GRAY CURVATURE IDENTITIES FOR
ALMOST CONTACT METRIC MANIFOLDS

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Abstract. Alfred Gray introduced in [8] three curvature identities for the class of almost Hermitian manifolds. Using the warped product construction and the Boothby-Wang fibration we will give an equivalent of these identities for the class of almost contact metric manifolds.

1. Introduction

In [8], A. Gray introduced three curvature identities, K_i, $i = 1, 2, 3$, for almost Hermitian manifolds. These identities have important applications in geometry and topology (for example, some other geometric objects can be constructed on manifolds satisfying K_i, see e.g. [12, 13]).

A naturally arising question would be: *which are the equivalent identities for the almost contact manifolds?*

Concerning this, A. Bonome, L. M. Hervella, and I. Rozas defined in [4] $K_{i\varphi}$-curvature identities ($i = 1, 2, 3$) for an almost contact metric manifold $(M, \varphi, \xi, \eta, g)$ by using the usual Hermitian structure on $M \times \mathbb{R}$ (the product manifold). Gray proved in [8] that Kaehlerian manifolds satisfy K_i, $i = 1, 2, 3$ (curvature identities for almost Hermitian manifolds). In the same spirit, in [4] it is shown that cosymplectic manifolds satisfy $K_{i\varphi}$-identities.

It is known that both cosymplectic and Sasakian manifolds are natural odd-dimensional versions of Kaehlerian manifolds. We ask what happens if the analogy with the Kaehlerian manifolds is extended to Sasakian manifolds? We bear in mind the fact that a Riemannian manifold (M, g) is Sasakian if the holonomy group of the metric cone on M: $(C(M) = \mathbb{R}_+ \times M, \widetilde{g} = dt^2 + t^2 g)$

reduces to a subgroup of $U\left(\frac{m+1}{2}\right)$, i.e., $(C(M), \widetilde{g})$ is Kaehlerian. (Here $m = \dim M$.)

Inspired by this definition and by [4], we will take a different approach to Gray curvature identities for almost contact metric manifolds. For this purpose we will use the warped product and the Boothby-Wang fibration, two different constructions which lead us to the same result.

1.1. Gray curvature identities

An almost Hermitian manifold (M, J, g) is said to satisfy Gray curvature identities $(K1)$, $(K2)$ and respectively $(K3)$, if its Riemann Christoffel curvature tensor satisfies

$(K1)$ $R(X, Y, Z, W) = R(X, Y, JZ, JW)$,

$(K2)$ $R(X, Y, Z, W) = R(JX, JY, Z, W) + R(JX, Y, JZ, W) + R(JX, Y, Z, JW)$,

$(K3)$ $R(X, Y, Z, W) = R(JX, JY, JZ, JW)$

for all vector fields X, Y, Z, W on $\chi(M)$. Throughout this paper, the curvature tensor is defined by $R_{XY}Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]}Z$ for all $X, Y, Z \in \chi(M)$ while the Riemann Christoffel curvature tensor is given by $R(X, Y, Z, W) = -g(R_{XY}Z, W)$. The identity $(K1)$ is often called *Kaehler identity*. It is known the fact that there exist non Kaehler manifolds satisfying $(K1)$ identity (e.g. para-Kaehler manifolds, [11]).

2. Warped product manifolds

Singly warped products or simply *warped products* were first defined by Bishop and O'Neill in [1] in order to construct Riemannian manifolds with negative sectional curvature. Let (B, g_B) and (F, g_F) be Riemannian manifolds and let $b : B \longrightarrow (0, \infty)$ be a smooth function. The warped product $\widetilde{M} = B \times_b F$ is the product manifold $B \times F$ endowed with the metric $\widetilde{g} = g_B \oplus b^2 g_F$. More precisely, if $\pi : B \times F \longrightarrow B$ and $\tau : B \times F \longrightarrow F$ are natural projections, the metric g is defined by

$$\widetilde{g} = \pi^* g_B + (b \circ \pi)^2 \tau^* g_F.$$

The function b is called *warping function*. If $b \equiv 1$, then we have a product manifold.

If X, Y are tangent to B and Z, W tangent to F, then the Levi-Civita connection $\widetilde{\nabla}$ of $\widetilde{M}$ is given by

$$\begin{cases} \widetilde{\nabla}_X Y = \nabla_X^B Y, \quad \widetilde{\nabla}_X Z = X(\ln b)Z \\ \widetilde{\nabla}_Z W = \nabla_Z^F W - b^2 \ g_F(Z, W)\nabla^B(\ln b), \end{cases}$$

where ∇^B and ∇^F are the Levi-Civita connections on B, respectively on F, and $\nabla^B(\ln b)$ is the gradient of $\ln b$ with respect to the metric g_B.

Let $(M, \varphi, \xi, \eta, g)$ be an almost contact metric manifold. Consider the warped product manifold $\widetilde{M} = \mathbb{R}_+ \times_t M$, where $t > 0$ is the global coordinate of $\mathbb{R}_+$, i.e., the metric $\widetilde{g}$ of $\widetilde{M}$ is defined by

$$\widetilde{g} = dt^2 + t^2 g.$$

Let an endomorphism on $\chi(\widetilde{M})$ defined by

$$J\partial_t = -\frac{1}{t}\, \xi \, , \quad JX = \varphi X + t\eta(X)\partial_t, \quad \forall X \in \chi(M),$$

where $\partial_t = \frac{d}{dt}$. For $\widetilde{X} = (a, X) \in \chi(\widetilde{M})$, $a \in C^\infty(\mathbb{R}_+), X \in \chi(M)$ we have

$$J\widetilde{X} = J(a, X) = \left(t\eta(X), \varphi X - \frac{a}{t}\, \xi\right).$$

The proofs of the following propositions are straightforward.

Proposition 2.1. *J is an almost complex structure compatible with the metric $\widetilde{g}$.*

Proposition 2.2. *The Levi-Civita connection $\widetilde{\nabla}$ of $\widetilde{g}$ is given by:*
$$\begin{cases} \widetilde{\nabla}_{\partial_t}\partial_t = 0, \quad \widetilde{\nabla}_X\partial_t = \widetilde{\nabla}_{\partial_t}X = \frac{1}{t}X \\ \widetilde{\nabla}_X Y = \nabla_X Y - tg(X,Y)\partial_t, \quad X, Y \in \chi(M). \end{cases}$$

Proposition 2.3. *The covariant derivative of J is given by:*
$$\begin{cases} (\widetilde{\nabla}_{\partial_t}J)\partial_t = (0,0), \quad (\widetilde{\nabla}_{\partial_t}J)X = (0,0) \\ (\widetilde{\nabla}_X J)\partial_t = (0, -\frac{1}{t}(\nabla_X\xi + \varphi X)) \\ (\widetilde{\nabla}_X J)Y = (t((\nabla_X\eta)(Y) - g(X,\varphi Y)), (\nabla_X\varphi)Y - g(X,Y)\xi + \eta(Y)X). \end{cases}$$

Corollary 2.4. *J is parallel if and only if*
$$\begin{cases} (\widetilde{\nabla}_X\varphi)Y = g(X,Y)\xi - \eta(Y)X, \quad (\nabla_X\eta)(Y) = g(X,\varphi Y) \\ \nabla_X\xi = -\varphi X, \quad X, Y \in \chi(M) \end{cases}$$

i.e., $(\widetilde{M}, J, \widetilde{g})$ is Kaehler if and only if $(M, \varphi, \xi, \eta, g)$ is Sasakian.

Proposition 2.5. *For the curvature of the manifold $\widetilde{M}$ we have*
$$\begin{cases} \widetilde{R}(\partial_t, X)\partial_t = 0, \ \widetilde{R}(X,Y)\partial_t = 0, \ \widetilde{R}(\partial_t, X)Y = 0 \\ \widetilde{R}(X,Y)Z = R(X,Y)Z - g(Y,Z)X + g(X,Z)Y, \end{cases}$$

where $\widetilde{R}$ (respectively R) are the curvature tensors for $\widetilde{g}$ (respectively for g).
  Moreover, the following relations hold:
$$\begin{cases} \widetilde{R}(\partial_t, X)(J\partial_t) = 0, \ \widetilde{R}(\partial_t, X)(JY) = 0 \\ \widetilde{R}(X,Y)(J\partial_t) = -\frac{1}{t}[R(X,Y)\xi - \eta(Y)X + \eta(X)Y] \\ \widetilde{R}(X,Y)(JZ) = R(X,Y)(\varphi Z) - g(Y,\varphi Z)X + g(X,\varphi Z)Y. \end{cases}$$

In the following we compute expressions of the form $\widetilde{g}(\widetilde{R}(A, B)(JC), JD)$. The useful following expressions are obtained in the cases:

1. $\quad \widetilde{g}(\widetilde{R}(X, Y)(J\partial_t), JW)$
$= -t\left[g(R(X, Y)\xi, \varphi W) - \eta(Y)g(X, \varphi W) + \eta(X)g(Y, \varphi W)\right],$

2. $\quad \widetilde{g}(\widetilde{R}(X, Y)(JZ), J\partial_t)$
$= -t\left[\eta(R(X, Y)(\varphi Z)) - \eta(X)g(Y, \varphi Z) + \eta(Y)g(X, \varphi Z)\right],$

3. $\quad \widetilde{g}(\widetilde{R}(X, Y)(JZ), JW)$
$= t^2\left[g(R(X, Y)\varphi Z, \varphi W) - g(Y, \varphi Z)g(X, \varphi W) + g(X, \varphi Z)g(Y, \varphi W)\right].$

Theorem 2.6. $\widetilde{M}$ *is* $(K1)$ *if and only if*

$$R(X, Y, Z, W)$$
$$(1) \qquad = R(X, Y, \varphi Z, \varphi W) - g(X, \varphi Z)g(Y, \varphi W) + g(Y, \varphi Z)g(X, \varphi W)$$
$$- g(Y, Z)g(X, W) + g(X, Z)g(Y, W).$$

Proof. $\widetilde{M}$ is $(K1)$ if and only if $\widetilde{g}(\widetilde{R}(A, B)(JC), JD) = \widetilde{g}(\widetilde{R}(A, B)C, D)$ for all $A, B, C, D \in \chi(\widetilde{M})$. We have:

1. $\qquad \widetilde{g}(\widetilde{R}(X, Y)(J\partial_t), JW) = \widetilde{g}(\widetilde{R}(X, Y)\partial_t, W)$
$\Longrightarrow -t[g(R(X, Y)\xi, \varphi W) - \eta(Y)g(X, \varphi W) + \eta(X)g(Y, \varphi W)] = 0$
$\Longrightarrow g(\varphi(R(X, Y)\xi - \eta(Y)X + \eta(X)Y), W) = 0$ for every W
$\Longrightarrow R(X, Y)\xi - \eta(Y)X + \eta(X)Y \in \ker(\varphi).$

Thus,

$$(2) \qquad\qquad R(X, Y)\xi = \eta(Y)X - \eta(X)Y$$

since both terms have no ξ-component.

2. $\qquad \widetilde{g}(\widetilde{R}(X, Y)(JZ), J\partial_t) = \widetilde{g}(\widetilde{R}(X, Y)Z, \partial_t)$
$\Longrightarrow \quad \widetilde{g}\left(R(X, Y)(\varphi Z) - g(Y, \varphi Z)X + g(X, \varphi Z)Y, -\frac{1}{t}\,\xi\right)$
$= \widetilde{g}\left(R(X, Y)Z - g(Y, Z)X + g(X, Z)Y, \partial_t\right)$
$\Longrightarrow -\frac{1}{t}\,t^2\eta(R(X, Y)(\varphi Z) - g(Y, \varphi Z)X + g(X, \varphi Z)Y) = 0.$

Thus we have obtained,

$$(3) \qquad R(X, Y)(\varphi Z) - g(Y, \varphi Z)X + g(X, \varphi Z)Y \in \ker(\eta).$$

3. $\qquad \widetilde{g}(\widetilde{R}(X, Y)(JZ), JW) = \widetilde{g}(\widetilde{R}(X, Y)Z, W)$
$\Longrightarrow \quad \widetilde{g}(R(X, Y)(\varphi Z) - g(Y, \varphi Z)X + g(X, \varphi Z)Y, \varphi W + t\eta(W)\partial_t)$
$= \widetilde{g}(R(X, Y)Z - g(Y, Z)X + g(X, Z)Y, W).$

We obtain directly

$$g(R(X,Y)(\varphi Z), \varphi W) - g(Y, \varphi Z)g(X, \varphi W) + g(X, \varphi Z)g(Y, \varphi W)$$
$$= g(R(X,Y)Z, W) - g(Y, Z)g(X, W) + g(X, Z)g(Y, W)$$

and consequently

$$R(\varphi W, \varphi Z, X, Y) - R(W, Z, X, Y)$$
(4)
$$= g(Y, \varphi Z)g(X, \varphi W) - g(X, \varphi Z)g(Y, \varphi W)$$
$$+ g(X, Z)g(Y, W) - g(Y, Z)g(X, W).$$

Note that (2) implies (3) and (4) implies (2). $\qquad\square$

As consequences we have

$$R(\xi, Y, \xi, W) = g(Y, W),$$
$$R(\xi, Y, Z, W) = R(\xi, Y, \varphi Z, \varphi W) = 0,$$
$$R(X, Y, Z, W) - g(Y, W)g(X, Z) + g(X, W)g(Y, Z)$$
$$= R(X, Y, \varphi Z, \varphi W) - g(Y, \varphi W)g(X, \varphi Z) + g(X, \varphi W)g(Y, \varphi Z),$$

where X, Y, Z and W are orthogonal to ξ.

Definition 2.7. We say that an almost contact metric manifold satisfies **(G1)**-identity if its curvature tensor fulfills (1).

Proposition 2.8. *The curvature tensor of a Sasakian manifold satisfies* **(G1)** (see also Lemma 7.1 in [3]).

Proposition 2.9. *Any contact manifold satisfying* **(G1)** *is Sasakian.*

Proof. It is known (e.g. Proposition 7.6 from [3]) that a contact manifold is Sasakian if and only if $R(X, Y)\xi = \eta(Y)X - \eta(X)Y$ for all X and Y. $\qquad\square$

Back to the cone manifold $\widetilde{M}$. We give:

Theorem 2.10. $\widetilde{M}$ *is* $(K2)$ *if and only if*

$$R(X, Y, Z, W)$$
(5)
$$= R(\varphi X, Y, Z, \varphi W) + R(X, \varphi Y, Z, \varphi W) + R(X, Y, \varphi Z, \varphi W)$$
$$+ g(X, Z)\eta(W)\eta(Y) - g(Z, Y)\eta(X)\eta(W).$$

Proof. $\widetilde{M}$ is $(K2)$ if and only if

$$\widetilde{R}(A, B, C, D) = \widetilde{R}(JA, B, C, JD) + \widetilde{R}(A, JB, C, JD) + \widetilde{R}(A, B, JC, JD),$$

where A, B, C and D are arbitrary vector fields on $\widetilde{M}$.

Three cases are essential:
1) $A = \partial_t$, $B = Y$, $C = \partial_t$, $D = W$ which is trivial.
2) $A = \partial_t$, $B = Y$, $C = Z$, $D = W$.

One has:

$$\widetilde{R}(J\partial_t, Y, Z, JW) = -\frac{1}{t}\widetilde{R}(\xi, Y, Z, \varphi W),$$

$$\widetilde{R}(\partial_t, JY, Z, JW) = 0,$$

$$\widetilde{R}(\partial_t, Y, JZ, JW) = 0.$$

It follows that the right side is equal to:

$$-tg(\xi, R(Z, \varphi W)Y - g(\varphi W, Y)Z + g(Z, Y)\varphi W).$$

Since the left side vanishes, in this case we obtain

(6) $R(\xi, Y, Z, \varphi W) = \eta(Z)g(\varphi W, Y)$ for every $Y, Z, W \in \chi(M)$.

3) $A = X$, $B = Y$, $C = Z$, $D = W$. One has

$$\widetilde{R}(JX, Y, Z, JW) = \widetilde{R}(\varphi X, Y, Z, \varphi W),$$

$$\widetilde{R}(X, JY, Z, JW) = \widetilde{R}(X, \varphi Y, Z, \varphi W),$$

$$\widetilde{R}(X, Y, JZ, JW) = \widetilde{R}(X, Y, \varphi Z, \varphi W).$$

It follows that the right side is equal to

$$t^2[R(\varphi X, Y, Z, \varphi W) + R(X, \varphi Y, Z, \varphi W) + R(X, Y, \varphi Z, \varphi W)]+$$

$$+t^2[-g(\varphi W, \varphi Y)g(X, Z) + g(Z, Y)g(\varphi X, \varphi W)]$$

while the left side is equal to:

$$t^2 R(X, Y, Z, W) + t^2[-g(W, Y)g(X, Z) + g(Z, Y)g(X, W)],$$

and hence we have (5). Since (5) implies (6), we get the statement.   □

As consequences one has

$R(\xi, Y, \xi, W) = g(Y, W),$

$R(\xi, Y, Z, W) = 0,$

$R(X, Y, Z, W) = R(\varphi X, Y, Z, \varphi W) + R(X, \varphi Y, Z, \varphi W) + R(X, Y, \varphi Z, \varphi W)$

for all X, Y, Z, W orthogonal to ξ.

Definition 2.11. We say that an almost contact metric manifold satisfies **(G2)**-identity if its curvature tensor fulfills (5).

Let us focus our attention to the third identity of Gray.

Theorem 2.12. *The manifold $\widetilde{M}$ is $(K3)$ if and only if*

$$R(X, Y, Z, W)$$

(7) $= R(\varphi X, \varphi Y, \varphi Z, \varphi W) + g(X, Z)\eta(W)\eta(Y) - g(Z, Y)\eta(X)\eta(W)$

$\qquad + g(Y, W)\eta(X)\eta(Z) - g(X, W)\eta(Y)\eta(Z)$

for all $X, Y, Z, W \in \chi(M)$.

Proof. $\widetilde{M}$ is $(K3)$ if and only if $\widetilde{R}(A, B, C, D) = \widetilde{R}(JA, JB, JC, JD)$ for all $A, B, C, D \in \chi(\widetilde{M})$.

The essential cases are:

1) $A = \partial_t$, $B = Y$, $C = \partial_t$, $D = W$.

The left member vanishes and the right member is equal to $R(\xi, \varphi Y, \xi, \varphi W) - g(\varphi W, \varphi Y)$. We get

$$(8) \qquad\qquad R(\xi, \varphi Y, \xi, \varphi W) = g(\varphi W, \varphi Y).$$

2) $A = \partial_t$, $B = Y$, $C = Z$, $D = W$.

The left hand vanishes and on the right hand we have $R(\xi, \varphi Y, \varphi Z, \varphi W)$. We get

$$(9) \qquad\qquad R(\xi, \varphi Y, \varphi Z, \varphi W) = 0.$$

3) $A = X$, $B = Y$, $C = Z$, $D = W$.

Again, the left part is equal to

$$t^2[R(X, Y, Z, W) - g(Z, X)g(W, Y) + g(Y, Z)g(X, W)]$$

and the right part is equal to

$$t^2[R(\varphi X, \varphi Y, \varphi Z, \varphi W) - g(\varphi W, \varphi Y)g(\varphi X, \varphi Z) + g(\varphi Y, \varphi Z)g(\varphi X, \varphi W)].$$

Hence (7) is proved. Note that (7) implies both (8) and (9). $\qquad\square$

Consequently

$$R(\xi, Y, \xi, W) = g(Y, W),$$
$$R(\xi, Y, Z, W) = 0,$$
$$R(X, Y, Z, W) = R(\varphi X, \varphi Y, \varphi Z, \varphi W)$$

for all $X, Y, Z, W \in \chi(M)$ orthogonal to ξ.

Definition 2.13. We say that an almost contact metric manifold satisfies $(\mathbf{G3})$-identity if its curvature fulfills the relation (7).

3. The Boothby-Wang fibration

In this section we use the Boothby-Wang fibration [5] in order to strengthen previous results.

Let M be a $(2n+1)$-dimensional smooth manifold. A *contact form* on M is a 1-form η satisfying

$$\eta \wedge (d\eta)^n \neq 0.$$

We say that η endows M with a *contact structure*. It is clear that η induces an orientation on M and hence there is a global non vanishing vector field ξ on M so that $\eta(\xi) = 1$. If ξ is *regular* in the sense of Palais (see [10]), then the contact structure (and also M) is called *regular*. If moreover M is compact, one can consider the space of all orbits of ξ, i.e., $N = M_{/\xi}$ thus obtaining a smooth manifold. We have:

Theorem A ([5]). *Let (M, η) be a compact, regular, contact manifold. Then M is a principal circle bundle over N and η is a connection form of this bundle. The curvature form Θ of η defines a symplectic form on N.*

This fibration $S^1 \longrightarrow M \xrightarrow{\pi} N$ is called the *Boothby-Wang fibration*.

Let Ω the symplectic 2-form of N. We denote by G the associated metric, i.e., $\Omega(X, Y) = G(X, JY)$ with J the almost complex structure.

In the following, by $X^\uparrow$ we denote the lift of a vector field $X \in \chi(N)$. $X^\uparrow$ is a horizontal vector field of M. On M a $(1,1)$ tensor field φ can be defined, namely

$$\varphi X^\uparrow = (JX)^\uparrow, \qquad \varphi \xi = 0.$$

We can easily see that

$$\varphi^2 = -I + \eta \otimes \xi.$$

In this way, (φ, ξ, η) becomes an almost contact structure. The metric G can be lifted and hence one defines g on M as follows:

$$g = \pi^* G + \eta \otimes \eta.$$

The metric g is compatible with the contact structure and $\xi = \eta^\#$.

Without loss of generality one can suppose $d\eta = \pi^*\Omega$ and thus we have

$$g(X^\uparrow, \varphi Y^\uparrow) = G(X, JY) \circ \pi = \Omega(X, Y) \circ \pi = \pi^*\Omega(X^\uparrow, Y^\uparrow) = d\eta(X^\uparrow, Y^\uparrow).$$

In this way, (φ, ξ, η, g) becomes a contact metric structure on M.

If the symplectic structure of N derives from a Kaehlerian structure (J, G), the obtained structure on M is Sasakian (i.e., contact and normal manifold). See e.g. [3]. But generally, a symplectic structure need not come from a Kaehlerian one. Yet, one can always find an almost Kaehlerian structure inducing it. In this case, the contact structure on the total space of a Boothby-Wang fibration is K-contact, i.e., the vector field ξ is Killing, namely $\mathcal{L}_\xi g = 0$. It easily follows that the integral curves of ξ are geodesics.

It is easy to prove the relation

$$[X^\uparrow, Y^\uparrow] = [X, Y]^\uparrow - 2G(X, JY)\xi$$

for all $X, Y \in \chi(N)$.

Denote by $\overset{M}{\nabla}$ and $\overset{N}{\nabla}$ the Levi-Civita connections on M and N, respectively. We have

$$g(\overset{M}{\nabla}_{X^\uparrow} Y^\uparrow, Z^\uparrow) \circ \pi = G(\overset{N}{\nabla}_X Y, Z)$$

for any $X, Y, Z \in \chi(N)$. For the vertical part we shall compute $\eta(\overset{M}{\nabla}_{X^\uparrow} Y^\uparrow)$:

$$\begin{aligned}
2g(\nabla^M_{X^\uparrow} Y^\uparrow, \xi) &= X^\uparrow g(Y^\uparrow, \xi) + Y^\uparrow g(X^\uparrow, \xi) - \xi g(X^\uparrow, Y^\uparrow) + g([X^\uparrow, Y^\uparrow], \xi) \\
&\quad + g([\xi, X^\uparrow], Y^\uparrow) + g(X^\uparrow, [\xi, Y^\uparrow]) \\
&= \eta([X^\uparrow, Y^\uparrow]) - (\mathcal{L}_\xi g)(X^\uparrow, Y^\uparrow) \\
&= -2d\eta(X^\uparrow, Y^\uparrow).
\end{aligned}$$

We obtain

$$\eta(\overset{M}{\nabla}_{X^\uparrow} Y^\uparrow) \circ \pi = -G(X, JY).$$

In the following, we will ignore π, due to the isomorphism between the horizontal distribution of $T(M)$ and $T(N)$. Hence

$$\overset{M}{\nabla}_{X^\uparrow} Y^\uparrow = (\overset{N}{\nabla}_X Y)^\uparrow - G(X, JY)\xi.$$

In the same way, one can show

$$\overset{M}{\nabla}_{X^\uparrow} \xi = -\varphi X^\uparrow.$$

Denote by R^M and R^N the curvature tensors of M and N, respectively. Then

$$R^M(X^\uparrow, Y^\uparrow)Z^\uparrow$$
$$= \left(R^N(X, Y)Z\right)^\uparrow + g(Y^\uparrow, \varphi Z^\uparrow)\varphi X^\uparrow - g(X^\uparrow, \varphi Z^\uparrow)\varphi Y^\uparrow - 2g(x^\uparrow, \varphi Y^\uparrow)\varphi Z^\uparrow$$
$$+ \left\{ g\left(X^\uparrow, \left(\overset{M}{\nabla}_{Y^\uparrow} \varphi\right)Z^\uparrow\right) - g\left(Y^\uparrow, \left(\overset{M}{\nabla}_{X^\uparrow} \varphi\right)Z^\uparrow\right) \right\} \xi$$

and hence

$$R^M(W^\uparrow, Z^\uparrow, X^\uparrow, Y^\uparrow) = R^N(W, Z, X, Y) \circ \pi - 2g(X^\uparrow, \varphi Y^\uparrow)g(W^\uparrow, \varphi Z^\uparrow)$$
$$+ g(Y^\uparrow, \varphi Z^\uparrow)g(W^\uparrow, \varphi X^\uparrow) - g(X^\uparrow, \varphi Z^\uparrow)g(W^\uparrow, \varphi Y^\uparrow).$$

Suppose that the base manifold N satisfies Gray identities. What are the corresponding curvature identities for the upstairs manifold M?

If N is (K_1), then

$$R^M(X^\uparrow, Y^\uparrow, \varphi Z^\uparrow, \varphi W^\uparrow) - R^M(X^\uparrow, Y^\uparrow, Z^\uparrow, W^\uparrow)$$
$$= -g(Y^\uparrow, W^\uparrow)g(Z^\uparrow, X^\uparrow) - g(Y^\uparrow, \varphi W^\uparrow)g(Z^\uparrow, \varphi X^\uparrow)$$
$$+ g(X^\uparrow, W^\uparrow)g(Z^\uparrow, Y^\uparrow) + g(X^\uparrow, \varphi W^\uparrow)g(Z^\uparrow, \varphi Y^\uparrow).$$

If N is (K_2), then

$$R^M(\varphi X^\uparrow, Y^\uparrow, Z^\uparrow, W^\uparrow) + R^M(X^\uparrow, \varphi Y^\uparrow, Z^\uparrow, W^\uparrow)$$
$$+ R^M(X^\uparrow, Y^\uparrow, \varphi Z^\uparrow, W^\uparrow) + R^M(X^\uparrow, Y^\uparrow, Z^\uparrow, \varphi W^\uparrow) = 0.$$

If N is (K_3), then

$$R^M(\varphi X^\uparrow, \varphi Y^\uparrow, \varphi Z^\uparrow, \varphi W^\uparrow) - R^M(X^\uparrow, Y^\uparrow, Z^\uparrow, W^\uparrow) = 0.$$

These relations are exactly the defined Gray identities for almost contact metric manifolds for vector fields orthogonal to ξ.

4. Properties and examples

In their paper [9], D. Janssens and L. Vanhecke have studied curvature tensors for almost contact metric structures and defined *almost $C(\alpha)$-manifolds*, namely those almost contact metric manifolds whose curvature tensor satisfies the following property:

$$\exists \alpha \in \mathbb{R} \text{ such that for all } X, Y, Z, W \in \chi(M)$$

$$R(X, Y, Z, W) = R(X, Y, \varphi Z, \varphi W) + \alpha \left\{ g(X, Z)g(Y, W) - g(X, W)g(Y, Z) \right.$$
$$\left. - g(X, \varphi Z)g(Y, \varphi W) + g(X, \varphi W)g(Y, \varphi Z) \right\}.$$

(In the original paper [9] different signs appear due the fact that the Riemann Christoffel curvature tensor is defined with the opposite sign.) This means that manifolds satisfying the first Gray identity $(K_{1\varphi})$ in the sense of Bonome et al. are in fact $C(0)$-manifolds, while that manifolds satisfying $(G1)$ are $C(1)$-manifolds. Note that cosymplectic, Sasakian and Kenmotsu manifolds are respectively $C(0)$, $C(1)$ and $C(-1)$ manifolds (see Theorem 2.3, in [9]).

Let us come back to Gray identities for an almost Hermitian manifold.

It is known that $K1 \Rightarrow K2 \Rightarrow K3$ (see [8], §5). Consequently we have:

Proposition 4.1. *For a class $\mathcal{L}$ of almost contact metric manifolds, denote by $\mathcal{L}_i$ the subclass of manifolds whose curvature satisfies Gi, $i = 1, 2, 3$. Then we have the following inclusions*

$$\mathcal{L}_1 \subseteq \mathcal{L}_2 \subseteq \mathcal{L}_3 \subseteq \mathcal{L}.$$

As Gray remarked for Kaehlerian manifolds, we can say that *as i decreases, a manifold in $\mathcal{L}_i$ resembles Sasakian manifold more closely.*

Proposition 4.2. *Let $(M, \varphi, \xi, \eta, g)$ be a K-contact manifold satisfying $G1$ curvature identity. Then the manifold M is Sasakian.*

Proof. By using Proposition 7.5 in [3, p. 94], a K-contact manifold whose curvature satisfies $R_{XY}\xi = \eta(Y)X - \eta(X)Y$ is Sasakian. But this last relation is a consequence of G1 identity. See also Proposition 2.9. □

Proposition 4.3. *Let M be a contact metric manifold for which ξ belongs to the (κ, μ)-nullity distribution, namely its curvature satisfies*

$$R_{XY}\xi = \kappa \left(\eta(Y)X - \eta(X)Y \right) + \mu \left(\eta(Y)hX - \eta(X)hY \right),$$

where $h = \frac{1}{2}\mathcal{L}_\xi\varphi$ and κ, μ are constants. Suppose M satisfies $(G1)$ identity. Then M is Sasakian.

Proof. If M is $(G1)$, then $R_{XY}\xi = \eta(Y)X - \eta(X)Y$ for all $X, Y \in \chi(M)$. Combining with the fact that ξ belongs to the (κ, μ)-nullity distribution we obtain

$$(\kappa - 1)(\eta(Y)X - \eta(X)Y) + \mu(\eta(Y)hX - \eta(X)hY) = 0$$

for all $X, Y \in \chi(M)$. If $\mu \neq 0$ this implies $hY = \frac{1-\kappa}{\mu} Y$ for all $Y \in \ker(\eta)$. We know that h anticommutes with φ and hence one gets $\kappa = 1$. But using [3, Theorem 7.7, p. 103], it follows that M is a Sasakian manifold. If $\mu = 0$ we immediately have $\kappa = 1$. $\qquad\square$

Proposition 4.4. *Let $(M, \varphi, \eta, \xi, g)$ be a contact metric manifold satisfying* (G3) *identity. Then M is K-contact.*

Proof. Choose a φ-adapted local orthonormal frame on M, namely $\{X_i, \varphi X_i, \xi\}$, $i = 1, \ldots, n$. Since M is (G3) the relation $R(X, \xi, Y, \xi) = g(X, Y)$ holds for all $X, Y \in \ker(\eta)$. Taking $X = Y = X_i$ (respectively $X = Y = \varphi X_i$) one immediately obtains $Ric(\xi, \xi) = 2n$, where Ric is the Ricci tensor on M. Now we use the fact that *a contact metric manifold is K-contact if and only if the Ricci tensor in the direction of the characteristic vector field ξ is equal to $2n$* ([2, Theorem, p. 65]). $\qquad\square$

As consequence, we can state:

Proposition 4.5. *If $(M, \varphi, \eta, \xi, g)$ is a contact metric manifold satisfying* (G1) *identity, then it is Sasakian.*

Proof. The statement follows from Propositions 4.2 and 4.4. $\qquad\square$

4.1. The generalized Heisenberg group $H(p, 1)$

It is defined as the set of matrices of real numbers having the form

$$a = \begin{bmatrix} 1 & A & c \\ 0 & I_p & {}^tB \\ 0 & 0 & 1 \end{bmatrix},$$

where I_p is the identity $p \times p$ matrix, $A = (a_1, \ldots, a_p)$, $B = (b_1, \ldots, b_p) \in \mathbb{R}^p$ and $c \in \mathbb{R}$. (Cf. [7].) $H(p, 1)$ is a connected, simply connected nilpotent Lie group of dimension $2p + 1$. We will consider $p = 2$. A global system of coordinates (x^1, x^2, y^1, y^2, z) on $H(2, 1)$ is defined by $x^i(a) = a_i$, $y^i(a) = b_i$ for $i = 1, 2$ and $z(a) = c$. The global vector fields

$$X_i = 2\frac{\partial}{\partial x^i}, \ Y_i = 2\left(\frac{\partial}{\partial y^i} + x^i\frac{\partial}{\partial z}\right) \text{ for } i = 1, 2, \text{ and } \xi = 2\frac{\partial}{\partial z}$$

are left invariant. We consider $\eta = \frac{1}{2}(dz - x^1 dy^1 - x^2 dy^2)$ and the metric

$$g = \frac{1}{4}(dx^1 \otimes dx^1 + dx^2 \otimes dx^2 + dy^1 \otimes dy^1 + dy^2 \otimes dy^2) + \eta \otimes \eta.$$

By direct computations we obtain that $d\eta = -\frac{1}{2}(dx^1 \wedge dy^1 + dx^2 \wedge dy^2)$ and ξ is the characteristic vector field, namely $\eta(\xi) = 1$ and $i_\xi d\eta = 0$. Moreover, the basis defined above is orthonormal: $g(X_i, X_j) = g(Y_i, Y_j) = \delta_{ij}$, $g(\xi, \xi) = 1$ and $g(X_i, Y_j) = g(X_i, \xi) = g(Y_i, \xi) = 0$. One has $[X_i, Y_i] = 2\xi$ for $i = 1, 2$

and the other brackets are equal to zero. Therefore it is easy to verify that the Levi-Civita connection is given by the following formulas:

$$\nabla_\xi X_i = -Y_i = \nabla_{X_i}\xi,$$

$$\nabla_\xi Y_i = X_i = \nabla_{Y_i}\xi,$$

$$\nabla_{X_i} Y_i = -\nabla_{Y_i} X_i = \xi$$

for $i = 1, 2$, the other derivatives being zero. We compute also the Riemann-Christoffel curvature tensor field:

$$R(X_1, X_2, Y_1, Y_2) = -1, \qquad R(X_1, Y_2, X_2, Y_1) = -1,$$

$$R(X_1, Y_1, X_2, Y_2) = -2, \qquad R(X_i, Y_i, X_i, Y_i) = -3,$$

$$R(X_i, \xi, X_i, \xi) = 1, \qquad R(Y_i, \xi, Y_i, \xi) = 1 \quad \text{for} \quad i = 1, 2.$$

The other values are zero or can be obtained from these ones.

Define φ by:

$$\varphi X_1 = \cos\theta Y_1 + \sin\theta Y_2, \ \ \varphi X_2 = \varepsilon\sin\theta Y_1 - \varepsilon\cos\theta Y_2,$$

$$\varphi Y_1 = -\cos\theta X_1 - \varepsilon\sin\theta X_2, \ \ \varphi Y_2 = -\sin\theta X_1 + \varepsilon\cos\theta X_2, \ \ \varphi\xi = 0, \ \ \varepsilon = \pm 1.$$

Hence $(H(2,1), \varphi, \xi, \eta, g)$ is an almost contact metric manifold.

Proposition 4.6. *The structure is quasi Sasakian. Moreover, it is K-contact if and only if $\varepsilon = -1$ and $\theta = 0$. In this case $H(2,1)$ becomes a Sasakian manifold.*

Proof. It can be proved that $\nabla_\xi\varphi = 0$. Then for every $X, Y \in \chi(M)$ we have

$$g(\nabla_X\xi, Y) + g(\nabla_Y\xi, X) = 0,$$

which means that ξ is Killing. The following relation holds on $H(2,1)$:

$$g(Y, (\nabla_X\varphi)Z) = \eta(Y)(\nabla_{\varphi X}\eta)(Z) + \eta(Z)(\nabla_Y\eta)(\varphi X)$$

which characterizes quasi Sasakian manifolds. Direct computations yield the second part of the statement. $\qquad\square$

Moreover, we can give the following:

Proposition 4.7. *On $H(2,1)$*
  (1) *the G3 identity holds if and only if $\cos\theta = 0$ or $\sin\theta = 0$ or $\varepsilon = 1$;*
  (2) *the G2 identity holds if and only if $\varepsilon = 1$ or $\sin\theta = 0$;*
  (3) *the G1 identity holds if and only if $\varepsilon = -1$ and $\sin\theta = 0$.*

Proof. Straightforward computations. $\qquad\square$

4.2. Other examples

Let $(N, \bar{g}, J)$ be an almost Hermitian manifold. Consider the warped product manifold $M = \mathbb{R} \times_f N$, where $f = f(\theta)$ is the warping function and θ is the global parameter on $\mathbb{R}$. Denote by $g = d\theta^2 + f^2(\theta)\bar{g}$ the Riemannian metric on M. Define the global vector field $\xi = \frac{\partial}{\partial \theta}$ and the 1-form $\eta = d\theta$. Define also the $(1,1)$ tensor field φ by $\varphi X = JX$ if X is tangent to N and $\varphi \frac{\partial}{\partial \theta} = 0$. Thus (φ, ξ, η, g) is an almost contact metric structure on M. If $\bar{\nabla}$ and ∇ are the Levi-Civita connections on N, respectively on M, we have

$$\nabla_\xi X = \nabla_X \xi = \frac{f'}{f} X, \quad \nabla_\xi \xi = 0, \quad \nabla_X Y = \bar{\nabla}_X Y - ff'\bar{g}(X,Y)\xi$$

for all X, Y tangent to N.

The Riemann Christoffel curvature tensor is given by

$$R(W, \xi, X, Y) = 0, \quad R(W, \xi, X, \xi) = -\frac{f''}{f} g(X, W),$$

$$R(W, Z, X, Y) = f^2 \left[\bar{R}(W, Z, X, Y) + (f')^2 \left(\bar{g}(X, Z)\bar{g}(Y, W) - \bar{g}(Y, Z)\bar{g}(X, W) \right) \right].$$

In order to have one of the three curvature identities we directly have

$$\frac{f''}{f} = -1$$

which implies that $f = \alpha \cos\theta + \beta \sin\theta$ with α and β real constants. Now one can state the following:

Proposition 4.8. *The manifold M is $G2$ (respectively $G3$) if and only if the almost Hermitian manifold N is $K2$ (respectively $K3$).*

Proof. One has the following relations:

$$R(\varphi W, Z, X, \varphi Y) + R(W, \varphi Z, X, \varphi Y) + R(W, Z, \varphi X, \varphi Y)$$
$$= f^2 \left[\bar{R}(JW, Z, X, JY) + \bar{R}(W, JZ, X, JY) + \bar{R}(W, Z, JX, JY) \right]$$
$$+ (f')^2 f^2 \left(\bar{g}(X, Z)\bar{g}(Y, W) - \bar{g}(Y, Z)\bar{g}(X, W) \right)$$

and

$$R(W, Z, X, Y) - R(\varphi W, \varphi Z, \varphi X, \varphi Y) = f^2 \left[\bar{R}(W, Z, X, Y) - \bar{R}(JW, JZ, JX, JY) \right].$$

Hence the statement. $\qquad \square$

Remark 4.9. If $\dim N \geq 4$, then the manifold M cannot be $G1$.

Proof. Suppose M satisfies $G1$ identity. A straightforward computation gives

$$\bar{R}(W, Z, JX, JY) - \bar{R}(W, Z, X, Y)$$
$$= \left(1 + (f')^2 \right) \left[\bar{g}(JX, W)\bar{g}(JY, Z) - \bar{g}(JX, Z)\bar{g}(JY, W) \right.$$
$$\left. + \bar{g}(Y, W)\bar{g}(X, Z) - \bar{g}(Y, Z)\bar{g}(X, W) \right].$$

Since f depends on θ (and it is not linear) while $\bar{g}$ and $\bar{R}$ do not, it follows that N is $K1$ and

$$\bar{g}(JX,W)\bar{g}(JY,Z)-\bar{g}(JX,Z)\bar{g}(JY,W)+\bar{g}(Y,W)\bar{g}(X,Z)-\bar{g}(Y,Z)\bar{g}(X,W) = 0$$

for all X, Y, Z, W tangent to N. This yields

$$(10) \qquad \bar{g}(JY,Z)JX - \bar{g}(JX,Z)JY + \bar{g}(X,Z)Y - \bar{g}(Y,Z)X = 0.$$

If $\dim N \geq 4$ we can choose X and Y so that X, Y, JX and JY are linearly independent, so, the previous equality is impossible. $\square$

Example 4.10. On $\mathbb{R}^4$ consider the global coordinates x, y, u and v respectively. Let $z \in I = (0, \pi/2)$. Define the warped product $M = I \times_f \mathbb{R}^4$, where the warping function $f : I \longrightarrow \mathbb{R}$ is given by $f(z) = \cos z$. More precisely, consider the Riemannian metric $g = dz^2 + \cos^2 z \left(dx^2 + dy^2 + du^2 + dv^2 \right)$. Let us define the almost contact structure by: $\xi = \partial_z$, $\eta = dz$, $\varphi\partial_x = \partial_y$, $\varphi\partial_y = -\partial_x$, $\varphi\partial_u = \partial_v$, $\varphi\partial_v = -\partial_u$ and $\varphi\partial_z = 0$. Then M is $G2$ but not $G1$.

The same result holds if on M consider the warped metric

$$g = dz^2 + \sin^2 z \left(dx^2 + dy^2 + du^2 + dv^2 \right).$$

This kind of structure is called *sine-cone* and gives way to construct many geometric objects (e.g. nearly Kaehler structures starting from a 5-dimensional Sasaki Einstein manifold). Cf. [6].

Proposition 4.11. *Let N be a surface and consider the warped product manifold $M = I \times N$, where I is an open interval. Then M satisfies $G1$.*

Proof. Being a surface, N is automatically Kaehler. The almost contact structure is defined as in the beginning of this section. The statement follows from the fact that a Kaehler manifold is $K1$ and the equation (10) is satisfied in dimension 2. $\square$

4.3. Hypersurfaces of almost Hermitian manifolds

Let $(\widetilde{M}, J, \widetilde{g})$ a $(2n+2)$-dimensional Kaehler manifold, and let M be a totally umbilical (real) hypersurface in $\widetilde{M}$. Denote by N the unit normal on M and let A, h be the Weingarten operator and the scalar-valued second fundamental form, respectively. As M is totally umbilical, we have that $AX = \beta X$ for all X tangent to M, with $\beta \in C^\infty(M)$.

It is well known the fact that on M we can define an almost contact metric structure (see e.g. [3]). More precisely, we take $\xi = -JN$ and for $X \in \chi(M)$ we decompose JX as:

$$JX = \varphi X + \eta(X)N.$$

Let g be the restriction of the metric $\widetilde{g}$ on M. Denote by $\widetilde{\nabla}$ (respectively ∇) the Levi-Civita connection on $\widetilde{M}$ (respectively on M). Then, by the formula of Gauss, one has

$$\widetilde{\nabla}_X \xi = \nabla_X \xi + h(X,\xi)N.$$

On the other hand, we have $\widetilde{\nabla}_X \xi = -J\widetilde{\nabla}_X N = JAX = \varphi AX + \eta(AX)N$. Hence
$$\nabla_X \xi = \varphi AX \quad \text{and} \quad h(X,\xi) = \eta(AX).$$
Suppose now that M satisfies the $(G3)$ identity. This implies

(11) $$R(X,\xi,Y,\xi) = g(X,Y) \quad \forall X,Y \in \ker(\eta).$$

We should compute $R(X,\xi)\xi = \nabla_X \nabla_\xi \xi - \nabla_\xi \nabla_X \xi - \nabla_{[X,\xi]}\xi$. Since M is totally umbilical, we have $\nabla_X \xi = \beta\varphi X$. Thus $\nabla_\xi \xi = 0$. Then
$$\nabla_\xi \nabla_X \xi = \xi(\beta)\varphi X + \beta(\nabla_\xi \varphi)X + \beta\varphi\nabla_\xi X.$$

But
$$\nabla_\xi X = \beta\varphi X - [X,\xi]$$

and so
$$\nabla_\xi \nabla_X \xi = \xi(\beta)\varphi X + \beta(\nabla_\xi \varphi)X + \beta^2\varphi^2 X - \beta\varphi[X,\xi].$$

It follows that
$$R(X,\xi)\xi = -\xi(\beta)\varphi X - \beta(\nabla_\xi \varphi)X + \beta^2 X.$$

Now, due to the fact M is Kaehler, we have
$$\widetilde{\nabla}(JY) = J\widetilde{\nabla}_X Y = J(\nabla_X Y + h(X,Y)N) = \varphi\nabla_X Y + \eta(\nabla_X Y)N - h(X,Y)\xi.$$

On the other hand
$$\widetilde{\nabla}(JY) = \widetilde{\nabla}_X(\varphi Y + \eta(Y)N) = \nabla(\varphi Y) + h(X,\varphi Y)N + X\eta(Y)N - \eta(Y)\beta X.$$

Identifying the tangent and the normal parts of $\widetilde{\nabla}(JY)$ we obtain

(12) $$(\nabla_X \varphi)Y = \beta\eta(Y)X - \beta g(X,Y)\xi,$$
$$(\nabla_X \eta)(Y) = -\beta g(X,\varphi Y),$$

respectively.

Putting $X = \xi$ in (12) we have $(\nabla_\xi \varphi)Y = \beta\eta(Y)\xi - \beta g(\xi,Y)\xi = 0$ which implies
$$\nabla_\xi \varphi = 0.$$

Then
$$R(X,\xi)\xi = -\xi(\beta)\varphi X + \beta^2 X.$$

From (11) we have
$$g(\beta^2 X - X - \xi(\beta)\varphi X, Y) = 0, \quad \forall Y \in \ker(\eta).$$

As X and φX are linearly independent (and belong to $\ker(\eta)$), we obtain $\beta = \pm 1$.

Consequently
$$AX = \pm X \quad \text{and} \quad h(X,Y) = \pm g(X,Y).$$

For $\beta = -1$ it follows that
$$(\nabla_X \varphi)Y = g(X,Y)\xi - \eta(Y)X.$$

According to Theorem 6.14 in [3] this implies that M is Sasakian.

Proposition 4.12. *Let M be a totally umbilical hypersurface of a Kaehler manifold $\widetilde{M}$ endowed with the usual almost contact metric structure. If M satisfies $G3$ identity, then M is a Sasakian manifold and hence M satisfies all Gi for $i = 1, 2, 3$.*

More generally, if the second fundamental form of M is given by

$$h(X, Y) = \lambda\eta(X)\eta(Y) + \mu g(X, Y), \quad \forall X, Y \in \chi(M)$$

with λ and μ smooth functions on M, i.e., M is *totally quasi umbilical*, and if M satisfies $(G3)$ identity, then it is Sasakian. As consequence, there is no *cylindrical submanifold* satisfying $(G3)$ and whose second fundamental form is $h(X, Y) = \lambda\eta(X)\eta(Y)$.

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