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A RESULT ON GENERALIZED DERIVATIONS WITH ENGEL CONDITIONS ON ONE-SIDED IDEALS

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Abstract. Let R be a non-commutative prime ring and I a non-zero left ideal of R. Let U be the left Utumi quotient ring of R and C be the center of U and k, m, n, r fixed positive integers. If there exists a generalized derivation g of R such that $[g(x^m)x^n, x^r]_k = 0$ for all $x \in I$, then there exists $a \in U$ such that $g(x) = xa$ for all $x \in R$ except when $R \cong M_2(GF(2))$ and $I[I, I] = 0$.

1. Introduction

Throughout this paper unless specially stated, R always denotes a prime ring with center $Z(R)$, extended centroid C, left Utumi quotient ring U, and two sided Martindale quotient ring Q. For any $x, y \in R$, we set $[x, y]_1 = [x, y] = xy - yx$ and $[x, y]_k = [[x, y]_{k-1}, y]$ for $k > 1$.

We mean by a derivation of R an additive mapping d from R into itself which satisfies the rule $d(xy) = d(x)y + xd(y)$ for all $x, y \in R$. A well-known result proved by Posner [21] states that R must be commutative if there exists a nonzero derivation d of R such that $[d(x), x] = 0$ for all $x \in R$. Many related generalizations have been obtained by a number of authors in the literature (e.g., see, [10], [14], [15], [16]).

An additive mapping $g : R \to R$ is called a generalized derivation of R if there exists a derivation d of R such that $g(xy) = g(x)y + xd(y)$ for all $x, y \in R$ [9]. Obviously any derivation is a generalized derivation. Moreover, another basic example of a generalized derivation is the mapping of the form $g(x) = ax + xb$ for $a, b \in R$. Many authors have studied generalized derivations in the context of prime and semiprime rings (see [1], [2], [3], [13], [9], [18]).

In [13], T. K. Lee extended the definition of a generalized derivation as follows. By a generalized derivation he means an additive mapping $g : J \to U$ such that $g(xy) = g(x)y + xd(y)$ for all $x, y \in J$, where U is the right Utumi

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quotient ring of R, J is a dense right ideal of R and d is a derivation from J to U. He also proved that every generalized derivation can be uniquely extended to a generalized derivation of U. In fact, there exists $a \in U$ and a derivation d of U such that $g(x) = ax + d(x)$ for all $x \in U$ [13, Theorem 3]. A corresponding form to dense left ideals as follows. Let I be a dense left ideal of R and U be the left Utumi quotient ring of R. An additive mapping $g : I \to U$ is called a generalized derivation if there exists a derivation $d : I \to U$ such that $g(xy) = xg(y) + d(x)y$ for all $x, y \in I$. Following the same methods in [13], one can extend g uniquely to a generalized derivation of U, which we will also denote by g, and g assumes the form $g(x) = xa + d(x)$ for all $x \in U$ and some $a \in U$, where d is a derivation of U. Notice that $g(x) = ax + (d - d_a)(x)$ for all $x \in U$, where d_a denotes the inner derivation induced by the element $a \in U$, i.e., $d_a(x) = [a, x]$. Setting $\delta = d - d_a$, we may always assume that a generalized derivation of a prime ring is of the form $g(x) = ax + \delta(x)$ for all $x \in U$, where $a \in U$ and δ is a derivation of U.

In [11], C. Lanski proved that if R is a prime ring with derivation d, I is a left ideal of R, and k, n are positive integers such that $[d(x^k), x^k]_n = 0$ for all $x \in I$, then either $d = 0$ or R is commutative. In [1], this result extended to generalized derivations.

In [17], T. K. Lee and W. K. Shiue showed that if R is a non-commutative prime ring, I is a nonzero left ideal of R and d is a derivation of R such that $[d(x^m)x^n, x^r]_k = 0$ for all $x \in I$, where k, m, n, r are fixed positive integers, then $d = 0$ except when $R \cong M_2(GF(2))$.

The aim of the present paper is to extend this result to generalized derivations. Precisely, we will prove the following.

Theorem 1. *Let R be a non-commutative prime ring and k, m, n, r fixed positive integers. If there exists a generalized derivation g of R such that $[g(x^m)x^n, x^r]_k = 0$ for all $x \in R$, then there exists an element $a \in U$ such that $g(x) = xa$ for all $x \in R$.*

Theorem 2. *Let R be a non-commutative prime ring, I a non-zero left ideal of R and k, m, n, r fixed positive integers. If there exists a generalized derivation g of R such that $[g(x^m)x^n, x^r]_k = 0$ for all $x \in I$, then there exists $a \in U$ such that $g(x) = xa$ for all $x \in R$ except when $R \cong M_2(GF(2))$ and $I[I, I] = 0$.*

2. Preliminaries

In what follows, unless stated otherwise, R will be a prime ring. The related object we need to mention is the left Utumi quotient ring U of R (sometimes, as in [4], U is called the maximal left ring of quotients).

The definitions, the axiomatic formulations and the properties of this quotient ring U can be found in [4].

In any case, when R is a prime ring, all we need about U is that

1) $R \subseteq U$;

2) U is a prime ring;

3) The center of U, denoted by C, is a field which is called the extended centroid of R.

We also frequently use the theory of generalized polynomial identities and differential identities (see [4], [10], [12], [20]). In particular we need to recall the following:

Remark 1 ([6]). If R is a prime ring and I is a non-zero left ideal of R, then I, RI and UI satisfy the same generalized polynomial identities.

Remark 2 ([10]). Let R be a prime ring, d a nonzero derivation of R and I a nonzero two-sided ideal of R. Let $f(x_1, \dots, x_n, d(x_1), \dots, d(x_n))$ be a differential identity in I, that is

$$f(r_1, \dots, r_n, d(r_1), \dots, d(r_n)) = 0 \text{ for all } r_1, \dots, r_n \in I.$$

Then one of the following holds:

1) Either d is an inner derivation in Q, the Martindale quotient ring of R, in the sense that there exists $q \in Q$ such that $d(x) = [q, x]$ for all $x \in R$, and I satisfies the generalized polynomial identity

$$f(r_1, \dots, r_n, [q, r_1], \dots, [q, r_n])$$

or

2) I satisfies the generalized polynomial identity

$$f(x_1, \dots, x_n, y_1, \dots, y_n).$$

3. Results

We need the following lemmas.

Lemma 1. *Let $R = M_t(F)$, where F is a field, $t \geq 2$ and $a, b \in R$. Suppose that*

$$[ax^{m+n} + [b, x^m]x^n, x^r]_k = 0 \quad \textit{for all } x \in R, \tag{1}$$

where k, m, n, r are fixed positive integers. Then $a + b \in F$.

Proof. Let e be an idempotent element in R. Setting $x = e$ in (1) and multiplying left side by $(1 - e)$, we see that $(1 - e)(a + b)e = 0$ for any idempotent element e. Thus $a + b$ is a diagonal matrix. Note that $u(a + b)u^{-1}$ must be diagonal for each invertible element $u \in R$, since

$$[(uau^{-1})x^{m+n} + [(ubu^{-1}), x^m]x^n, x^r]_k = 0$$

for all $x \in R$. Write $a + b = \sum_{i=1}^{t} \beta_i e_{ii}$, where $\beta_i \in F$. Then for each $j > 1$, we see that $\beta_j - \beta_1$, the $(1, j)$-entry of $(1 + e_{1j})(a + b)(1 + e_{1j})^{-1}$, equals 0. That is, $\beta_j = \beta_1$ for $j > 1$ and hence $a + b \in F$. □

Lemma 2. *Let R be a non-commutative prime ring and $a, b \in R$ such that $[ax^{m+n} + [b, x^m]x^n, x^r]_k = 0$ for all $x \in R$, where k, m, n, r are fixed positive integers. Then $a + b \in Z(R)$.*

Proof. Suppose on the contrary that $a+b \notin C$. Then

$$f(X) = [(a+b)X^{m+n} - X^m bX^n, X^r]_k$$

is a nontrivial generalized polynomial identity (GPI) for R. By [6], $f(X)$ is also a GPI for Q. Denote by F either the algebraic closure of C or C according to the cases where C is either infinite or finite, respectively. Then, by a standard argument (e.g., see [19, Proposition]), $f(X)$ is also a GPI for $Q \otimes_C F$. Since $Q \otimes_C F$ is centrally closed prime F-algebra [7, Theorems 2.5 and 3.5], by replacing R, C with $Q \otimes_C F, F$, respectively we may assume R is centrally closed and C is either finite or algebraically closed. In view of Martindale's theorem [20], R is a primitive ring having a non-zero socle H with C as its associated division ring.

Since $a+b \notin C$, we have $[a+b, h] \neq 0$ for some $h \in H$. By Litoff's theorem [8], there exists an idempotent $e \in H$ such that $h, ah, ha, hb, bh \in eRe$. Note that $ef(eXe)e$ is a GPI for R. Thus, $[(eae)X^{m+n} + [ebe, X^m]X^n, X^r]_k$ is a GPI for eRe. Since $eRe \cong M_s(C)$ for some $s \geq 1$ then $eae + ebe$ is central in eRe by Lemma 1. Then there exists $c \in C$ such that $ce = eae + ebe$. Hence $ch = eaeh + ebeh = eah + ebh = ah + bh = (a+b)h$. Similarly $hc = heae + hebe = hae + hbe = ehae + ehbe = ha + hb = h(a+b)$. So $[a+b, h] = 0$, a contradiction. Therefore $a+b \in Z(R)$. □

Corollary 1. *Let R be a prime ring and $a \in R$ such that $[ax^m, x^n]_k = 0$ for all $x \in R$, where k, m, n are fixed positive integers. Then $a \in Z(R)$.*

Proof of Theorem 1. As we have already noted that every generalized derivation g on a dense left ideal of R can be uniquely extended to U and assumes the form $g(x) = ax + d(x)$ for some $a \in U$ and a derivation d on U. If $d = 0$, then $[ax^{m+n}, x^r]_k = 0$ for all $x \in R$. By Remark 1, U satisfies the above generalized identity. Moreover, since U remains prime by the primeness of R, replacing R with U, we may assume that $a \in R$ and C is just the center of R. By Corollary 1, we have $a \in Z(R)$. Thus $g(x) = ax = xa$ for all $x \in R$. So we may assume that $d \neq 0$.

In the light of Remark 2, we divide the proof into two cases:

Case 1. Let d be the inner derivation induced by the element $b \in U - C$, that is, the $d(x) = [b, x]$ for all $x \in U$. Then R satisfies the nontrivial generalized polynomial identity

$$[ax^{m+n} + [b, x^m]x^n, x^r]_k.$$

By Remark 1, U satisfies the above generalized polynomial identity. Moreover, since U remains prime by the primeness of R, replacing R with U, we may assume that $a, b \in R$ and C is just the center of R. Then by Lemma 2, we have that $a+b \in Z(R)$. Therefore $g(x) = ax + [b, x] = (a+b)x - xb = x(a+b-b) = xa$ for all $x \in R$.

Case 2. Let now d be an outer derivation of U. To continue the proof, we set $G(Y, X) = \sum_{i=0}^{m-1} X^i Y X^{m-1-i}$, a non-commuting polynomial in variables

X and Y. Note that $d(x^m) = G(d(x), x)$. Then R satisfies the following differential identity

$$[ax^{m+n} + G(d(x), x)x^n, x^r]_k.$$

Now by Remark 2, R satisfies the identity

$$[ax^{m+n} + G(y, x)x^n, x^r]_k.$$

Taking $y = 0$ in the above identity, we get

$$[ax^{m+n}, x^r]_k = 0 \text{ for all } x \in R.$$

So we have

$$[d(x^m)x^n, x^r]_k = 0 \text{ for all } x \in R.$$

Therefore by [17, Theorem 1], we must have $d = 0$, a contradiction. This proves the result. $\square$

By using almost the same argument in [17], we have the following.

Lemma 3. *Let $R = M_l(F)$, where F is a field, $l \geq 2$, and I a minimal left ideal of R. Suppose $[ax^{m+n} + [b, x^m]x^n, x^r]_k = 0$ for all $x \in I$, where m, n, r, k are fixed positive integers. Then $a + b \in F$ except when $R \cong M_2(GF(2))$.*

Proof. Suppose that $a + b \notin F$. Since I is a minimal left ideal, it is clear that we may assume $I = Re_{11}$. Let $e = e^2 \in I$. By the hypothesis, we have $[ae + [b, e]e, e]_k = 0$. Left multiplying by $1 - e$, we see that

$$(1 - e)(a + b)e = 0 \quad \text{for all } e \in I. \tag{2}$$

Let $\beta \in F$ and $x \in R$. Then $f = e + (1 - e)xe$ and $g = e + \beta(1 - e)xe$ are idempotents in I. Set $c = a + b$, then $c \notin F$ and $(1 - e)ce = 0$. Thus by (2) we have $(1 - f)cf = 0 = (1 - g)cg$. Therefore we see that

$$((1 - e) - (1 - e)xe)c(e + (1 - e)xe) = 0$$

and

$$((1 - e) - \beta(1 - e)xe)c(e + \beta(1 - e)xe) = 0.$$

Using (2) we arrive at the following equations:

$$(1 - e)cxe - (1 - e)xce - (1 - e)xec(1 - e)xe = 0$$

and

$$\beta(1 - e)cxe - \beta(1 - e)xce - \beta^2(1 - e)xec(1 - e)xe = 0.$$

Multiplying first equation by β and comparing the last two equations we see that $(\beta^2 - \beta)(1 - e)xec(1 - e)xe = 0$ for all $x \in R$. Then either $\beta \in \{0, 1\}$ or $ec(1 - e) = 0$ for any idempotent $e \in I$. Suppose that the second possibility holds. In particular, $e_{11}c(1 - e_{11}) = 0$. Let $x \in R$. Then we have $xe_{11}c = xe_{11}ce_{11} = \mu xe_{11}$ for some $\mu \in F$. Thus we see that $I(c - \mu) = Re_{11}(c - \mu) = 0$ for some $\mu \in F$. On the other hand, in view of (2) we get $[c, e] = 0$ for any idempotent $e \in I$. Then $0 = [c, e] = [c - \mu, e] = (c - \mu)e$ for all idempotent

$e \in I$. Note that $e_{11} + (1 - e_{11})xe_{11} \in I$ is also an idempotent for every $x \in R$. Hence $(c - \mu)(e_{11} + (1 - e_{11})xe_{11}) = 0$ for all $x \in R$. Since $(c - \mu)e_{11} = 0$, it is clear that $(c - \mu)Re_{11} = 0$. Therefore $c = \mu \in F$, a contradiction. Thus we get $F = GF(2)$.

Now we prove that $l = 2$. Suppose on the contrary that $l > 2$. Let i, j be two distinct positive integers such that $2 \leq i, j \leq l$. Then $e_{11}, e_{11}+e_{i1}, e_{11}+e_{j1}$ and $e_{11} + e_{i1} + e_{j1}$ are idempotents in I. In view of (2) we obtain that

$$ce_{11} = e_{11}ce_{11},$$

$$c(e_{11} + e_{i1}) = (e_{11} + e_{i1})c(e_{11} + e_{i1}),$$

$$c(e_{11} + e_{j1}) = (e_{11} + e_{j1})c(e_{11} + e_{j1})$$

and

$$c(e_{11} + e_{i1} + e_{j1}) = (e_{11} + e_{i1} + e_{j1})c(e_{11} + e_{i1} + e_{j1}). \tag{3}$$

Using $ce_{11} = e_{11}ce_{11}$ and comparing the other equations in (3), we arrive at $e_{i1}ce_{j1}+e_{j1}ce_{i1} = 0$. Set $c = \sum_{1\leq i,j\leq l} \beta_{ij}e_{ij}$, where $\beta_{ij} \in F$. Then this implies that $\beta_{1j} = 0 = \beta_{1i}$. Hence the second equation in (3) reduces to $ce_{i1} = \beta_{11}e_{i1}$, and so $\beta_{pi} = 0$ for $p \neq i$ and $\beta_{ii} = \beta_{11}$. Thus we get $c = a + b \in F$, a contradiction. This proves the lemma. □

Lemma 4. *Let R be a prime ring, I a non-zero left ideal of R and $a \in R$ such that $[ax^m, x^n]_k = 0$ for all $x \in I$, where k, m, n are fixed positive integers. Then $a \in Z(R)$ except when $R \cong M_2(GF(2))$ and $I[I, I] = 0$.*

Proof. Assume that $[ax^m, x^n]_k = 0$ for all $x \in I$. Then

$$[[a, x^n]x^m, x^n]_k = [ax^m, x^n]_{k+1} = 0$$

for all $x \in I$. Now by [17, Lemma 3] we have $a \in Z(R)$ except when $R \cong M_2(GF(2))$ and $I[I, I] = 0$. □

Lemma 5. *Let R be a non-commutative prime ring and I a non-zero left ideal and $a, b \in R$ such that*

$$[ax^{m+n} + [b, x^m]x^n, x^r]_k = 0 \quad \text{for all } x \in I, \tag{4}$$

where k, m, n, r are fixed positive integers. Then $a + b \in Z(R)$ except when $R \cong M_2(GF(2))$ and $I[I, I] = 0$.

Proof. Assume that $a + b \notin C$. If $I(b - \beta) = 0$ for some $\beta \in C$, then setting $b' = b - \beta$ we have $Ib' = 0$. Moreover by (4) it is clear that

$$[ax^{m+n} + [b', x^m]x^n, x^r]_k = 0 \quad \text{for all } x \in I.$$

Thus we get

$$[(a + b')x^{m+n}, x^r]_k = 0 \quad \text{for all } x \in I. \tag{5}$$

By Remark 1, $[(a + b')x^{m+n}, x^r]_k = 0$ for all $x \in UI$. Moreover $UIb' = 0$ if and only if $Ib' = 0$. Now I and UI satisfy the same basic conditions. Hence replacing R, I with U, UI, respectively, we may assume that $a, b' \in R$ and C is just the center of R. Thus we get the conclusion $R \cong M_2(GF(2))$ and $I[I, I] = 0$ since $a + b' \notin C$.

So we may assume that $I(b - \beta) \neq 0$ for all $\beta \in C$. Hence, in view of [14, Lemma 3], either R is a PI-ring or there exists an element $u \in I$ such that ub and u are C-independent. For the latter case,

$$[a(Xu)^{m+n} + [b, (Xu)^m](Xu)^n, (Xu)^r]_k$$

is a non-trivial GPI for R.

On the other hand we have $[ax^{m+n} + [b, x^m]x^n, x^r]_k = 0$ for all $x \in QI$ by [6]. Thus applying the same argument in Lemma 2 and replacing I by QI, we may assume that R is a centrally closed prime ring having a non-zero socle H, with C as its associated division ring and $I = IC$. Moreover C is either algebraically closed or finite. If R contains no non-trivial idempotents, then R is a division ring and $I = R$. Then by the proof of Theorem 1 we obtain that $a + b \in C$, a contradiction. So we may assume that R contains a non-trivial idempotent. On the other hand we have

- $I[I, I] = 0$ if and only if $HI[HI, HI] = 0$ by [6],
- $I(b - \mu) = 0$ if and only if $HI(b - \mu) = 0$ for some $\mu \in C$.

So replacing I by HI we may assume $I \subseteq H$. Suppose first that $I[I, I] \neq 0$. Then I always contains an idempotent with rank 2 or greater that 2. Let e be such an idempotent in I.

Now choose x as exe in (4), then

$$[a(exe)^{m+n} + [b, (exe)^m](exe)^n, (exe)^r]_k = 0 \quad \text{for all } x \in R,$$

and left-side multiplying by e yields

$$[(eae)(exe)^{m+n} + [ebe, (exe)^m](exe)^n, (exe)^r]_k = 0 \quad \text{for all } x \in R.$$

But $eRe \cong M_l(C)$, where $l = \text{rank}(e) \geq 2$. By Lemma 2 we have $e(a+b)e \in Ce$. Choosing $x = e$ in (4), we get $(a + b)e - e(a + b)e = 0$. Then $ae + be \in Ce$ for every idempotent $e \in I$ with $\text{rank}(e) \geq 2$. Note that $e + (1 - e)xe \in I$ is also an idempotent for all $x \in R$ and $\text{rank}(e + (1 - e)xe) = \text{rank}(e) \geq 2$. Set $c = a + b$, so $ce \in Ce$ for all idempotent $e \in I$ with $\text{rank}(e) \geq 2$. In particular we have

$$c(e + (1 - e)xe) \in C(e + (1 - e)xe)$$

for all $x \in R$. Left-side multiplying by e yields that

$$ece + ecxe - ecexe \in Ce.$$

Since $ece = ce \in Ce$, we get $[e, c]xe \in Ce$ for all $x \in R$. Suppose for the moment that $[e, c] \neq 0$. Choose $x_0 \in R$ such that $[e, c]x_0e = \beta e \neq 0$ for some $\beta \in C$. Then we have $\beta exe = [e, c]x_0exe \in Ce$. Therefore $eRe = Ce$, because $\beta \neq 0$. But $eRe = Ce$ implies that $\text{rank}(e) = 1$, a contradiction. Hence $[e, c] = 0$. Now since I is completely reducible left H-module, each element of I is contained in Hf for some $f^2 = f \in I$ with $\text{rank}(f) \geq 2$. But $fc = cf \in Cf$.

Let $x \in I$. Then $x = hf$ for some $h \in H$. We see that $xc = hfc \in Chf = Cx$, and so $[xc, x] = 0$ for all $x \in I$. Linearizing this last equation, we get

$$[xc, y] + [yc, x] = 0 \quad \text{for all } x, y \in I. \tag{6}$$

Replacing $y = e$ in (6) and using the fact that $[e, c] = 0$, we obtain

$$0 = [xc, e] + [ec, x] = e[c, x] \quad \text{for all } x \in I.$$

Hence we have $0 = e[c, xy] = ex[c, y]$ for all $x, y \in I$. Therefore we get $eRI[c, I] = (0)$, and so $I[c, I] = 0$. In particular, $[x[c, x], x] = 0$ for all $x \in I$. So in view of [17, Lemma 3(ii)] one obtains $I(c - \lambda) = 0$ for some $\lambda \in C$. Let $x \in R$, then it is clear that $f = e + (1 - e)xe \in I$ is an idempotent with $\text{rank}(f) = \text{rank}(e) \geq 2$. Since $[c, e] = 0$ for all $e = e^2 \in I$ with $\text{rank}(e) \geq 2$, in particular we have

$$[c - \lambda, e + (1 - e)xe] = 0 \quad \text{for all } x \in R.$$

Hence we get $(c - \lambda)e + (c - \lambda)(1 - e)xe = 0$. On the other hand we have $(c - \lambda)e = [c - \lambda, e] = 0$. So $(c - \lambda)xe = 0$ for all $x \in R$. Thus the primeness of R implies that $c = \lambda \in C$, and hence $a + b = c \in Z(R)$, a contradiction. This proves that $I[I, I] = 0$.

If now $H \cong M_l(C)$ for some $l \geq 2$, then in view of Lemma 3, we are done. Thus we may assume $H \ncong M_l(C)$ for all $l \geq 2$. Since $c \notin C$, it is clear that $ch \neq hc$ for some $h \in I$. It follows from Litoff's theorem [8] that there exists $e = e^2 \in H$, $\text{rank}(e) \geq 3$, such that $ch, hc, h \in eHe$. Note that $ece \notin Ce$. Indeed, if $ece \in Ce$, then $eceh = hece$, and hence $ch = hc$, a contradiction. On the other hand, $0 \neq h \in I \cap eRe$. Since R is centrally closed, $IC = I$ and $I[I, I] = 0$, it is clear that I is a minimal left ideal of R by [5, Lemma 5.1]. We also have that $I \cap eRe$ is still a minimal left ideal of eRe and $eRe \cong M_l(C)$, where $l = \text{rank}(e) \geq 3$. Indeed, if J is a left ideal of eRe such that $J \subsetneqq I \cap eRe$, then $RJ \subsetneqq RI \subsetneqq I$. Using the fact that RJ is a left ideal of R such that $RJ \subsetneqq I$ and I is a minimal ideal of R, we get $RJ = 0$. Hence $J = 0$ by the primeness of R. Now by the hypothesis, we have

$$[(eae)(exe)^{m+n} + [ebe, (exe)^m](exe)^n, (exe)^r]_k = 0 \quad \text{for all } x \in R,$$

and so

$$[(eae)x^{m+n} + [ebe, x^m]x^n, x^r]_k = 0 \quad \text{for all } x \in I \cap eRe.$$

In view of Lemma 3 this yields that $eRe \cong M_2(GF(2))$, a contradiction. This proves the result. $\square$

Example 1. Let $R = M_s(F), s > 1$, the $s \times s$ matrices over a field F and $I = Re_{11}$. If we set $a = 1 - e_{s1}$ and $b = e_{s1}$, then $[ax^{m+n} + [b, x^m]x^n, x^r]_k = 0$ for all $x \in I$, where k, m, n, r are fixed positive integers and $a + b \in Z(R)$.

Proof of Theorem 2. As we have already noted that every generalized derivation g on a dense left ideal of R can be uniquely extended to U, we may assume that g has the form $g(x) = ax + d(x)$ for some $a \in U$ and a derivation d on U. If $d = 0$, then $[ax^{m+n}, x^r]_k = 0$ for all $x \in I$. Then by Lemma 4 we have $a \in C$

except when $R \cong M_2(GF(2))$ and $I[I,I] = 0$. If $a \in C$, then $g(x) = ax = xa$ for all $x \in R$. So we may assume that $d \neq 0$.

In the light of Remark 2, we divide the proof into two cases:

Case 1. Let d be the inner derivation induced by the element $b \in U - C$, that is, $d(x) = [b, x]$ for all $x \in U$. Then I satisfies the nontrivial generalized polynomial identity

$$[aX^{m+n} + [b, X^m]X^n, X^r]_k.$$

By Remark 1, RI satisfies the above generalized identity. Since by [4], R and U satisfy the same GPIs, we have that UI satisfies above identity. Then applying Lemma 5 to UI, we have that $a + b \in C$ except when $U \cong M_2(GF(2))$ and $UI[UI, UI] = 0$. Moreover as in the proof of Lemma 5 we may replace R, I by U, UI, respectively. Then in particular, $a+b \in C$ except when $R \cong M_2(GF(2))$ and $I[I, I] = 0$. If $a + b \in C$, then $g(x) = ax + [b, x] = (a + b)x - xb = x(a + b - b) = xa$ for all $x \in R$.

Case 2. Let now d be an outer derivation of U. To continue the proof we set $G(Y, X) = \sum_{i=0}^{m-1} X^i Y X^{m-1-i}$, a non-commuting polynomial in variables X and Y. Note that $d(x^m) = G(d(x), x)$. Since

$$[ax^{m+n} + G(d(x), x)x^n, x^r]_k.$$

is an identity for I, then for any $u \in I - C$

$$[a(xu)^{m+n} + G(d(xu), xu)(xu)^n, (xu)^r]_k$$

is an identity for R. Thus R satisfies the following

$$[a(xu)^{m+n} + G(d(x)u + xd(u), xu)(xu)^n, (xu)^r]_k.$$

From Remark 2, since d is an outer derivation R satisfies the following identity

$$[a(xu)^{m+n} + G(yu + xd(u), xu)(xu)^n, (xu)^r]_k. \tag{7}$$

Taking $y = 0$ in (7) we get

$$[a(xu)^{m+n} + G(xd(u), xu)(xu)^n, (xu)^r]_k = 0. \tag{8}$$

By the linearity of $g(Y, X)$ on Y, subtracting equation (7) from (8) yields that R satisfies

$$[G(yu, xu)(xu)^n, (xu)^r]_k = 0,$$

which means that R satisfies

$$\begin{aligned} 0 &= \left[\sum_{i+j=m-1} (xu)^i (yu)(xu)^{j+n}, (xu)^r \right]_k \\ &= \sum_{i+j=m-1} (xu)^i [(yu), (xu)^r]_k (xu)^{j+n}. \end{aligned} \tag{9}$$

Clearly (9) is a nontrivial GPI for R, since $u \notin C$. So RC is a primitive ring with a non-zero socle H ([20]). $J = HI$ is a non-zero left ideal of H. Note that H is simple, $J = HJ$ and J satisfies the same basic conditions as I ([12]). Now replace R by H and I by J, then, without loss of generality, R is simple and equal to its own socle and $RI = I$. Let $e^2 = e$ be some non-trivial idempotent in I. So for all $x, y \in R$, we have

$$\sum_{i+j=m-1} (xe)^i[(ye), (xe)^r]_k (xe)^{j+n} = 0$$

and choosing $y = (1-e)r \in R$ we get

$$(1-e)(re)(xe)^{kr+m+n-1} = 0.$$

This leads to the contradiction that either $e = 0$ or $e = 1$. Thus any idempotent element of I is trivial, that is, $I = R$. Therefore we have to consider the condition

$$\sum_{i+j=m-1} x^i[y, x^r]_k x^{j+n} = 0$$

for all $x, y \in R$, which is a polynomial identity. From Lemma 2 in [11], it follows that there exists a suitable field F such that $R \subseteq M_s(F)$, the ring of all $s \times s$ matrices over F, and moreover $M_s(F)$ satisfies the same polynomial identity of R. In particular $M_s(F)$ satisfies

$$\sum_{i+j=m-1} x^i[y, x^r]_k x^{j+n} = 0. \tag{10}$$

Suppose $s \geq 2$ and choose $x = e_{11}$ and $y = e_{21}$ in (10). Then we have $e_{21} = 0$. Thus $s = 1$ and R is commutative, a contradiction. □

The following example shows our results do not hold in semiprime rings:

Example 2. Let F be any field. Consider the semiprime ring

$$R = \begin{pmatrix} GF(2) & GF(2) & 0 \\ GF(2) & GF(2) & 0 \\ 0 & 0 & F \end{pmatrix}.$$

Let

$$I = \begin{pmatrix} GF(2) & 0 & 0 \\ GF(2) & 0 & 0 \\ 0 & 0 & F \end{pmatrix}$$

be the left ideal of R. If $a = \left(\begin{smallmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right)$ and $b = \left(\begin{smallmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \alpha \end{smallmatrix}\right)$ for $\alpha \in F$ fixed, one can easily see that $[ax^2 + [b, x]x, x] = 0$ for all $x \in I$, since $uv(u+v) = 0$ for all $u, v \in GF(2)$. Then $g(x) = ax + [b, x] = (a+b)x - xb$ is a generalized derivation such that $[g(x)x, x] = 0$ for all $x \in I$. But

$$a + b \notin C = \left\{ \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \mu \end{pmatrix} \mid \lambda \in GF(2), \mu \in F \right\}.$$

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