ & +(-1)^{m+1} \lambda \int_0^{2\pi} g(t, x(t)) x'(t) dt + (-1)^m \lambda \int_0^{2\pi} p(t) x'(t) dt. \end{aligned}$$

By the assumption (H_4), we can see

$$\begin{aligned}
&\int_0^{2\pi} \left|x^{(m+1)}(t)\right|^2 dt \\
\leq\ & \sum_{i=1}^{m-1} |a_{2i+1}| \left|\int_0^{2\pi} x^{(2i+1)}(t)x'(t)dt\right| + \int_0^{2\pi} |g(t,x(t))x'(t)|\,dt \\
&+ \int_0^{2\pi} |p(t)x'(t)|\,dt \\
\leq\ & \sum_{i=1}^{m-1} |a_{2i+1}| \int_0^{2\pi} \left|x^{(i+1)}(t)\right|^2 dt + \int_0^{2\pi} |g(t,x(t))x'(t)|\,dt \\
&+ \int_0^{2\pi} |p(t)x'(t)|\,dt.
\end{aligned}$$

A similar argument with Theorem 3.1, we can prove that there exists at least one anti-periodic solution of equation (3). □

Similar with Theorem 4.1, we can obtain the following result.

Theorem 4.2. *Assume that*
(H_6) *there is* $M \geq 0$ *such that*

$$|f(x)| \leq M, \quad \forall x \in \mathbb{R};$$

(H_7) $1 - \sum_{i=1}^{m-1} |a_{2i+1}| - M - \alpha > 0$
and the assumptions $(H_1), (H_2)$ *are true. Then there exists at least one anti-periodic solution of equation* (3).

When $g(t,x) = g(x)$, we can remove the assumption (H_2) in Theorem 4.1 and obtain the following result.

Theorem 4.3. *Assume that*
(H_8) $1 - \sum_{i=1}^{m-1} |a_{2i+1}| > 0$
and the assumptions $(H_1), (H_4)$ *are true. Then there exists at least one anti-periodic solution of equation* (3) $(g(t,x) = g(x))$.

Proof. We consider the auxiliary equation of (3)

$$\begin{aligned}
x^{(2m+1)} &= -\lambda \sum_{i=2}^{2m} a_i x^{(i)} - \lambda f(x)x' - \lambda g(x) + \lambda p(t) \\
&:= \lambda q_3\left(x^{(2m)}, \dots, x', x, t\right), \qquad (11)
\end{aligned}$$

where $\lambda \in [0,1]$. Obviously, $q_3\left(x^{(2m)}, \dots, x', x, t\right)$ is continuous.

Similar with the proof of Theorem 3.1, we need only prove that there exists a prior bound in $C^{2m+1,\pi}$, for the possible solution $x(t)$ of equation (11).

Similar with the proof of Theorem 4.1, multiplying equation (11) with $x'(t)$ and integrating it over $[0, 2\pi]$, we get

$$\begin{aligned} & \int_0^{2\pi} \left| x^{(m+1)}(t) \right|^2 dt \\ = \;& (-1)^{m+1} \lambda \sum_{i=1}^{m-1} a_{2i+1} \int_0^{2\pi} x^{(2i+1)}(t) x'(t) dt \\ & + (-1)^{m+1} \lambda \int_0^{2\pi} f(x(t)) \left(x'(t)\right)^2 dt \\ & + (-1)^{m+1} \lambda \int_0^{2\pi} g(x(t)) x'(t) dt + (-1)^m \lambda \int_0^{2\pi} p(t) x'(t) dt. \end{aligned}$$

Noting

$$\int_0^{2\pi} g(x(t)) x'(t) dt = \int_0^{2\pi} g(x(t)) d(x(t)) = \int_{x(0)}^{x(2\pi)} g(\tau) d\tau = 0$$

and by hypothesis (H_4), we get

$$\int_0^{2\pi} \left| x^{(m+1)}(t) \right|^2 dt \le \sum_{i=1}^{m-1} |a_{2i+1}| \int_0^{2\pi} \left| x^{(i+1)}(t) \right|^2 dt + \int_0^{2\pi} |p(t) x'(t)| \, dt.$$

A similar argument with Theorem 3.1, we can prove that there exists at least one anti-periodic solution of equation (3) $(g(t,x) = g(x))$. □

Similar with Theorem 4.3, we can obtain the following result.

Theorem 4.4. *Assume that*

(H_9) $1 - \sum_{i=1}^{m-1} |a_{2i+1}| - M > 0$

and the assumptions $(H_1), (H_6)$ *are true. Then there exists at least one anti-periodic solution of equation* (3) $(g(t,x) = g(x))$.

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