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# ON GENERALIZED $(\sigma,\tau)$-DERIVATIONS II

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Abstract. This paper continues a line investigation in [1]. Let $A$ be a $K$-algebra and $M$ an $A/K$-bimodule. In [5] Hamaguchi gave a necessary and sufficient condition for $gDer(A,M)$ to be isomorphic to $BDer(A,M)$. The main aim of this paper is to establish similar relationships for generalized $(\sigma,\tau)$-derivations.

## 1. Introduction

The notion of generalized derivations on a ring $A$ which was introduced by M. Brešar [2] is related to a derivation of $A$. We denote by $BDer(A,M)$ the set of Brešar generalized derivations from a $K$-algebra $A$ to $A/K$-bimodule $M$ over a commutative ring $K$. In [9], A. Nakajima defined another type of generalized derivations without using derivations, and gave some categorical properties of that generalized derivations. We denote the set of generalized derivations in the sense of Nakajima from $A$ to $M$ by $gDer(A,M)$. When $A$ has an identity element, these two notions coincide. The results in [10] were extended to generalized $(\sigma,\tau)$-derivations, $(\sigma,\tau)$-Jordan derivations, $(\sigma,\tau)$-Lie derivations in [1]. In [5], N. Hamaguchi gave a necessary and sufficient condition for $gDer(A,M)$ to be isomorphic to $BDer(A,M)$ as $K$-modules.

In this paper, we introduce a notion of Brešar generalized Lie $(\sigma,\tau)$-derivation and extend some properties in [5] to generalized $(\sigma,\tau)$-derivations and Brešar generalized $(\sigma,\tau)$-derivations.

Furthermore we give a necessary and sufficient condition for any generalized Lie $(\sigma,\tau)$-derivation (res. generalized Jordan $(\sigma,\tau)$-derivation) to be a Brešar generalized Lie $(\sigma,\tau)$-derivation (res. Brešar generalized Jordan $(\sigma,\tau)$-derivation).

Finally we also refer to the extendability of these generalized $(\sigma,\tau)$-derivations to a ring having a unit element.

In the following $A$ be an algebra over a commutative ring $K$ and let $M$ be an $A/K$-bimodule, i.e., $M$ is a left and right $A$-module and moreover the relations

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$s(mt) = (sm)t$, $a(sm) = (as)m$, and $am = ma$ hold for all $s, t \in A$, $a \in K$, and $m \in M$.

## 2. Preliminaries

In this section we remind some definitions and lemmas from [1], [9], [10] and [11], which we shall use frequently in the other sections.

A $K$-homomorphism $d : A \longrightarrow M$ is called a $K$-*derivation* if $d(ab) = d(a)b + ad(b)$ for all $a, b \in A$. We denote the set of $K$-derivations from $A$ to $M$ by $Der(A, M)$. In particular, the map $d$ is called *Jordan derivation* if

$$d(a^2) = d(a)a + ad(a)$$

for all $a \in A$. We denote the set by $JDer(A, M)$.

Let $d : A \longrightarrow M$ be a $K$-homomorphism. $d$ is called *Lie derivation* if $d([a, b]) = [d(a), b] + [a, d(b)]$ for all $a, b \in A$. We denote by Lie$Der(A, M)$ the set of Lie derivations from $A$ to $M$. Here $[a, b] = ab - ba$.

Let $d : A \longrightarrow M$ be a $K$-derivation and $f : A \longrightarrow M$ a $K$-homomorphism. Then a pair $(f, d)$ is said to be a *Brešar generalized derivation* or *generalized d-derivation* if $f(ab) = f(a)b + ad(b)$ for all $a, b \in A$. We denote by $BDer(A, M)$ the set of Brešar generalized derivation from $A$ to $M$.

$f : A \longrightarrow M$ is called a *Brešar generalized Jordan derivation* if

$$f(a^2) = f(a)a + ad(a)$$

for all $a \in A$. Here $d$ is a Jordan derivation. We denote the set of Brešar generalized Jordan derivations from $A$ to $M$ by $BJDer(A, M)$.

The $K$-homomorphism $f : A \longrightarrow M$ is said to be *Brešar generalized Lie derivation* if the identity

$$f([a, b]) = [f(a), b] + [a, d(b)]$$

holds for all $a, b \in A$. Here $d$ is a Lie derivation. We denote the set by $BLieDer(A, M)$.

A pair $(f, m)$, with $f : A \longrightarrow M$ a $K$-module homomorphism and $m \in M$, is called a *generalized derivation* provided that the relation

$$f(st) = f(s)t + sf(t) + smt$$

holds for $s, t \in A$. We denote by $gDer(A, M)$ the set of generalized derivations from $A$ to $M$.

In [9] Nakajima proved a series of categorical properties of the set of all generalized derivations from a $K$-algebra $A$ into an $A/K$-bimodule $M$. In [10] he introduced the following notions. Let $A$, $M$ and $f$ be as about and assume that $m \in M$. The pair $(f, m)$ is called a *generalized Jordan derivation* if

$$f(a^2) = f(a)a + af(a) + ama$$

for all $a \in A$.

We now denote the set of generalized Jordan derivations from $A$ to $M$ by $gDer(A, M)$. The pair $(f, m)$ is called a *generalized Lie derivation* if the relation

$$f([a, b]) = [f(a), b] + [a, f(b)] + amb - bma$$

holds for all $a, b \in A$ and the set of generalized Lie derivations from $A$ to $M$ can be denoted by $gLieDer(A, M)$.

If $m = 0$, then these definitions lead to the conventional notions of Jordan and Lie derivations (see [4], [6], [7]).

Let $\sigma$ and $\tau$ be arbitrary $K$-endomorphisms of a ring $A$ and let $M$ be an $A/K$-bimodule.

A map $d : A \longrightarrow M$ is called a $(\sigma, \tau)$-*derivation* if the identity

$$d(xy) = d(x)\tau(y) + \sigma(x)d(y)$$

holds for all $x, y \in A$. Recall that a Jordan $(\sigma, \tau)$-derivation was defined in [3] as a map $d : A \longrightarrow M$ satisfying the identity

$$d(x^2) = d(x)\tau(x) + \sigma(x)d(x).$$

A $K$-module homomorphism $d : A \longrightarrow M$ is called a *Lie* $(\sigma, \tau)$-*derivation* if

$$d([x, y]) = [d(x), y]_{\sigma,\tau} - [d(y), x]_{\sigma,\tau}.$$

Here $[x, y]_{\sigma,\tau} = x\tau(y) - \sigma(y)x$ for all $x, y \in A$.

A $K$-module homomorphism $f : A \longrightarrow M$ is called a *Brešar generalized* $(\sigma, \tau)$-*derivation* if there is a $(\sigma, \tau)$-derivation $d : A \longrightarrow M$ such that the identity

$$f(xy) = f(x)\tau(y) + \sigma(x)d(y)$$

holds for all $x, y \in A$. This derivation is denoted by $(f, d)$. If $f$ satisfies the relation

$$f(x^2) = f(x)\tau(x) + \sigma(x)d(x)$$

for all $x \in A$, then it is called a *Brešar generalized Jordan* $(\sigma, \tau)$-derivation.

Finally we recall the following notion of [1], a $K$-module homomorphism $f : A \longrightarrow M$ is called a *generalized* $(\sigma, \tau)$-*derivation* if

$$f(xy) = f(x)\tau(y) + \sigma(x)f(y) + \sigma(x)m\tau(y)$$

for all $x, y \in A$ and some $m \in M$. This derivation is denoted by $(f, m)$. We call $f$ a *generalized Jordan* $(\sigma, \tau)$-*derivation* if

$$f(x^2) = f(x)\tau(x) + \sigma(x)f(x) + \sigma(x)m\tau(x)$$

for all $x \in A$. We say that $(f, m)$ is a *generalized Lie* $(\sigma, \tau)$-*derivation* if

$$f([x, y]) = [f(x), y]_{\sigma,\tau} - [f(y), x]_{\sigma,\tau} + \sigma(x)m\tau(y) - \sigma(y)m\tau(x)$$

for all $x, y \in A$.

In the following $A$ is a $K$-algebra over a commutative ring $K$ and $M$ is a $A/K$-bimodule.

Throughout this paper we use the following notions:

$Der^{(\sigma,\tau)}(A, M)$, the set of $(\sigma, \tau)$-derivations;

$BDer^{(\sigma,\tau)}(A,M)$, the set of Brešar generalized $(\sigma,\tau)$-derivations;
$BJDer^{(\sigma,\tau)}(A,M)$, the set of Brešar generalized Jordan $(\sigma,\tau)$-derivations;
$gDer^{(\sigma,\tau)}(A,M)$, the set of generalized $(\sigma,\tau)$-derivations;
$JDer^{(\sigma,\tau)}(A,M)$, the set of Jordan $(\sigma,\tau)$-derivations;
$gJDer^{(\sigma,\tau)}(A,M)$, the set of generalized Jordan $(\sigma,\tau)$-derivations;
$LieDer^{(\sigma,\tau)}(A,M)$, the set of Lie $(\sigma,\tau)$-derivations;
$gLieDer^{(\sigma,\tau)}(A,M)$, the set of generalized Lie $(\sigma,\tau)$-derivations.

Below $f : A \longrightarrow M$ is always a $K$-module homomorphism. Given $m \in M$, we put $m_l : x \ni A \longrightarrow mx \ni M$.

**Lemma 2.1** ([1, Lemma 3.1]). *Let $(f,m)$ be a generalized $(\sigma,\tau)$-derivation. Then there is a $(\sigma,\tau)$-derivation $d : A \longrightarrow M$ such that $f(st) = f(s)\tau(t) + \sigma(s)d(t)$ for all $s,t \in A$. Moreover, if $\{m \in M \mid Am = 0\} = 0$ and the map $\sigma : A \longrightarrow A$ is surjective, then $d$ is determined from $f$ uniquely.*

**Lemma 2.2** ([1, Lemma 3.2]). *Let $d : A \longrightarrow M$ be a $(\sigma,\tau)$-derivation. Then for every nonzero $m \in M$ the map $(f = d + m_l\tau, -m)$ is a generalized $(\sigma,\tau)$-derivation; moreover, $f \neq d$ and $d$ is associated with $f$.*

**Lemma 2.3** ([10, Theorem 2.3]). *Suppose that $\{m \in M \mid Am = 0\} = 0$ and the map $\sigma : A \longrightarrow A$ is surjective. Then the following sequence of $K$-modules is exact and splitting*:

$$0 \longrightarrow M \xrightarrow{\Psi_M} gLieDer(A,M) \xrightarrow{\Phi_M} LieDer(A,M) \longrightarrow 0.$$

*Here $\Psi_M(m) = (m_l, -m)$ and $\Phi_M((f,m)) = f + m_l$.*

**Lemma 2.4** ([1, Theorem 3.1]). *Suppose that $\{m \in M \mid Am = 0\} = 0$ and the map $\sigma : A \longrightarrow A$ is surjective. Then the following sequence of $K$-modules is exact and splitting*:

$$0 \longrightarrow M \xrightarrow{\Psi_M} gDer^{(\sigma,\tau)}(A,M) \xrightarrow{\Phi_M} Der^{(\sigma,\tau)}(A,M) \longrightarrow 0.$$

*Here $\Psi_M(m) = (m_l\tau, -m)$ and $\Phi_M((f,m)) = f + m_l\tau$.*

**Lemma 2.5** ([10, Lemmas 2.1, 2.5]). *Let $f : A \longrightarrow M$ be a $K$-linear map. Then the followings hold for every $m \in M$*:

(i) *If $(f,m)$ is a generalized Jordan (Lie) $(\sigma,\tau)$-derivation, then $f + m_l\tau$ and $f + m_r\tau$ are Jordan (Lie) $(\sigma,\tau)$-derivations.*

(ii) *If $f$ is a Jordan (Lie) $(\sigma,\tau)$-derivation, then $(f + m_l\tau, -m)$ and $(f + m_r\tau, -m)$ are generalized Jordan (Lie) $(\sigma,\tau)$-derivations.*

**Lemma 2.6** ([1, Theorem 4.1]). *Suppose that $\{m \in M \mid Am = 0\} = 0$ and $\sigma$ is surjective. Then the following sequence of $K$-modules is exact and splitting*:

$$0 \longrightarrow M \xrightarrow{\Psi_M} gJDer^{(\sigma,\tau)}(A,M) \xrightarrow{\Phi_M} JDer^{(\sigma,\tau)}(A,M) \longrightarrow 0.$$

*Here $\Psi_M(m) = (m_l\tau, -m)$ and $\Phi_M((f,m)) = f + m_l\tau$.*

**Lemma 2.7** ([1, Theorem 4.2])**.** *Suppose that* $\{m \in M \mid Am = 0\} = 0$ *and* $\sigma$ *is surjective. Then the following sequence of* $K$*-modules is exact and splitting*:

$$0 \longrightarrow M \xrightarrow{\Psi_M} gLieDer^{(\sigma,\tau)}(A,M) \xrightarrow{\Phi_M} LieDer^{(\sigma,\tau)}(A,M) \longrightarrow 0.$$

*Here* $\Psi_M(m) = (m_l\tau, -m)$ *and* $\Phi_M((f,m)) = f + m_l\tau$*.*

**Lemma 2.8** ([8, Five Lemma])**.** *Let*

$$\begin{array}{ccccccccc} A_1 & \to & A_2 & \to & A_3 & \to & A_4 & \to & A_5 \\ \downarrow \alpha_1 & & \downarrow \alpha_2 & & \downarrow \alpha_3 & & \downarrow \alpha_4 & & \downarrow \alpha_5 \\ B_1 & \to & B_2 & \to & B_3 & \to & B_4 & \to & B_5 \end{array}$$

*be a commutative diagram of* $R$*-modules and* $R$*-module homomorphisms, with exact rows, the followings hold*:

(i) *If* $\alpha_1$ *is an epimorphism and* $\alpha_2$, $\alpha_4$ *are monomorphisms, then* $\alpha_3$ *is a monomorphism*;

(ii) *If* $\alpha_5$ *is a monomorphism and* $\alpha_2$, $\alpha_4$ *are epimorphisms, then* $\alpha_3$ *is an epimorphism.*

## 3. Generalized $(\sigma,\tau)$-derivations

In [5], Hamaguchi gave a necessary and sufficient condition for $gDer(A,M)$ to be isomorphic to $BDer(A,M)$ as $K$-modules. Moreover he applied the result to $gJDer(A,M)$ and $BJDer(A,M)$.

In this section our goal is to give a relationship between $gDer^{(\sigma,\tau)}(A,M)$ and $BDer^{(\sigma,\tau)}(A,M)$.

A $K$-homomorphism $f : A \longrightarrow M$ is said to be a $\tau$*-left multiplier* if $f(xy) = f(x)\tau(y)$ for all $x, y \in A$. We denote by $Mul^{(\tau)}(A,M)$ the set of all $\tau$-left multipliers from $A$ to $M$. If $f$ and $g$ are $\tau$-left multipliers and $\alpha \in K$, then $f+g$ and $\alpha f$ are also $\tau$-left multipliers and hence $Mul^{(\tau)}(A,M)$ is a $K$-module.

**Theorem 3.1.** *Let* $\phi : gDer^{(\sigma,\tau)}(A,M) \longrightarrow BDer^{(\sigma,\tau)}(A,M)$ *and* $\psi : M \longrightarrow Mul^{(\tau)}(A,M)$ *be* $K$*-homomorphisms such that* $\phi((f,m)) = (f, f + m_l\tau)$ *and* $\psi(m) = m_l\tau$ *for all* $m \in M$*. Then* $\phi$ *is a* $K$*-isomorphism if and only if* $\psi$ *is a* $K$*-isomorphism.*

*Proof.* Let $\psi_1 : Mul^{(\tau)}(A,M) \longrightarrow BDer^{(\sigma,\tau)}(A,M)$ and $\psi_2 : BDer^{(\sigma,\tau)}(A,M) \longrightarrow Der^{(\sigma,\tau)}(A,M)$ are $K$-homomorphisms such that $\psi_1(g) = (g, 0)$ and $\psi_2((f, d)) = d$. So we obtain the following split and exact sequence:

$$0 \longrightarrow Mul^{(\tau)}(A,M) \xrightarrow{\psi_1} BDer^{(\sigma,\tau)}(A,M) \xrightarrow{\Psi_2} Der^{(\sigma,\tau)}(A,M) \longrightarrow 0.$$

Let $\iota_2 : Der^{(\sigma,\tau)}(A,M) \longrightarrow BDer^{(\sigma,\tau)}(A,M)$ be a $K$-homomorphism such that $\iota_2(d) = (d, d)$. Using Lemma 2.4, we have the following split exact sequence of $K$-modules:

$$0 \longrightarrow M \xrightarrow{\varphi_1} gDer^{(\sigma,\tau)}(A,M) \xrightarrow{\varphi_2} Der^{(\sigma,\tau)}(A,M) \longrightarrow 0.$$

Here $\varphi_1(m) = (m_l\tau, -m)$ and $\varphi_2((f,m)) = f + m_l\tau$. Then we get the following commutative diagram:

$$\begin{array}{ccccccccc}
0 & \rightarrow & M & \xrightarrow{\varphi_1} & gDer^{(\sigma,\tau)}(A,M) & \xrightarrow{\varphi_2} & Der^{(\sigma,\tau)}(A,M) & \rightarrow & 0 \\
 & & \downarrow \psi & & \downarrow \phi & & \downarrow id & & \\
0 & \rightarrow & Mul^{(\tau)}(A,M) & \xrightarrow{\psi_1} & BDer^{(\sigma,\tau)}(A,M) & \xrightarrow{\psi_2} & Der^{(\sigma,\tau)}(A,M) & \rightarrow & 0
\end{array}$$

In view of Lemma 2.8 (Five Lemma), the theorem is thereby proved. □

Now we introduce the set of Jordan $\tau$-left multipliers from $A$ to $M$ as follows.

A $K$-homomorphism $g : A \longrightarrow M$ is said to be a *Jordan $\tau$-left multiplier* if $g(x^2) = g(x)\tau(x)$ for all $x \in A$. We denote by $JMul^{(\tau)}(A, M)$ the set of all Jordan $\tau$-left multipliers from $A$ to $M$. It can be easily seen that $JMul^{(\tau)}(A, M)$ is a $K$-module.

Furthermore we can define the set of Lie $(\sigma, \tau)$-left multiplier from $A$ to $M$ as follows.

A $K$-homomorphism $g : A \longrightarrow M$ is said to be a *Lie $(\sigma, \tau)$-left multiplier* if $g([x, y]) = [-g(y), x]_{\sigma,\tau}$ for all $x, y \in A$. We denote by $LieMul^{(\sigma,\tau)}(A, M)$ the set of all Lie $(\sigma, \tau)$-left multipliers from $A$ to $M$. The set $LieMul^{(\sigma,\tau)}(A, M)$ is a $K$-module.

Now we introduce the set $\mathcal{M}^{(\sigma,\tau)}(A)$ as follows:

$$\mathcal{M}^{(\sigma,\tau)}(A) = \{m \in M \mid [m, x]_{\sigma,\tau} = 0 \text{ for all } x \in A\}.$$

Finally we introduce the notion of Brešar generalized Lie $(\sigma, \tau)$-derivation as follows:

The pair $(f, L)$, with $f : A \longrightarrow M$ a $K$-module homomorphism is called a *Brešar generalized Lie $(\sigma, \tau)$-derivation* if there exists a Lie $(\sigma, \tau)$-derivation $L$ such that

$$f([x, y]) = [L(x), y]_{\sigma,\tau} - [f(y), x]_{\sigma,\tau}$$

for all $x, y \in A$. The set of Brešar generalized Lie $(\sigma, \tau)$-derivations from $A$ to $M$ will be denoted by $BLieDer^{(\sigma,\tau)}(A, M)$.

**Corollary 3.1.** *$\phi : gJDer^{(\sigma,\tau)}(A, M) \longrightarrow BJDer^{(\sigma,\tau)}(A, M)$ is a $K$-isomorphism if and only if $\psi : M \longrightarrow JMul^{(\tau)}(A, M)$ is a $K$-isomorphism.*

**Theorem 3.2.** *Let $\phi : gLieDer^{(\sigma,\tau)}(A, M) \longrightarrow BLieDer^{(\sigma,\tau)}(A, M)$ and $\psi : M \longrightarrow LieMul^{(\sigma,\tau)}(A, M)$ be $K$- homomorphisms such that $\phi((f, m)) = (f, f + m_l\tau)$ and $\psi(m) = m_l\tau$ for all $x, y \in A$. Let us define the set $\mathcal{M}^{(\sigma,\tau)}(A) = \{m \in M \mid [m, x]_{\sigma,\tau} = 0$ for all $x \in A\}$ as well.*

*If $\mathcal{M}^{(\sigma,\tau)}(A) = M$, then $\phi$ is a $K$-isomorphism if and only if $\psi$ is a $K$-isomorphism.*

*Proof.* Let us define the maps $\psi_1 : LieMul^{(\sigma,\tau)}(A, M) \longrightarrow BLieDer^{(\sigma,\tau)}(A, M)$ and $\psi_2 : BLieDer^{(\sigma,\tau)}(A, M) \longrightarrow LieDer^{(\sigma,\tau)}(A, M)$. These maps are $K$-homomorphisms such that $\psi_1(g) = (g, 0)$ and $\psi_2((f, L)) = L$. Also let $\iota_2 : LieDer^{(\sigma,\tau)}(A, M) \longrightarrow BLieDer^{(\sigma,\tau)}(A, M)$ be a $K$-homomorphism such that

$\iota_2(L) = (L, L)$. Then $\psi_2\iota_2 = id_{LieDer^{(\sigma,\tau)}(A,M)}$, henceforth, we have the following split and exact sequence of $K$-modules:

$$0 \to LieMul^{(\sigma,\tau)}(A,M) \xrightarrow{\psi_1} BLieDer^{(\sigma,\tau)}(A,M) \xrightarrow{\psi_2} LieDer^{(\sigma,\tau)}(A,M) \to 0.$$

However we have already had the following split and exact sequence by Lemma 2.7:

$$0 \longrightarrow M \xrightarrow{\varphi_1} gLieDer^{(\sigma,\tau)}(A,M) \xrightarrow{\varphi_2} LieDer^{(\sigma,\tau)}(A,M) \longrightarrow 0.$$

Here $\varphi_1(m) = (m_l\tau, -m)$ and $\varphi_2((f,m)) = f + m_l\tau$. Since $\mathcal{M}^{(\sigma,\tau)}(A) = M$, we get $\psi(m) = m_l\tau \in LieMul^{(\sigma,\tau)}(A,M)$. So we obtain the following commutative diagram:

$$\begin{array}{ccccccccc}
0 & \to & M & \overset{\varphi_1}{\to} & gLieDer^{(\sigma,\tau)}(A,M) & \overset{\varphi_2}{\to} & LieDer^{(\sigma,\tau)}(A,M) & \to & 0 \\
 & & \downarrow\psi & & \downarrow\phi & & \downarrow id & & \\
0 & \to & LieMul^{(\sigma,\tau)}(A,M) & \overset{\psi_1}{\to} & BLieDer^{(\sigma,\tau)}(A,M) & \overset{\psi_2}{\to} & LieDer^{(\sigma,\tau)}(A,M) & \to & 0
\end{array}$$

By using Lemma 2.8 (Five Lemma), we complete the proof of Theorem 3.2. □

## 4. Extension of Brešar generalized $(\sigma,\tau)$-derivations

Some extensions of Brešar generalized derivations and its applications to generalized Jordan derivations are given in [5].

Let $A$ be a $K$-algebra without a unit element and the set

$$\hat{A} = \{(n,a) \mid n \in K, a \in A\}$$

be a direct product $K \times A$ with multiplication $(n_1,a_1)\circ(n_2,a_2) = (n_1n_2, n_1a_2 + n_2a_1 + a_1a_2)$ for $n_1, n_2 \in K$, $a_1, a_2 \in A$. Then $\hat{A}$ is a $K$-algebra with a unit element $(1,0)$. Let $M$ be an $A/K$ -bimodule. Then $M$ is an $\hat{A}/K$-bimodule with $(n_1,a_1)\cdot m_1 = n_1m_1 + a_1m_1$ and $m_2\cdot(n_2,a_2) = n_2m_2 + m_2a_2$ for all $n_i \in K$, $a_i \in A$ and $m_i \in M$; $i = 1,2$.

Let $d : A \longrightarrow M$ be a $K$-derivation. Then there exists a unique $K$-derivation $\tilde{d} : \hat{A} \longrightarrow M$ such that its restriction $\tilde{d}|_A$ is equal to $d$. This is defined by $\tilde{d}((n,a)) = d(a)$.

Now we want to give similar results for Brešar generalized $(\sigma,\tau)$-derivations.

**Theorem 4.1.** *Let $A$ be a $K$-algebra, $M$ an $A/K$-bimodule and $\hat{A}$ a $K$-algebra defined as above. Let $(f,d)$ be a Brešar generalized $(\sigma,\tau)$-derivation from $A$ to $M$. Then the following conditions are equivalent*:

(i) *A Brešar generalized $(\sigma,\tau)$-derivation $(f,d)$ from $A$ to $M$ can be extended to a Brešar generalized $(\sigma,\tau)$-derivation from $\hat{A}$ to $M$.*

(ii) *A left multiplier $f-d$ from $A$ to $M$ can be extended to a $\tau$-left multiplier from $\hat{A}$ to $M$.*

(iii) *There exists an element $m \in M$ such that $f-d = m_l\tau$.*

*Proof.* (i) $\Longrightarrow$ (ii) Let $\tau'$ be an automorphism from $\hat{A}$ to $\hat{A}$ such that its restriction $\tau'|_A$ is equal to $\tau$. This $\tau'$ defined by $\tau'((n,a)) = (n,\tau(a))$ for all $(n,a) \in \hat{A}$. Let $(f,d) \in BDer^{(\sigma,\tau)}(A,M)$ and $(F,D)$ be an extension of $(f,d)$ from $\hat{A}$ to $M$. We claim that $(f-d) \in Mul^{(\tau)}(A,M)$. Then for all $(n_1,a_1),(n_2,a_2) \in \hat{A}$, $(F-D)((n_1,a_1)\circ(n_2,a_2)) = (F-D)((n_1,a_1))\cdot\tau'((n_2,a_2))$, this would tell us that $(F-D) \in Mul^{(\tau)}(\hat{A},M)$. Furthermore $(F-D)|_A = f-d$.

(ii) $\Longrightarrow$ (iii) For $(f-d) \in Mul(A,M)$, suppose that there exists its extension $G \in Mul^{(\tau)}(\hat{A},M)$. Then the assumption leads to

$$\begin{aligned}(f-d)(a) &= G((0,a)) = G((1,0)\circ(0,a))\\ &= G((1,0))\cdot\tau'((0,a))\\ &= G((1,0))\cdot(0,\tau(a))\\ &= G((1,0))\tau(a)\end{aligned}$$

for all $a \in A$. By putting $G((1,0)) = m \in M$, this is our desired result.

(iii) $\Longrightarrow$ (i) Let $D \in Der(\hat{A},M)$ be a unique extension of $d \in Der(A,M)$. Let $F : \hat{A} \longrightarrow M$ be a $K$-homomorphism defined by $F((n,a)) = nm + f(a)$ for all $(n,a) \in \hat{A}$ and $\tau'$ and $\sigma'$ be automorphisms from $\hat{A}$ to $\hat{A}$ such that their restrictions are $\tau'|_A$ and $\sigma'|_A$ are equal to $\tau$, $\sigma$ respectively. These maps are defined by $\tau'((n,a)) = (n,\tau(a))$ and $\sigma'((n,a)) = (n,\sigma(a))$ for all $(n,a) \in \hat{A}$.

Now we get $F((n_1,a_1)\circ(n_2,a_2)) = F((n_1n_2, n_1a_2 + n_2a_1 + a_1a_2))$ for all $(n_1,a_1),(n_2,a_2) \in \hat{A}$. Using the definition of $F$, we arrive at

$$n_1n_2m + n_1f(a_2) + n_2f(a_1) + f(a_1)\tau(a_2) + \sigma(a_1)d(a_2).$$

In other words we have

$$n_1n_2m + n_1f(a_2) + n_2f(a_1) + f(a_1)\tau(a_2) + \sigma(a_1)d(a_2) + n_1d(a_2) - n_1d(a_2).$$

It means that

$$n_2(n_1m + f(a_1)) + n_1(f-d)(a_2) + f(a_1)\tau(a_2) + \sigma(a_1)d(a_2) + n_1d(a_2).$$

By the hypothesis, the above relation equals to

$$n_2(n_1m + f(a_1)) + n_1m\tau(a_2) + f(a_1)\tau(a_2) + \sigma(a_1)d(a_2) + n_1d(a_2).$$

Using the definition of $D$ and, the operation between $M$ and $\hat{A}$, we get

$$(n_1m + f(a_1))\cdot(n_2,\tau(a_2)) + n_1D((n_2,a_2)) + \sigma(a_1)D((n_2,a_2)).$$

This implies that

$$(n_1m + f(a_1))\cdot(n_2,\tau(a_2)) + (n_1,\sigma(a_1))\cdot D((n_2,a_2)).$$

The definitions of $F$ and $\tau'$ give us

$$F((n_1,a_1))\cdot\tau'((n_2,a_2)) + \sigma'((n_1,a_1))\cdot D((n_2,a_2)).$$

Therefore $(F,D) \in BDer^{(\sigma,\tau)}(\hat{A},M)$ and it can be easily seen that $(F,D)|_A = (f,d)$. $\square$

**Corollary 4.1.** *Let $A$ be a $K$-algebra, $M$ an $A/K$-bimodule and $\hat{A}$ a $K$-algebra defined as above. Let $(f,d)$ be a Brešar generalized Jordan $(\sigma,\tau)$-derivation from $A$ to $M$. Then the following conditions are equivalent*:

(i) *A Brešar generalized Jordan $(\sigma,\tau)$-derivation $(f,d)$ from $A$ to $M$ can be extended to a Brešar generalized Jordan $(\sigma,\tau)$-derivation from $\hat{A}$ to $M$.*

(ii) *A Jordan left multiplier $f-d$ from $A$ to $M$ can be extended to a Jordan left multiplier from $\hat{A}$ to $M$.*

(iii) *There exists an element $m \in M$ such that $f-d = m_l\tau$.*

**Theorem 4.2.** *Let $A$ be a $K$-algebra, $M$ an $A/K$-bimodule and $\hat{A}$ a $K$-algebra defined as above. Let $(f,d)$ be a Brešar generalized Lie $(\sigma,\tau)$-derivation from $A$ to $M$ and $(F,D)$ be an extension of $(f,d)$ from $\hat{A}$ to $M$. If $f-d$ is a Lie $(\sigma,\tau)$-left multiplier from $A$ to $M$, then it can be extended to a Lie $(\sigma,\tau)$-left multiplier from $\hat{A}$ to $M$.*

*Proof.* Let $(f,d)$ be a Brešar generalized Lie $(\sigma,\tau)$-derivation and $(F,D)$ be an extension of $(f,d)$ from $\hat{A}$ to $M$. Assume that $(f-d) \in LieMul^{(\sigma,\tau)}(A,M)$. Then $(F-D)[(n_1,a_1),(n_2,a_2)] = [-(F-D)((n_2,a_2)),(n_1,a_1)]$ for all $(n_1,a_1)$, $(n_2,a_2) \in \hat{A}$ and $(F-D) \in LieMul^{(\sigma,\tau)}(\hat{A},M)$. It is easy to see that $(F-D)|_A = f-d$ as well. $\square$

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