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WEAK α-SKEW ARMENDARIZ RINGS

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Abstract. For an endomorphism α of a ring R, we introduce the weak α-skew Armendariz rings which are a generalization of the α-skew Armendariz rings and the weak Armendariz rings, and investigate their properties. Moreover, we prove that a ring R is weak α-skew Armendariz if and only if for any n, the $n \times n$ upper triangular matrix ring $T_n(R)$ is weak $\bar{\alpha}$-skew Armendariz, where $\bar{\alpha} : T_n(R) \to T_n(R)$ is an extension of α. If R is reversible and α satisfies the condition that $ab = 0$ implies $a\alpha(b)$=0 for any $a, b \in R$, then the ring $R[x]/(x^n)$ is weak $\bar{\alpha}$-skew Armendariz, where (x^n) is an ideal generated by x^n, n is a positive integer and $\bar{\alpha} : R[x]/(x^n) \to R[x]/(x^n)$ is an extension of α. If α also satisfies the condition that $\alpha^t = 1$ for some positive integer t, the ring $R[x]$ (resp, $R[x; \alpha]$) is weak $\bar{\alpha}$-skew (resp, weak) Armendariz, where $\bar{\alpha} : R[x] \to R[x]$ is an extension of α.

1. Introduction

Throughout this paper R denotes an associative ring with identity, nil(R) denotes the set of all the nilpotent elements of R and α always means the endomorphism of R. Rege and Chhawchharia [9] introduced the notion of an Armendariz ring. They defined a ring R to be an Armendariz ring if whenever polynomials $f(x) = a_0 + a_1x + \cdots + a_mx^m$, $g(x) = b_0 + b_1x + \cdots + b_nx^n \in R[x]$ satisfy $f(x)g(x) = 0$, then $a_ib_j = 0$ for each i and j. The name "Armendariz ring" was chosen because Armendariz [2] had noted that every reduced ring satisfies this condition. Some properties of the Armendariz rings were studied in Rege and Chhawchharia [9], Armendariz [2], Anderson and Camillo [1], Huh et al. [5], and Kim and Lee [6]. For an endomorphism α of a ring R, Hong, Kim, and Kwak [4] called R an α-skew Armendariz ring if whenever polynomials $f(x) = a_0 + a_1x + \cdots + a_mx^m$, $g(x) = b_0 + b_1x + \cdots + b_nx^n \in R[x; \alpha]$ satisfy $f(x)g(x) = 0$, then $a_i\alpha^i(b_j) = 0$ for each i and j, which is a generalization

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of the Armendariz rings. They showed that if a ring R is α-rigid (That is, $a\alpha(a) = 0$ implies $a = 0$ for $a \in R$), then $R[x]/(x^2)$ is $\bar{\alpha}$-skew Armendariz. They also showed that if $\alpha^t = 0$ for some positive integer t, then R is α-skew Armendariz if and only if $R[x]$ is $\bar{\alpha}$-skew Armendariz. Liu and Zhao [8] called a ring R weak Armendariz if whenever polynomials $f(x) = a_0+a_1x+\cdots+a_mx^m$, $g(x) = b_0+b_1x+\cdots+b_nx^n \in R[x]$ satisfy $f(x)g(x) = 0$, then a_ib_j is a nilpotent element of R for each i and j. They showed that the semicommutative rings are weak Armendariz, and R is weak Armendariz if and only if the $n\times n$ upper triangular matrix ring over R is weak Armendariz. Moreover, they also showed that for a semicommutative ring R, $R[x]/(x^n)$ is weak Armendariz.

Motivated by the above results, for an endomorphism α of a ring R, we investigate a generalization of the α-skew Armendariz rings and the weak Armendariz rings which we call a weak α-skew Armendariz ring and discuss the relationship between reversible rings and weak α-skew Armendariz rings.

2. Weak α-skew Armendariz rings

Definition 2.1. Let R be a ring and α be an endomorphism of R. R is said to be weak α-skew Armendariz if whenever polynomials $f(x) = a_0 + a_1x + \cdots + a_mx^m$, $g(x) = b_0 + b_1x + \cdots + b_nx^n \in R[x;\alpha]$ satisfy $f(x)g(x) = 0$, then $a_i\alpha^i(b_j) \in \text{nil}(R)$ for each i and j.

Let α be an endomorphism of a ring R and $M_n(R)$ be the $n\times n$ full matrix ring over R, and $\bar{\alpha}$: $M_n(R) \longrightarrow M_n(R)$ defined by $\bar{\alpha}((a_{ij})) = (\alpha(a_{ij}))$. Then $\bar{\alpha}$ is an endomorphism of $M_n(R)$. Clearly, $\bar{\alpha}|_{T_n(R)}$, the restriction of $\bar{\alpha}$ to $T_n(R)$, is an endomorphism of $T_n(R)$, where $T_n(R)$ is the $n\times n$ upper triangular matrix ring over R. We also denote $\bar{\alpha}|_{T_n(R)}$ by $\bar{\alpha}$.

For an α-skew Armendariz ring R, the $T_n(R)$ $(n \geq 2)$ need not be $\bar{\alpha}$-skew Armendariz by [3, Example 14]. However, we have the following result.

Proposition 2.2. *Let α be an endomorphism of a ring R. Then R is a weak α-skew Armendariz ring if and only if, for any n, $T_n(R)$ is a weak $\bar{\alpha}$-skew Armendariz ring.*

Proof. Note that any invariant subring of weak α-skew Armendariz rings is a weak α-skew Armendariz ring. Thus if $T_n(R)$ is a weak $\bar{\alpha}$-skew Armendariz ring, then R is a weak α-skew Armendariz ring.

Conversely, let $f(x) = A_0+A_1x+\cdots+A_px^p$, and $g(x) = B_0+B_1x+\cdots+B_qx^q$ be elements of $T_n(R)[x;\bar{\alpha}]$ satisfying $f(x)g(x) = 0$, where

$$A_i = \begin{pmatrix} a_{11}^{(i)} & a_{12}^{(i)} & a_{13}^{(i)} & \cdots & a_{1n}^{(i)} \\ 0 & a_{22}^{(i)} & a_{23}^{(i)} & \cdots & a_{2n}^{(i)} \\ 0 & 0 & a_{33}^{(i)} & \cdots & a_{3n}^{(i)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_{nn}^{(i)} \end{pmatrix} \text{ and } B_j = \begin{pmatrix} b_{11}^{(j)} & b_{12}^{(j)} & b_{13}^{(j)} & \cdots & b_{1n}^{(j)} \\ 0 & b_{22}^{j} & b_{23}^{j} & \cdots & b_{2n}^{j} \\ 0 & 0 & b_{33}^{(j)} & \cdots & b_{3n}^{(j)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & b_{nn}^{(j)} \end{pmatrix}.$$

Then from $f(x)g(x) = 0$, it follows that

$$\left(\sum_{i=0}^{p} a_{ss}^{(i)} x^i\right)\left(\sum_{j=0}^{q} b_{ss}^{(j)} x^j\right) = 0$$

in $R[x;\alpha]$ for each s with $1 \le s \le n$. Since R is weak α-skew Armendariz, there exists $m_{ijs} \in \mathbb{N}$ such that $(a_{ss}^{(i)}\alpha^i(b_{ss}^{(j)}))^{m_{ijs}} = 0$ for any s, i and j. Let $m_{ij} = \max\{m_{ij1}, m_{ij2}, \ldots, m_{ijn}\}$. Then

$$(A_i\bar{\alpha}^i(B_j))^{m_{ij}} = \begin{pmatrix} a_{11}^{(i)}\alpha^i(b_{11}^{(j)}) & * & * & \cdots & * \\ 0 & a_{22}^{(i)}\alpha^i(b_{22}^{(j)}) & * & \cdots & * \\ 0 & 0 & a_{33}^{(i)}\alpha^i(b_{33}^{(j)}) & \cdots & * \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_{nn}^{(i)}\alpha^i(b_{nn}^{(j)}) \end{pmatrix}^{m_{ij}}$$

$$= \begin{pmatrix} 0 & * & * & \cdots & * \\ 0 & 0 & * & \cdots & * \\ 0 & 0 & 0 & \cdots & * \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

Thus $((A_i\bar{\alpha}^i(B_j))^{m_{ij}})^n = 0$. This shows that $T_n(R)$ is a weak $\bar{\alpha}$-skew Armendariz ring. $\square$

Corollary 2.3 ([8, Proposition 2.2])**.** *A ring R is a weak Armendariz ring if and only if for any n, $T_n(R)$ is a weak Armendariz ring.*

Corollary 2.4. *If a ring R is an α-skew Armendariz ring, then for any n, $T_n(R)$ is a weak $\bar{\alpha}$-skew Armendariz ring.*

Liu and Zhao [8, Example 2.5] showed that $M_n(R)$ $(n \ge 2)$ over a weak 1_R-skew Armendariz ring R need not be weak $\bar{1}_R$-skew Armendariz ring. In general, for any ring R and any endomorphism α of R, $M_n(R)$ $(n \ge 2)$ over R need not be weak $\bar{\alpha}$-skew Armendariz rings.

Example 2.5. Let R be a ring and α be an endomorphism of R. Let $S = M_2(R)$. For $f(x) = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right) + \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right) x$ and $g(x) = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 0 \end{smallmatrix}\right) + \left(\begin{smallmatrix} 0 & 0 \\ -1 & -1 \end{smallmatrix}\right) x$ in $S[x;\bar{\alpha}]$, we have $f(x)g(x) = 0$. But $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)\alpha(\left(\begin{smallmatrix} 1 & 1 \\ 0 & 0 \end{smallmatrix}\right)) = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ is not nilpotent. Thus S is not weak $\bar{\alpha}$-skew Armendariz.

We note that the α-skew Armendariz ring is weak α-skew Armendariz, but the converse is not always true by the following example.

Example 2.6. Let α be an endomorphism of a ring R and R be an α-rigid ring. Let

$$S_4 = \left\{ \begin{pmatrix} a & a_{12} & a_{13} & a_{14} \\ 0 & a & a_{23} & a_{24} \\ 0 & 0 & a & a_{34} \\ 0 & 0 & 0 & a \end{pmatrix} | a, a_{ij} \in R \right\}.$$

Since R is an α-rigid ring, it is α-skew Armendariz by [4, Corollary 4]. Hence R is weak α-skew Armendariz. Thus S_4 is weak $\bar{\alpha}$-skew Armendariz by Proposition 2.2. However, S_4 is not $\bar{\alpha}$-skew Armendariz by [4, Example 18].

Given a ring R and a bimodule ${}_RM_R$, the trivial extension of R by M is the $T(R, M) = R \bigoplus M$ with the usual addition and the multiplication: $(r_1, m_1)(r_2, m_2) = (r_1r_2, r_1m_2 + m_1r_2)$. This is isomorphic to the ring of all matrices $\left(\begin{smallmatrix} r & m \\ 0 & r \end{smallmatrix}\right)$, where $r \in R$ and $m \in M$ and the usual matrix operations are used.

Proposition 2.7. *Let α be an endomorphism of a ring R. Then R is a weak α-skew Armendariz ring if and only if the trivial extension $T(R, R)$ is a weak $\bar{\alpha}$-skew Armendariz ring.*

Proof. It follows from Proposition 2.2. □

There exist an abelian ring R and an endomorphism α such that $\alpha(e) \neq e$ for some $e^2 = e \in R$ by Example 3.7. In the following, we provide a characterization of an abelian ring R.

Proposition 2.8. *Let R be an abelian ring and α be an endomorphism with $\alpha(e) = e$ for every $e^2 = e \in R$. Then R is weak α-skew Armendariz if and only if eR and $(1 - e)R$ are weak α-skew Armendariz for some $e^2 = e \in R$*

Proof. If R is weak α-skew Armendariz, eR and $(1 - e)R$ are weak α-skew Armendariz since they are the invariant subrings of R. Conversely, let $f(x) = a_0 + a_1x + \cdots + a_mx^m$, $g(x) = b_0 + b_1x + \cdots + b_nx^n$ in $R[x; \alpha]$ with $f(x)g(x) = 0$. Let $f_1(x) = ef(x)$, $f_2(x) = (1 - e)f(x)$, $g_1(x) = eg(x)$ and $g_2(x) = (1 - e)g(x)$. Then $f_1(x)g_1(x) = 0$ and $f_2(x)g_2(x) = 0$. Since eR and $(1 - e)R$ are weak α-skew Armendariz, there exist m_{ij} and n_{ij} such that $e(a_i\alpha^i(b_j))^{m_{ij}} = ((ea_i)\alpha^i(eb_j))^{m_{ij}} = 0$ and $(1-e)(a_i\alpha^i(b_j))^{n_{ij}} = (((1-e)a_i)\alpha^i((1-e)b_j))^{m_{ij}} = 0$. Let $k_{ij} = \max\{m_{ij}, n_{ij}\}$. Then $e(a_i\alpha^i(b_j))^{k_{ij}} = 0$ and $(1 - e)(a_i\alpha^i(b_j))^{k_{ij}} = 0$. Hence $(a_i\alpha^i(b_j))^{k_{ij}} = 0$. This means that R is weak α-skew Armendariz. □

Let I be an ideal of R. If $\alpha(I) \subseteq I$, then $\bar{\alpha} : R/I \to R/I$ defined by $\bar{\alpha}(a + I) = \alpha(a) + I$ for $a \in R$ is an endomorphism of the factor ring R/I

Proposition 2.9. *Let α be an endomorphism of a ring R and I be an ideal of R with $\alpha(I) \subseteq I$. If $I \subseteq \mathrm{nil}(R)$ and R/I is weak $\bar{\alpha}$-skew Armendariz, then R is weak α-skew Armendariz.*

Proof. Let $f(x) = a_0 + a_1x + \cdots + a_mx^m$, $g(x) = b_0 + b_1x + \cdots + b_nx^n$ in $R[x;\alpha]$ with $f(x)g(x) = 0$. Then $(\sum_{i=0}^{m} \bar{a}_i x^i)(\sum_{j=0}^{n} \bar{b}_j x^j) = 0$. Thus $(\bar{a}_i\bar{\alpha}^i(\bar{b}_j))^{n_{ij}} = 0$ for some positive integer n_{ij}. Hence $a_i\alpha^i(b_j) \in \mathrm{nil}(R)$. Therefore R is weak α-skew Armendariz. □

Let D be a ring and C a subring of D with $1_D \in C$. Let $R = \mathbb{R}[D,C] = \{(d_1, \ldots, d_n, c, c, \ldots) \mid d_i \in D, c \in C, n \geq 1\}$. With addition and multiplication defined componentwise, R is a ring (see [3]). Let α be an endomorphism of D. Then $\bar{\alpha} : R \longrightarrow R$ defined by $\bar{\alpha}((d_1, \ldots, d_n, c, c, \ldots)) = (\alpha(d_1), \ldots, \alpha(d_n), \alpha(c), \alpha(c), \ldots)$ for $(d_1, \ldots, d_n, c, c, \ldots) \in R$ is an endomorphism of R.

Proposition 2.10. *D is weak α-skew Armendariz if and only if R is weak $\bar{\alpha}$-skew Armendariz.*

Proof. Suppose that D is weak α-skew Armendariz. Let $f(x) = \sum_{i=0}^{p} \xi_i x^i$, $g(x) = \sum_{j=0}^{q} \delta_j x^j \in R[x;\bar{\alpha}]$ be such that $f(x)g(x) = 0$. Without loss of generality, we can assume that there exists n such that $\xi_i = (a_{1i}, \ldots, a_{ni}, c_i, c_i, \ldots)$, $\delta_j = (b_{1j}, \ldots, b_{nj}, d_j, d_j, \ldots) \in R$ for all i, j. Let $f_s(x) = \sum_{i=0}^{p} a_{si}x^i$, $g_s(x) = \sum_{j=0}^{q} b_{sj}x^j$ with $1 \leq s \leq n$ and $f'(x) = \sum_{i=0}^{p} c_i x^i$, $g'(x) = \sum_{j=0}^{q} d_j x^j$. From $f(x)g(x) = 0$, we obtain $f_s(x)g_s(x) = 0$ and $f'(x)g'(x) = 0$ in $D[x;\alpha]$ for all s. Hence $a_{si}\alpha^i(b_{sj}) \in \mathrm{nil}(R)$ and $c_i\alpha^i(d_j) \in \mathrm{nil}(R)$ for all i, j, s. Suppose that $(a_{si}\alpha^i(b_{sj}))^{t_{sij}} = 0$ and $(c_i\alpha^i(d_j))^{t'_{ij}} = 0$ for $1 \leq s \leq n$. Set $t_{ij} = \max\{t_{1ij}, t_{2ij}, \ldots, t_{nij}, t'_{ij}\}$. Then we have $(\xi_i\bar{\alpha}^i(\delta_j))^{t_{ij}} = 0$ for all i, j. This means R is weak $\bar{\alpha}$-skew Armendariz.

Conversely, since D is a invariant subring of R, the assertion holds. □

Let R be a ring, α an automorphism of R and Ω a multiplicatively closed subset of R consisting of central regular elements. We define $\bar{\alpha} : \Omega^{-1}R \to \Omega^{-1}R$ by $\bar{\alpha}(b^{-1}a) = (\alpha(b))^{-1}\alpha(a)$ for any $b^{-1}a \in \Omega^{-1}R$. Then $\bar{\alpha}$ is an automorphism of $\Omega^{-1}R$.

Proposition 2.11. *A ring R is weak α-skew Armendariz if and only if $\Omega^{-1}R$ is weak $\bar{\alpha}$-skew Armendariz.*

Proof. The proof is similar to that of Proposition 3.11. □

The ring of Laurent polynomials in x coefficients in a ring R consists of all formal sums $\sum_{i=k}^{n} a_i x^i$ with obvious addition and multiplication, where $a_i \in R$ and k, n are (possibly negative) integers. We denote this ring by $R[x;x^{-1}]$. For an automorphism α of R, $\bar{\alpha} : R[x;x^{-1}] \to R[x;x^{-1}]$ defined by $\bar{\alpha}(f(x)) = \sum_{i=k}^{n} \alpha(a_i)x^i$ is an automorphism of $R[x;x^{-1}]$. $\bar{\alpha}|_{R[x]}$, the restriction of $\bar{\alpha}$ to $R[x]$, is also denoted by $\bar{\alpha}$.

Corollary 2.12. *For a ring R and an automorphism α of R, $R[x]$ is weak $\bar{\alpha}$-skew Armendariz if and only if $R[x;x^{-1}]$ is weak $\bar{\alpha}$-skew Armendariz.*

Proof. Suppose that $R[x]$ is weak $\bar{\alpha}$-skew Armendariz. Let $\Omega = \{1, x, x^2, \ldots, \}$, then clearly Ω is a multiplicatively closed subset of $R[x]$. Since $R[x;x^{-1}] = \Omega^{-1}R[x]$, the proof is completed by Proposition 2.10. The converse is clear. □

3. Reversible rings and weak α-skew Armendariz rings

A ring R is called reversible if for any $a, b \in R$, $ab = 0$ implies $ba = 0$. A ring R is called semicommutative if for any $a, b \in R$, $ab = 0$ implies $aRb = 0$. Kim and Lee [7] showed that the reversible rings are semicommutative and the converse may not be true. Moreover, Rege and Chhawchharia showed that commutative (hence reversible) rings need not to be Armendariz in [9, Example 3.2]. Liu and Zhao showed that the semicommutative rings are weak Armendariz, so are the reversible rings. However, there exists an endomorphism α of a reversible ring R such that R is not weak α-skew Armendariz by the following example.

Example 3.1. Let $R = \mathbb{Z}_2 \bigoplus \mathbb{Z}_2$, where $\mathbb{Z}_2$ is the ring of integer module 2. Then R is a commutative reduced ring. So it is reversible. Let $\alpha : R \longrightarrow R$ be an endomorphism defined by $\alpha((a, b)) = (b, a)$. Then for $p = (1, 0) + (1, 0)x$, $q = (0, 1)+(1, 0)x$ in $R[x; \alpha]$, $pq = 0$, but $(1, 0)\alpha((0, 1)) = (1, 0)$ is not nilpotent. Therefore R is not weak α-skew Armendariz.

Example 3.2 also shows that weak α-skew Armendariz rings need not be reversible.

Example 3.2. In Example 2.6, S_4 is weak $\bar{\alpha}$-skew Armendariz, but S_4 is not semicommutative by [6, Example 1.3], so it is not reversible.

Lemma 3.3. *Let R be a reversible ring and α be an endomorphism of R such that $a\alpha(b) = 0$ whenever $ab = 0$ for any $a, b \in R$. If $ab \in \mathrm{nil}(R)$, then $a\alpha^k(b) \in \mathrm{nil}(R)$ for any positive integer k.*

Proof. Suppose that $(ab)^t = 0$ for $a, b \in R$ and some positive integer t. Then $(ab)^{t-1}ab = 0$, so $(ab)^{t-1}a\alpha^k(b) = 0$ for any positive integer k by the hypothesis. Thus, $a\alpha^k(b)(ab)^{t-1} = 0$ since R is reversible. That is, $a\alpha^k(b)(ab)^{t-2}ab = 0$. Similarly, we have $a\alpha^k(b)(ab)^{t-2}a\alpha^k(b) = 0$, $(a\alpha^k(b))^2(ab)^{t-2} = 0$. Continuing this process, we obtain that $(a\alpha^k(b))^t = 0$. □

Lemma 3.4. *Let R be a reversible ring and α be an endomorphism of R such that $a\alpha(b) = 0$ whenever $ab = 0$ for any $a, b \in R$. If $a_0, a_1, \ldots, a_n \in \mathrm{nil}(R)$, then $a_0 + a_1x + \cdots + a_nx^n \in \mathrm{nil}(R[x; \alpha])$.*

Proof. First we claim that $\alpha^{k_1}(a)\alpha^{k_2}(a)\cdots\alpha^{k_m}(a) = 0$ if $a^m = 0$, where $k_1, k_2, \ldots, k_m$ are any nonnegative integers. Since $a^{m-1}a = 0$, $a^{m-1}\alpha^{k_m}(a) = 0$ by the hypothesis. Thus, $\alpha^{k_m}(a)a^{m-1} = 0$ since R is reversible. We have $\alpha^{k_m}(a)a^{m-2}a = 0$. Similarly $\alpha^{k_m}(a)a^{m-2}\alpha^{k_{m-1}}(a) = 0$. It follows that $\alpha^{k_{m-1}}(a)\alpha^{k_m}(a)a^{m-2} = 0$. Continuing this process, we obtain the above result.

Suppose that $a_i^{m_i} = 0$, $i = 0, 1, \ldots, n$. Let $k = m_0 + m_1 + \cdots + m_n + 1$. Then

$$(a_0 + a_1x + \cdots + a_nx^n)^k = \sum_{s=0}^{nk}\Big(\sum_{i_1+i_2+\cdots+i_k=s}\alpha^{t_1}(a_{i_1})\alpha^{t_2}(a_{i_2})\cdots\alpha^{t_k}(a_{i_k})\Big)x^s,$$

where $t_j \geq 0$, $a_{i_j} \in \{a_0, a_1, \ldots, a_n\}$, $j = 1, 2, \ldots, k$. If the number of a_0 $'s$ in $\alpha^{t_1}(a_{i_1})\alpha^{t_2}(a_{i_2}) \cdots \alpha^{t_k}(a_{i_k})$ is more than m_0, then we write $\alpha^{t_1}(a_{i_1})\alpha^{t_2}(a_{i_2}) \cdots \alpha^{t_k}(a_{i_k})$ as $b_0\alpha^{p_1}(a_0)b_1\alpha^{p_2}(a_0) \cdots b_{q-1}\alpha^{p_q}(a_0)b_q$, where $p_1, p_2, \ldots, p_q \geq 0$ and b_i is a product of some elements choosing from $\{\alpha^{t_j}(a_{i_j}) | a_{i_j} \neq a_0, j = 1, 2, \ldots, k\}$ or is equal to 1. Since $a_0^{m_0} = 0$, $\alpha^{p_1}(a_0)\alpha^{p_2}(a_0) \cdots \alpha^{p_q}(a_0) = 0$. Thus

$$\alpha^{t_1}(a_{i_1})\alpha^{t_2}(a_{i_2}) \cdots \alpha^{t_k}(a_{i_k}) = 0$$

since R is reversible. If the number of a_i $'s$ in $\alpha^{t_1}(a_{i_1})\alpha^{t_2}(a_{i_2}) \cdots \alpha^{t_k}(a_{i_k})$ is more than m_i, then a similar discussion yields that

$$\alpha^{t_1}(a_{i_1})\alpha^{t_2}(a_{i_2}) \cdots \alpha^{t_k}(a_{i_k}) = 0.$$

Hence

$$\sum_{i_1+i_2+\cdots+i_k=s} \alpha^{t_1}(a_{i_1})\alpha^{t_2}(a_{i_2}) \cdots \alpha^{t_k}(a_{i_k}) = 0,$$

which implies that $(a_0 + a_1x + \cdots + a_nx^n)^k = 0$ in $R[x; \alpha]$. □

Proposition 3.5. *Let R be a reversible ring and α be an endomorphism of R such that $a\alpha(b) = 0$ whenever $ab = 0$ for any $a, b \in R$. Then R is weak α-skew Armendariz.*

Proof. Suppose that $f(x)g(x) = 0$, where $f(x) = a_0 + a_1x + \cdots + a_mx^m$ and $g(x) = b_0 + b_1x + \cdots + b_nx^n \in R[x; \alpha]$. Then we have the following equations:

$$a_0b_0 = 0 \tag{1}$$

$$a_0b_1 + a_1\alpha(b_0) = 0 \tag{2}$$

$$a_0b_2 + a_1\alpha(b_1) + a_2\alpha^2(b_0) = 0 \tag{3}$$

$$\cdots$$

$$a_0b_k + a_1\alpha(b_{k-1}) + \cdots + a_{k-1}\alpha^{k-1}(b_1) + a_k\alpha^k(b_0) = 0 \tag{4}$$

$$\cdots$$

We will show that $a_i\alpha^i(b_j) \in \mathrm{nil}(R)$ by induction on $i + j$.

If $i + j = 0$, then $a_0b_0 = 0 \in \mathrm{nil}(R)$.

Now suppose that $k \leq m + n$ is such that $a_i\alpha^i(b_j) \in \mathrm{nil}(R)$ when $i + j < k$. We will show that $a_i\alpha^i(b_j) \in \mathrm{nil}(R)$ when $i+j = k$. By Lemma 3.3, $a_i\alpha^k(b_0) \in \mathrm{nil}(R)$ for any $i < k$. Since R is reversible, $a_ir\alpha^k(b_0) \in \mathrm{nil}(R)$ for any $r \in R$. Multiplying the equation (4) on the right side by $\alpha^k(b_0)$, then the equation (4) becomes

$$a_0b_k\alpha^k(b_0) + a_1\alpha(b_{k-1})\alpha^k(b_0) + \cdots + a_{k-1}\alpha^{k-1}(b_1)\alpha^k(b_0) + a_k\alpha^k(b_0)\alpha^k(b_0) = 0.$$

It follows that

$$a_k\alpha^k(b_0)\alpha^k(b_0) = -(a_0b_k\alpha^k(b_0) + a_1\alpha(b_{k-1})\alpha^k(b_0) + \cdots + a_{k-1}\alpha^{k-1}(b_1)\alpha^k(b_0)).$$

Since R is reversible, by [8, Lemma 3.1], $a_k\alpha^k(b_0)\alpha^k(b_0) \in \mathrm{nil}(R)$. Thus, $a_k\alpha^k(b_0) \in \mathrm{nil}(R)$. Multiplying the equation (4) on the right side by $\alpha^{k-1}(b_1)$. Similarly we have $a_{k-1}\alpha^{k-1}(b_1) \in \mathrm{nil}(R)$. Continuing this process, we have

$a_i\alpha^i(b_j) \in \text{nil}(R)$ when $i + j = k$. Therefore $a_i\alpha^i(b_j) \in \text{nil}(R)$ for all i, j, and R is weak α-skew Armendariz. □

We note that R is reversible in Example 3.1, but R is not weak α-skew Armendariz. Thus the condition that $ab = 0$ implies $a\alpha(b) = 0$ in Proposition 3.5 is not superfluous.

Recall that for an endomorphism α of a ring R, α is rigid if $a\alpha(a) = 0$ implies $a = 0$ for any $a \in R$. R is α-rigid if there exists a rigid endomorphism α of R. If R is α-rigid, then R is reversible and satisfies the condition that $ab = 0$ implies $a\alpha(b) = 0$ for any $a, b \in R$, but the converse is not true by the following examples.

Example 3.6. Let $R = \mathbb{Z}_4$ and $\alpha = 1_R$. Then R is reversible, and $ab = 0$ implies $\alpha(ab) = 0$. But R is not α-rigid.

Example 3.7. Let $R = \{\left(\begin{smallmatrix} a & t \\ 0 & a \end{smallmatrix}\right) | a \in \mathbb{Z}, t \in \mathbb{C}\}$, where $\mathbb{Z}$ and $\mathbb{C}$ are the set of all integers and all complex numbers, respectively. Then R is a commutative ring, so it is reversible. Let $\alpha : R \longrightarrow R$ be defined by $\alpha(\left(\begin{smallmatrix} a & t \\ 0 & a \end{smallmatrix}\right)) = \left(\begin{smallmatrix} a & \bar{t} \\ 0 & a \end{smallmatrix}\right)$, where $\bar{t}$ denotes the conjugate of t. Then

(1) R is not α-rigid: $\left(\begin{smallmatrix} 0 & t \\ 0 & 0 \end{smallmatrix}\right) \alpha(\left(\begin{smallmatrix} 0 & t \\ 0 & 0 \end{smallmatrix}\right)) = 0$, but $\left(\begin{smallmatrix} 0 & t \\ 0 & 0 \end{smallmatrix}\right) \neq 0$ if $t \neq 0$.
(2) $AB = 0$ implies $A\alpha(B) = 0$ for any A, $B \in R$.
Let $A = \left(\begin{smallmatrix} a & s \\ 0 & a \end{smallmatrix}\right)$ and $B = \left(\begin{smallmatrix} b & t \\ 0 & b \end{smallmatrix}\right)$. If $AB = 0$, $ab = 0$ and $at + sb = 0$.
(i) $a \neq 0$, then $b = 0$, $t = 0$. So $A\alpha(B) = 0$.
(ii) $b \neq 0$, then $a = 0$, $s = 0$. So $A\alpha(B) = 0$.
(iii) $a = 0$, $b = 0$, then $A\alpha(B) = 0$.

For a ring R and an endomorphism α of R, $\bar{\alpha} : R[x] \to R[x]$ defined by $\bar{\alpha}(f(x)) = \sum_{i=0}^{m} \alpha(a_i)x^i$ for any $f(x) = \sum_{i=0}^{m} a_i x^i \in R[x]$ is an endomorphism of $R[x]$. Moreover, the endomorphism of $R[x]/(x^n)$ induced by $\bar{\alpha}$ is also denoted by $\bar{\alpha}$. Hong, Kim and Kwak [4, Proposition 8] showed that if R is an α-rigid ring, then $R[x]/(x^2)$ is $\bar{\alpha}$-skew Armendariz. For weak α-skew Armendariz rings, we have the following results.

Theorem 3.8. *Let R be a reversible ring and α be an endomorphism of R such that $a\alpha(b) = 0$ whenever $ab = 0$ for any $a, b \in R$. Then $R[x]/(x^n)$ is a weak $\bar{\alpha}$-skew Armendariz ring for any positive integer n.*

Proof. Denote $\bar{x}$ in $R[x]/(x^n)$ by u, so $R[x]/(x^n) = R[u] = R + Ru + \cdots + Ru^{n-1}$, where u commutes with elements of R and $u^n = 0$. Let f, $g \in R[u][y; \bar{\alpha}]$ be such that $fg = 0$. Suppose that $f = \sum_{i=0}^{p} f_i y^i$ and $g = \sum_{j=0}^{q} g_j y^j$, where $f_i = \sum_{s=0}^{n-1} a_s^{(i)} u^s$, $g_j = \sum_{t=0}^{n-1} b_t^{(j)} u^t$ for $0 \leq i \leq p$ and $0 \leq j \leq q$. From $fg = 0$, we have the following equation

$$\sum_{s+t=k} u_s v_t = 0$$

in $R[y; \alpha]$, $k = 0, 1, \ldots, n-1$, where $u_s = a_s^{(0)} + a_s^{(1)} y + \cdots + a_s^{(p)} y^p$ and $v_t = b_t^{(0)} + b_t^{(1)} y + \cdots + b_t^{(q)} y^q$. We will show by induction on $s + t$ that

$a_s^{(i)}\alpha^i(b_t^{(j)}) \in \mathrm{nil}(R)$ for any $0 \le i \le p$, $0 \le j \le q$, and any s, t with $s+t = 0, 1, \ldots, n-1$. If $s+t=0$, then $s=t=0$. Thus $u_0v_0 = 0$. By Proposition 3.5, R is weak α-skew Armendariz, so $a_0^{(i)}\alpha^i(b_0^{(j)}) \in \mathrm{nil}(R)$ for $0 \le i \le p$, $0 \le j \le q$. Now suppose that $k \le n-1$ is such that $a_s^{(i)}\alpha^i(b_t^{(j)}) \in \mathrm{nil}(R)$ for any $0 \le i \le p$, $0 \le j \le q$ and any s, t with $s+t<k$. We will show that $a_s^{(i)}\alpha^i(b_t^{(j)}) \in \mathrm{nil}(R)$ for any $0 \le i \le p$, $0 \le j \le q$ and any s, t with $s+t=k$. From the equation

$$u_0v_k + u_1v_{k-1} + \cdots + u_kv_0 = 0,$$

we have

$$\sum_{s+t=k} a_s^{(0)}b_t^{(0)} = 0 \tag{1}$$

$$\sum_{s+t=k} a_s^{(0)}b_t^{(1)} + \sum_{s+t=k} a_s^{(1)}\alpha(b_t^{(0)}) = 0 \tag{2}$$

$$\sum_{s+t=k} a_s^{(0)}b_t^{(2)} + \sum_{s+t=k} a_s^{(1)}\alpha(b_t^{(1)}) + \sum_{s+t=k} a_s^{(2)}\alpha^2(b_t^{(0)}) = 0 \tag{3}$$

$$\cdots$$

(4)

$$\sum_{s+t=k} a_s^{(0)}b_t^{(p+q)} + \sum_{s+t=k} a_s^{(1)}\alpha(b_t^{(p+q-1)}) + \cdots + \sum_{s+t=k} a_s^{(p+q)}\alpha^{p+q}(b_t^{(0)}) = 0.$$

If $s \ge 1$, then $k-s<k$. Thus, by the induction hypothesis, $a_0^{(0)}b_{k-s}^{(0)} \in \mathrm{nil}(R)$. Since R is reversible, $b_{k-s}^{(0)}a_0^{(0)} \in \mathrm{nil}(R)$, and $a_1^{(0)}b_{k-1}^{(0)}a_0^{(0)} + a_2^{(0)}b_{k-2}^{(0)}a_0^{(0)} + \cdots + a_k^{(0)}b_0^{(0)}a_0^{(0)} = a_1^{(0)}(b_{k-1}^{(0)}a_0^{(0)}) + a_2^{(0)}(b_{k-2}^{(0)}a_0^{(0)}) + \cdots + a_k^{(0)}(b_0^{(0)}a_0^{(0)}) \in \mathrm{nil}(R)$ by [8, Lemma 3.1]. Therefore, if we multiply $\sum_{s+t=k} a_s^{(0)}b_t^{(0)} = 0$ on the right side by $a_0^{(0)}$, then it follows that $a_0^{(0)}b_k^{(0)}a_0^{(0)} \in \mathrm{nil}(R)$ and, so $a_0^{(0)}b_k^{(0)} \in \mathrm{nil}(R)$. If we multiply $\sum_{s+t=k} a_s^{(0)}b_t^{(0)} = 0$ on the right side by $a_1^{(0)}$, then, by [8, Lemma 3.1],

$$\begin{aligned} a_1^{(0)}b_{k-1}^{(0)}a_1^{(0)} &= -a_0^{(0)}b_k^{(0)}a_1^{(0)} - (a_2^{(0)}b_{k-2}^{(0)}a_1^{(0)} + \cdots + a_k^{(0)}b_0^{(0)}a_1^{(0)}) \\ &= -(a_0^{(0)}b_k^{(0)})a_1^{(0)} - (a_2^{(0)}(b_{k-2}^{(0)}a_1^{(0)}) + \cdots + a_k^{(0)}(b_0^{(0)}a_1^{(0)})) \in \mathrm{nil}(R). \end{aligned}$$

Thus $a_1^{(0)}b_{k-1}^{(0)} \in \mathrm{nil}(R)$. Similarly, we can show that $a_2^{(0)}b_{k-2}^{(0)} \in \mathrm{nil}(R), \ldots$, $a_k^{(0)}b_0^{(0)} \in \mathrm{nil}(R)$. So we have $a_s^{(i)}\alpha^i(b_t^{(j)}) \in \mathrm{nil}(R)$ for any s, t with $s+t=k$ and any i, j with $i+j=0$. Suppose that $l \le p+q$ is such that $a_s^{(i)}\alpha^i(b_t^{(j)}) \in \mathrm{nil}(R)$ for any s, t with $s+t=k$ and any i, j with $i+j<l$. We will show that $a_s^{(i)}\alpha^i(b_t^{(j)}) \in \mathrm{nil}(R)$ for any s, t with $s+t=k$ and any i, j with $i+j=l$. If $t<k$, then by the induction hypothesis, $a_0^{(0)}b_t^{(j)} \in \mathrm{nil}(R)$, so $a_0^{(0)}\alpha^r(b_t^{(j)}) \in \mathrm{nil}(R)$ for any nonnegative integer r by Lemma 3.3. Hence $\alpha^r(b_t^{(j)})a_0^{(0)} \in \mathrm{nil}(R)$. If $i \ge 1$, then $l-i<l$. Thus, by the induction hypothesis on $p+q$, $a_0^{(0)}(b_k^{(l-i)}) \in \mathrm{nil}(R)$ for any $i \ge 1$, which implies $a_0^{(0)}\alpha^r(b_k^{(l-i)}) \in \mathrm{nil}(R)$ for any nonnegative integer r. Hence $\alpha^r(b_k^{(l-i)})a_0^{(0)} \in \mathrm{nil}(R)$. Multiplying

$\sum_{s+t=k} a_s^{(0)} b_t^{(l)} + \sum_{s+t=k} a_s^{(1)} \alpha(b_t^{(l-1)}) + \cdots + \sum_{s+t=k} a_s^{(l)} \alpha^l(b_t^{(0)}) = 0$ on the right side by $a_0^{(0)}$. We have $a_0^{(0)} b_k^{(l)} a_0^{(0)} \in \mathrm{nil}(R)$ by [8, Lemma 3.1] and Lemma 3.3. Thus $a_0^{(0)} b_k^{(1)} \in \mathrm{nil}(R)$. Similarly we can show that $a_s^{(i)} \alpha^i(b_t^{(j)}) \in \mathrm{nil}(R)$ for any s, t with $s+t=k$ and any i, j with $i+j=l$. Therefore, by induction, we have $a_s^{(i)} \alpha^i(b_t^{(j)}) \in \mathrm{nil}(R)$ for any $0 \le i \le p$, any $0 \le j \le q$ and any s, t with $s+t = 0, 1, \ldots, n-1$. Now $f_i \bar{\alpha}^i(g_j) = (\Sigma_{s=0}^{n-1} a_s^{(i)} u^s) \bar{\alpha}^i (\Sigma_{t=0}^{n-1} b_t^{(j)} u^t) = \Sigma_{k=0}^{2n-2} (\Sigma_{s+t=k} a_s^{(i)} \alpha^i(b_t^{(j)})) u^k = \Sigma_{k=0}^{n-1} (\Sigma_{s+t=k} a_s^{(i)} \alpha^i(b_t^{(j)})) u^k$. Since R is reversible, by [8, Lemma 3.1], $\Sigma_{s+t=k} a_s^{(i)} \alpha^i(b_t^{(j)}) \in \mathrm{nil}(R)$. Thus by [8, Lemma 3.7], $f_i \bar{\alpha}^i(g_j) \in \mathrm{nil}(R[u])$. This shows that $R[u]$ is weak $\bar{\alpha}$-skew Armendariz. □

Note that the weak Armendariz ring is weak 1_R-skew Armendariz. Liu and Zhao [8, Theorem 3.8] showed that if a ring R is semicommutative, then $R[x]$ is weak Armendariz. For the case of weak α-skew Armendariz, we have the following result.

Theorem 3.9. *Let R be a reversible ring and α be an endomorphism of R such that $a\alpha(b) = 0$ whenever $ab = 0$ for any $a, b \in R$. If for some positive integer t, $\alpha^t = 1_R$, then $R[x]$ is weak $\bar{\alpha}$-skew Armendariz.*

Proof. Let $p(y) = f_0(x) + f_1(x)y + \cdots + f_m(x)y^m$ and $q(y) = g_0(x) + g_1(x)y + \cdots + g_n(x)y^n$ be in $(R[x])[y; \bar{\alpha}]$ with $p(y)q(y) = 0$. We also let $f_i(x) = a_{i0} + a_{i1}x + \cdots + a_{iw_i}x^{w_i}$ and $g_j(x) = b_{j0} + b_{j1}x + \cdots + b_{jv_j}x^{v_j}$ for any $0 \le i \le m$ and $0 \le j \le n$, where $a_{i0}, a_{i1}, \ldots, a_{iw_i}, b_{j0}, b_{j1}, \ldots, b_{jv_j} \in R$. We claim that $f_i(x)\bar{\alpha}^i(g_j(x)) \in \mathrm{nil}(R[x])$ for all $0 \le i \le m$ and $0 \le j \le n$. Take a positive integer k such that $k > \deg(f_0(x)) + \deg(f_1(x)) + \cdots + \deg(f_m(x)) + \deg(g_0(x)) + \deg(g_1(x)) + \cdots + \deg(g_n(x))$, where the degrees of $f_i(x)$ and $g_j(x)$ are as the polynomials in $R[x]$ and the degree of zero polynomial is taken to be 0 for all $0 \le i \le m$ and $0 \le j \le n$. Let $f(x) = f_0(x^t) + f_1(x^t)x^{tk+1} + f_2(x^t)x^{2tk+2} + \cdots + f_m(x^t)x^{mtk+m}$ and $g(x) = g_0(x^t) + g_1(x^t)x^{tk+1} + g_2(x^t)x^{2tk+2} + \cdots + g_n(x^t)x^{ntk+n} \in R[x]$. Then the set of coefficients of the $f_i(x)$ (respectively, $g_j(x)$) equals the set of coefficients of $f(x)$ (respectively, $g(x)$). Since $p(y)q(y) = 0$, x commutes with elements of R in the polynomial ring $R[x]$, and $\alpha^t = 1_R$, we have $f(x)g(x) = 0$ in $R[x; \alpha]$. By Proposition 3.5, R is weak α-skew Armendariz, so $a_{il}\alpha^i(b_{js}) \in \mathrm{nil}(R)$ for any $0 \le i \le m$, $0 \le j \le n$, $l \in \{0, 1, \ldots, w_0, \ldots, w_m\}$ and $s \in \{0, 1, \ldots, v_0, \ldots, v_n\}$. Since R is reversible, $\sum_{l+s=k} a_{il}\alpha^i(b_{js}) \in \mathrm{nil}(R)$, $k = 0, 1, \ldots, w_i + v_j$ by [8, Lemma 3.1]. So $f_i(x)\alpha^i(g_j(x)) \in \mathrm{nil}(R[x])$ by [8, Lemma 3.7] for all i and j, and hence $R[x]$ is weak $\bar{\alpha}$-skew Armendariz. □

Hong, Kim, and Kwak [4, Proposition 3] showed that if a ring R is α-rigid, then $R[x; \alpha]$ is reduced. Hence $R[x; \alpha]$ is Armendariz. Moreover, we note that even if α satisfies the condition "$\alpha^2 = 1_R$" in Example 3.7, R still need not be α-rigid. However, for the weak Armendariz rings, the following result holds.

Theorem 3.10. *Let R be a reversible ring and α be an endomorphism of R such that $a\alpha(b) = 0$ whenever $ab = 0$ for any $a, b \in R$. If, for some positive integer t, $\alpha^t = 1_R$, then $R[x;\alpha]$ is weak Armendariz.*

Proof. Let $p(y)$, $q(y)$ and k be the same as in the proof of Theorem 3.9. We claim that $f_i(x)g_j(x) \in \mathrm{nil}(R[x;\alpha])$ for all $0 \le i \le m$ and $0 \le j \le n$. Let $p(x^{tk}) = f_0(x) + f_1(x)x^{tk} + \cdots + f_m(x)x^{mtk}$ and $q(x^{tk}) = g_0(x) + g_1(x)x^{tk} + \cdots + g_n(x)x^{ntk} \in R[x;\alpha]$. Then the set of coefficients of the $f_i(x)$ (respectively, $g_j(x)$) equals the set of coefficients of $p(x^{tk})$ (respectively, $q(x^{tk})$). Since $p(y)q(y) = 0$ and $\alpha^t = 1_R$, we have $p(x^{tk})q(x^{tk}) = 0$ in $R[x;\alpha]$. Since R is weak α-skew Armendariz by Proposition 3.5, we have $a_{il}\alpha^l(b_{js}) \in \mathrm{nil}(R)$ for any $0 \le i \le m$, $0 \le j \le n$, $0 \le l \le w_i$ and $0 \le s \le v_j$. Thus $f_ig_j \in \mathrm{nil}(R[x;\alpha])$ for all $0 \le i \le m$, $0 \le j \le n$ by [8, Lemma 3.1] and Lemma 3.4, and hence $R[x;\alpha]$ is weak Armendariz. □

Note that weak Armendariz rings are weak 1_R-skew Armendariz rings. But Example 3.1 shows that there exists an endomorphism α of R such that weak Armendariz rings need not be weak α-skew Armendariz. We do not know whether the converse is true. However, if $R[x;\alpha]$ is weak Armendariz, then R is weak Armendariz since it is a invariant subring of $R[x;\alpha]$. Thus, By Proposition 3.5 and Theorem 3.10, we can obtain the conditions that weak α-skew Armendariz rings are weak Armendariz rings.

Let α be an automorphism of a ring R. Suppose that there exists the classical left quotient Q of R. Then for any $b^{-1}a \in Q$, where $a, b \in R$ with b regular, the induced map $\bar{\alpha} : Q(R) \to Q(R)$ defined by $\bar{\alpha}(b^{-1}a) = (\alpha(b))^{-1}\alpha(a)$ is also an automorphism.

Proposition 3.11. *Suppose that there exists the classical left quotient Q of a ring R. If R is reversible, then R is weak α-skew Armendariz if and only if Q is weak $\bar{\alpha}$-skew Armendariz.*

Proof. Suppose that R is weak α-skew Armendariz. Let $f(x) = s_0^{-1}a_0 + s_1^{-1}a_1x + \cdots + s_m^{-1}a_mx^m$ and $g(x) = t_0^{-1}b_0 + t_1^{-1}b_1x + \cdots + t_n^{-1}b_nx^n \in Q[x;\bar{\alpha}]$ such that $f(x)g(x) = 0$. Let C be a left denominator set. There exist s, $t \in C$ and a'_i, $b'_j \in R$ such that $s_i^{-1}a_i = s^{-1}a'_i$ and $t_j^{-1}b_j = t^{-1}b'_j$ for $i = 0, 1, \ldots, m$ and $j = 0, 1, \ldots, n$. Then $s^{-1}(a'_0 + a'_1x + \cdots + a'_mx^m)t^{-1}(b'_0 + b'_1x + \cdots + b'_nx^n) = 0$. It follows that $(a'_0 + a'_1x + \cdots + a'_mx^m)t^{-1}(b'_0 + b'_1x + \cdots + b'_nx^n) = 0$. Thus $(a'_0t^{-1} + a'_1(\alpha(t))^{-1}x + \cdots + a'_m(\alpha^m(t))^{-1}x^m)(b'_0 + b'_1x + \cdots + b'_nx^n) = 0$. For $(a'_i(\alpha^i(t))^{-1}$, $i = 0, 1, \ldots, n$, there exist $t' \in C$ and $a''_i \in R$ such that $a'_i(\alpha^i(t))^{-1} = t'^{-1}a''_i$. Hence $t'^{-1}(a''_0 + a''_1x + \cdots + a''_mx^m)(b'_0 + b'_1x + \cdots + b'_nx^n) = 0$. We have that $(a''_0 + a''_1x + \cdots + a''_mx^m)(b'_0 + b'_1x + \cdots + b'_nx^n) = 0$. Since R is weak α-skew Armendariz, $a''_i\alpha^i(b'_j) \in \mathrm{nil}(R)$ for all i and j. Suppose that $(a''_i\alpha^i(b'_j))^{n_{ij}} = 0$. Since R is reversible, Q is semicommutative. Then $(t'^{-1}(a''_i\alpha^i(b'_j)))^{n_{ij}} = 0$. So $(a'_i\bar{\alpha}^i(t^{-1}b'_j))^{n_{ij}} = (a'_i(\alpha^i(t))^{-1}\alpha^i(b'_j))^{n_{ij}} = ((t'^{-1}a''_i)\alpha^i(b'_j))^{n_{ij}} = 0$. Similarly we have $(s_i^{-1}a_i')(\bar{\alpha}^i(t_j^{-1}b_j'))^{n_{ij}} = (s^{-1}a_i')(\bar{\alpha}^i(t^{-1}b_j'))^{n_{ij}} = 0$. Therefore Q is weak $\bar{\alpha}$-skew Armendariz. The converse is clear. □

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