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DUAL PRESENTATION AND LINEAR BASIS OF THE TEMPERLEY-LIEB ALGEBRAS

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Abstract. The braid group B_n maps homomorphically into the Temperley-Lieb algebra TL_n. It was shown by Zinno that the homomorphic images of simple elements arising from the dual presentation of the braid group B_n form a basis for the vector space underlying the Temperley-Lieb algebra TL_n. In this paper, we establish that there is a dual presentation of Temperley-Lieb algebras that corresponds to the dual presentation of braid groups, and then give a simple geometric proof for Zinno's theorem, using the interpretation of simple elements as non-crossing partitions.

1. Introduction

Since Jones [7, 8] discovered the Jones polynomial for links by investigating representations of braid groups into Hecke algebras and Temperley-Lieb algebras, Temperley-Lieb algebras have played important roles in the quantum invariants of links and 3-manifolds. The Temperley-Lieb algebra TL_n is defined on non-invertible generators $e_1, \ldots, e_{n-1}$ with the relations:

$$e_ie_j = e_je_i \text{ for } |i-j| \geq 2; \quad e_i^2 = e_i; \quad e_ie_{i\pm1}e_i = \tau e_i$$

along with a complex number τ (see [8, 9]). Setting $\tau = 1/\delta^2$ and $e_i = (1/\delta)d_i$, we get an equivalent presentation, which is easily understood by diagrams, with non-invertible generators $d_1, \ldots, d_{n-1}$ and defining relations

$$d_id_j = d_jd_i \text{ for } |i-j| \geq 2; \quad d_i^2 = \delta d_i; \quad d_id_{i\pm1}d_i = d_i$$

along with a complex number δ (see [10, 12]). It is well-known that the dimension of TL_n is the nth Catalan number $\mathcal{C}_n = \frac{1}{n+1}\binom{2n}{n}$. Setting t such that $\tau^{-1} = 2 + t + t^{-1}$, and then setting $h_i = (t+1)e_i - 1$, we get an alternative presentation of TL_n with invertible generators $h_1, \ldots, h_{n-1}$ satisfying the

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relations:

(1) $$h_ih_j = h_jh_i \quad \text{if } |i-j| \geq 2;$$

(2) $$h_ih_{i+1}h_i = h_{i+1}h_ih_{i+1};$$

(3) $$h_i^2 = (t-1)h_i + t;$$

(4) $$h_ih_{i+1}h_i + h_ih_{i+1} + h_{i+1}h_i + h_i + h_{i+1} + 1 = 0.$$

The braid group B_n is defined by the Artin presentation, where the generators are $\sigma_1, \ldots, \sigma_{n-1}$ and the defining relations are

$$\sigma_i\sigma_j = \sigma_j\sigma_i \quad \text{if } |i-j| \geq 2;$$
$$\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \quad \text{for } i = 1, \ldots, n-2.$$

The braid group B_n maps homomorphically into the Temperley-Lieb algebra TL_n under $\pi : \sigma_i \mapsto h_i$. There is another presentation [4] with generators a_{ji} $(1 \leq i < j \leq n)$ and defining relations

$$a_{lk}a_{ji} = a_{ji}a_{lk} \quad \text{if } (l-j)(l-i)(k-j)(k-i) > 0;$$
$$a_{kj}a_{ji} = a_{ji}a_{ki} = a_{ki}a_{kj} \quad \text{for } i < j < k.$$

The generators a_{ji}'s are related to the σ_i's by

$$a_{ji} = \sigma_{j-1}\sigma_{j-2}\cdots\sigma_{i+1}\sigma_i\sigma_{i+1}^{-1}\cdots\sigma_{j-2}^{-1}\sigma_{j-1}^{-1}.$$

Bessis [1] showed that there is a similar presentation, called the *dual presentation*, for Artin groups of finite Coxeter type.

Both the Artin and dual presentations of the braid group B_n determine a *Garside monoid,* as defined by Dehornoy and Paris [6], where the *simple elements* play important roles. Nowadays, it becomes more and more popular to describe simple elements arising from the dual presentation via non-crossing partitions. Non-crossing partitions are useful in diverse areas [1, 5, 2, 3, 11], because they have beautiful combinatorial structures.

Let $P_1, \ldots, P_n$ be the points in the complex plane given by $P_k = \exp(-\frac{2k\pi}{n}i)$. See Figure 1. Recall that a partition of a set is a collection of pairwise disjoint subsets whose union is the entire set. Those subsets (in the collection) are called blocks. A partition of $\{P_1, \ldots, P_n\}$ is called a *non-crossing partition* if the convex hulls of the blocks are pairwise disjoint.

A positive word of the form $a_{i_1i_2}a_{i_2i_3}\cdots a_{i_{k-1}i_k}$, $i_1 > i_2 > \cdots > i_k$, is called a *descending cycle* and denoted $[i_1, i_2, \ldots, i_k]$. Two descending cycles $[i_1, \ldots, i_k]$ and $[j_1, \ldots, j_l]$ are said to be *parallel* if the convex hulls of $\{P_{i_1}, \ldots, P_{i_k}\}$ and of $\{P_{j_1}, \ldots, P_{j_l}\}$ are disjoint. The simple elements are the products of parallel descending cycles.

We remark that the definition of simple elements depends on the presentations. For example, the simple elements arising from the Artin presentation are in one-to-one correspondence with permutations. Throughout this note, we consider only the simple elements arising from the dual presentation of braid groups as above.

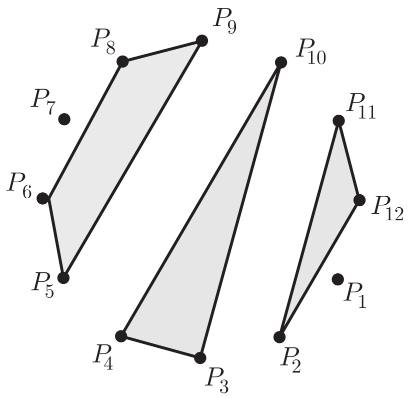

FIGURE 1. The shaded regions show the blocks in the non-crossing partition corresponding to the simple element $[12, 11, 2]\,[10, 4, 3]\,[9, 8, 6, 5]$ in B_{12}.

Note that simple elements are in one-to-one correspondence with non-crossing partitions. Our convention is that if a block in a non-crossing partition consists of a single point, then the corresponding descending cycle is the identity (i.e., the descending cycle of length 0). In particular, the number of the simple elements is the nth Catalan number $\mathcal{C}_n$, which is the dimension of TL_n. Zinno [13] established the following result.

Theorem 1 (Zinno [13])**.** *The homomorphic images of the simple elements arising from the dual presentation of B_n form a linear basis for the Temperley-Lieb algebra TL_n.*

We explain briefly Zinno's proof. It is known that the *ordered reduced words*

$$(h_{j_1}h_{j_1-1}\cdots h_{k_1})(h_{j_2}h_{j_2-1}\cdots h_{k_2})\cdots(h_{j_p}h_{j_p-1}\cdots h_{k_p}),$$

where $j_i \geq k_i$, $j_{i+1} > j_i$ and $k_{i+1} > k_i$, form a linear basis of TL_n, and Zinno showed that the matrix for writing the images of simple elements as the linear combination of the ordered reduced words is invertible. Because the number of the simple elements is equal to the dimension of TL_n, this proves the theorem.

In this note, we first establish that there is a dual presentation of TL_n. We are grateful to David Bessis for pointing out that the relation (4) in the Temperley-Lieb algebra presentation is equivalent to the forth relation in the dual presentation in the following theorem.

Theorem 2 (Dual presentation of TL_n)**.** *The Temperley-Lieb algebra TL_n has a presentation with invertible generators g_{ji} $(1 \leq i < j \leq n)$ satisfying the*

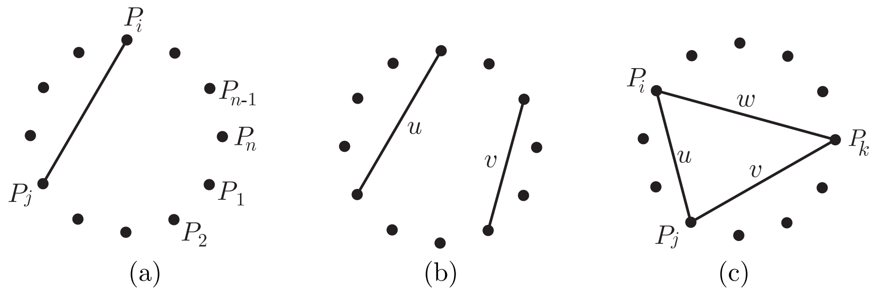

FIGURE 2

relations:

$$
\begin{aligned}
&g_{lk}g_{ji} = g_{ji}g_{lk} \quad \textit{if } (l-j)(l-i)(k-j)(k-i) > 0;\\
&g_{kj}g_{ji} = g_{ji}g_{ki} = g_{ki}g_{kj} \quad \textit{for } i < j < k;\\
&g_{ji}^2 = (t-1)g_{ji} + t \quad \textit{for } i < j;\\
&g_{ji}g_{kj} + tg_{kj}g_{ji} + g_{kj} + g_{ji} + tg_{ki} + 1 = 0 \quad \textit{for } i < j < k.
\end{aligned}
$$

The new generators are related to the old ones by

$$g_{ji} = h_{j-1}h_{j-2}\cdots h_{i+1}h_i h_{i+1}^{-1}\cdots h_{j-2}^{-1}h_{j-1}^{-1}.$$

Using the above presentation, we give a new proof of Zinno's theorem in §3. We exploit non-crossing partitions so as to make the proof easy and intuitive. For the proof, we show that any monomial in the $h_i^{\pm 1}$'s can be written as a linear combination of the images of simple elements. Therefore the images of simple elements span TL_n. As a result, they form a linear basis of TL_n because the number of simple elements is equal to the dimension of TL_n.

We remark that it seems possible to prove the linear independence of the images of the simple elements directly from the relations in the dual presentation of TL_n (without using the fact that the dimension of TL_n is the same as the number of simple elements), but that would be beyond the scope of this note because it would require repeating all the arguments used in the proof for the embedding of the positive braid monoid in the braid group.

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2. Dual presentation of the Temperley-Lieb algebras

Let D^2 be the disc in the complex plane with radius 2 and $P_1, \ldots, P_n$ be the points in D^2 given by $P_k = \exp(-\frac{2k\pi}{n}i)$. Let $D_n = D^2 \setminus \{P_1, \ldots, P_n\}$. The braid group B_n can be regarded as the group of self-homeomorphisms of D_n that fix the boundary pointwise, modulo isotopy relative to the boundary. The generators σ_i and a_{ji} correspond to the positive half Dehn-twists along the arcs P_iP_{i+1} and P_iP_j, respectively.

Let $\mathcal{L}_n = \{P_iP_j \mid i \neq j\}$ be the set of line segments as in Figure 2 (a). We say that a pair $(u, v) \in \mathcal{L}_n^2$ is *parallel* if u and v are disjoint as in Figure 2 (b), and *admissible* if $u = P_iP_j$ and $v = P_jP_k$ for some pairwise distinct points P_i, P_j and P_k which are in counterclockwise order on the unit circle as in Figure 2 (c). A triple $(u, v, w) \in \mathcal{L}_n^3$ is said to be *admissible* if so are all the pairs (u, v), (v, w) and (w, u). For $u = P_iP_j$ with $i < j$, let a_u denote the generator a_{ji} of the dual presentation of B_n. Then the dual presentation of B_n can be written as follows:

$$B_n = \left\langle \begin{matrix} a_u \\ (u \in \mathcal{L}_n) \end{matrix} \,\middle|\, \begin{matrix} a_ua_v = a_va_u & \text{if } u \text{ and } v \text{ are parallel} \\ a_ua_v = a_va_w = a_wa_u & \text{if } (u,v,w) \text{ is admissible} \end{matrix} \right\rangle.$$

It is easy to see the following: (i) if $u \cap v = \{P_i\}$ for some P_i in the unit circle, then exactly one of (u, v) and (v, u) is admissible; (ii) if (u, v) is admissible, then a_ua_v can be written in three ways as in the presentation, but a_va_u is not equivalent to any other positive word on the a_u's; (iii) a_ua_v is a simple element if and only if (u, v) is parallel or admissible.

Now we prove Theorem 2. For $u = P_iP_j$ with $i < j$, let g_u denote the generator g_{ji} of the presentation in that theorem. Then the presentation can be reformulated as follows. Its proof is elementary. However, we present it for completeness.

Theorem 3 (Dual presentation of TL_n—reformulated)**.** *TL_n has a presentation with invertible generators g_u $(u \in \mathcal{L}_n)$ satisfying the relations*:

(5) $\quad g_ug_v = g_vg_u$ *if u and v are parallel*;

(6) $\quad g_ug_v = g_vg_w = g_wg_u$ *if (u, v, w) is admissible*;

(7) $\quad g_u^2 = (t-1)g_u + t$ *for $u \in \mathcal{L}_n$*;

(8) $\quad g_vg_u + tg_ug_v + g_u + g_v + tg_w + 1 = 0$ *if (u, v, w) is admissible.*

Proof. From the results on the dual presentation of B_n in [4], it follows that the relations (1) and (2) are equivalent to the relations (5) and (6).

Assume the relations (1), (2), and hence (5), (6).
(7) ⇒ (3) It is clear since (3) is a special case of (7).
(3) ⇒ (7) It is clear since each g_u is conjugate to h_i (for some i) by a monomial in the h_j's.

Now assume the relations (1), (2), (3), and hence (5), (6), (7)
(8) ⇒ (4) Let $u = P_{i+2}P_{i+1}$, $v = P_{i+1}P_i$ and $w = P_iP_{i+2}$. Then $g_u = h_{i+1}$, $g_v = h_i$ and (u, v, w) is admissible. Since $h_ih_{i+1}h_i = g_vg_ug_v = g_vg_vg_w = ((t-1)g_v + t)g_w = (t-1)g_vg_w + tg_w = (t-1)g_ug_v + tg_w$,

$$\begin{aligned} & h_ih_{i+1}h_i + h_ih_{i+1} + h_{i+1}h_i + h_i + h_{i+1} + 1 \\ = \; & ((t-1)g_ug_v + tg_w) + g_vg_u + g_ug_v + g_v + g_u + 1 \\ = \; & g_vg_u + tg_ug_v + g_u + g_v + tg_w + 1 \; = \; 0. \end{aligned}$$

(4) $\Rightarrow$ (8) Note that for each admissible triple (u',v',w'), there is a self-homeomorphism of D_n sending (u',v',w') to (u,v,w). Therefore, there is a monomial x in the h_j's such that $xg_{u'}x^{-1}=g_u$, $xg_{v'}x^{-1}=g_v$ and $xg_{w'}x^{-1}=g_w$, simultaneously. Let $u'=P_{i+2}P_{i+1}$, $v'=P_{i+1}P_i$ and $w'=P_iP_{i+2}$. In the same way as in (8) $\Rightarrow$ (4), we obtain

$$\begin{aligned} & g_vg_u+tg_ug_v+g_u+g_v+tg_w+1 \\ = \;& x\,(g_{v'}g_{u'}+tg_{u'}g_{v'}+g_{u'}+g_{v'}+tg_{w'}+1)\,x^{-1} \\ = \;& x\,(h_ih_{i+1}h_i+h_ih_{i+1}+h_{i+1}h_i+h_i+h_{i+1}+1)\,x^{-1} \\ = \;& 0. \end{aligned}$$

□

3. A new proof of Zinno's theorem

Before starting the proof of Zinno's theorem, let us observe the relations $g_u^2=(t-1)g_u+t$ and $g_vg_u+tg_ug_v+g_u+g_v+tg_w+1=0$ in the dual presentation of TL_n. Among the monomials in the relations, all except g_u^2 and g_vg_u are images of simple elements. Therefore the relations can be interpreted as instructions for converting a product of two generators into a linear combination of the images of simple elements:

$$\begin{aligned} g_u^2 &= (t-1)\pi(a_u)+t; \\ g_vg_u &= -t\pi(a_ua_v)-\pi(a_u)-\pi(a_v)-t\pi(a_w)-1. \end{aligned}$$

Generalizing this idea, we will show in Proposition 4 that for a simple element A and an Artin generator σ_i, the homomorphic image $\pi(A\sigma_i)$ in TL_n can be written as a linear combination of the images of simple elements.

Recall that the simple elements are in one-to-one correspondence with non-crossing partitions. For a simple element A, take union of the convex hulls of the blocks in the non-crossing partition of A, and then remove those containing only one point. The resulting set is called the *underlying space* of A and denoted $\bar{A}$.

It is known that for a simple element A and $u\in\mathcal{L}_n$, Aa_u is a simple element if and only if for any $w\in\mathcal{L}_n$ with $w\subset\bar{A}$, the product a_wa_u is a simple element, in other words, (w,u) is parallel or admissible [4, Corollary 3.6]. Figure 3 shows typical cases of $(\bar{A},u)$ such that Aa_u becomes a simple element, and Figure 4 shows some cases of $(\bar{A},u)$ such that Aa_u is not a simple element.

It is easy to see that if $\bar{A}$ and u satisfy one of the following conditions, then Aa_u is a simple element and its underlying space is the union of the convex hulls of components of $\bar{A}\cup u$.

- $\bar{A}$ and u are disjoint.
- $\bar{A}$ and u intersect at the boundary of u as in the left hand sides of Figure 3. Intuitively, when we stand at an intersection point, with u on the right and the component of $\bar{A}$ containing the intersection point on the left, we are facing towards the inside of the unit circle.

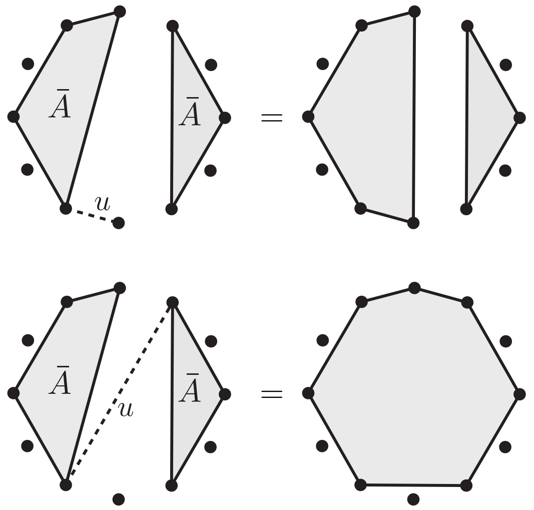

FIGURE 3. In the left hand sides $\bar{A}$ and u are depicted as shaded regions and dotted lines. The right hand sides show the underlying spaces of Aa_u's.

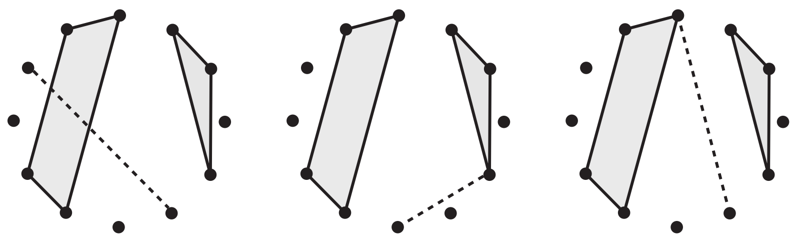

FIGURE 4. The shaded regions and the dotted lines represent the underling space of a simple element A and an element $u \in \mathcal{L}_n$, respectively. In this case, Aa_u is not a simple element.

Proposition 4. *For a simple element A and an Artin generator σ_i, $\pi(A\sigma_i)$ can be expressed as a linear combination of the images of simple elements.*

Proof. Let $u = P_iP_{i+1}$. Then $\sigma_i = a_u$ and $\pi(\sigma_i) = g_u$. We prove the assertion in three cases.

Case 1. If $u \subset \bar{A}$, then $\bar{A}$ and u are as in Figure 5 (a). Let B be the simple element whose underlying space is as in Figure 5 (b). More precisely, the noncrossing partition of B is obtained from that of A by making $\{P_i\}$ a new block.

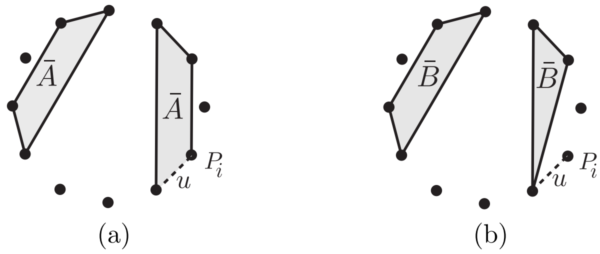

FIGURE 5. $A = Ba_u$ if $\bar{A}$, u and $\bar{B}$ are as above.

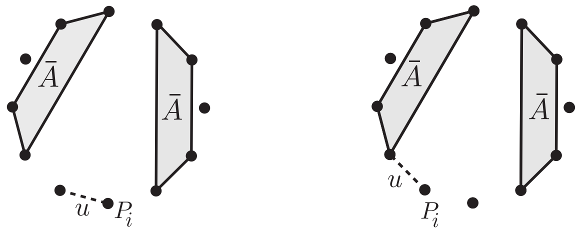

FIGURE 6. Aa_u is a simple element.

Then $A = Ba_u$ and

$$\begin{aligned}\pi(Aa_u) &= \pi(Ba_u^2) = \pi(B)g_u^2 = \pi(B)((t-1)g_u + t)\\ &= (t-1)\pi(Ba_u) + t\pi(B) = (t-1)\pi(A) + t\pi(B).\end{aligned}$$

Case 2. If $u \not\subset \bar{A}$ and $P_i \notin \bar{A}$, then $\bar{A}$ and u are as in Figure 6. In this case, Aa_u itself is a simple element.

Case 3. If $u \not\subset \bar{A}$ and $P_i \in \bar{A}$, then $\bar{A}$ and u are as in either (a) or (b) of Figure 7, depending on whether P_{i+1} belongs to $\bar{A}$ or not. Let v be the line segment containing P_i such that $Ba_v = A$ for some simple element B as in (c) and (d) of Figure 7. (More precisely, $v = P_iP_j$ for some P_j such that $P_iP_j \subset \bar{A}$ and the interior of $P_{i+1}P_j$ does not intersect $\bar{A}$.) Let w be the line segment connecting the endpoints of u and v other than P_i. Then (u, v, w) is admissible and

$$\begin{aligned}\pi(Aa_u) &= \pi(Ba_va_u) = \pi(B)g_vg_u\\ &= -\pi(B)(tg_ug_v + g_u + g_v + tg_w + 1)\\ &= -t\pi(Ba_ua_v) - \pi(Ba_u) - \pi(Ba_v) - t\pi(Ba_w) - \pi(B).\end{aligned}$$

Note that Ba_ua_v, Ba_u, Ba_v and Ba_w are simple elements. □

Proof of Theorem 1. Let V_n be the subspace (of TL_n) spanned by the images of simple elements. Since the number of simple elements is equal to the dimension

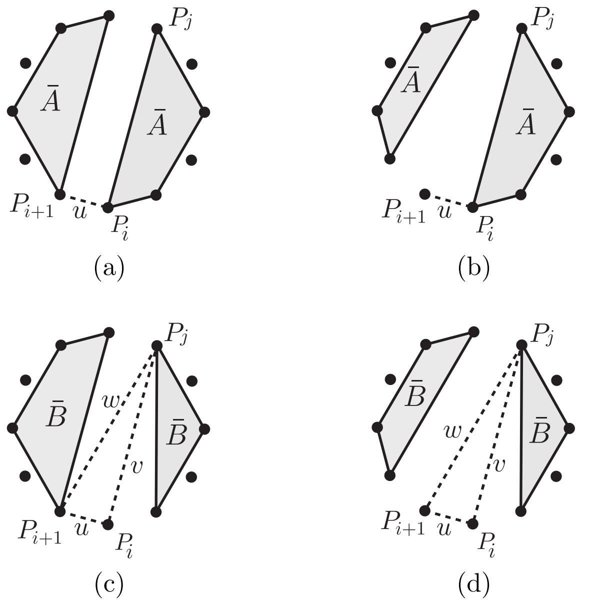

FIGURE 7

of TL_n, the images of simple elements form a linear basis of TL_n if we show that $V_n = TL_n$ (i.e., every monomial in the $h_i^{\pm 1}$'s belongs to V_n).

Observe that $h_i^{-1} = t^{-1}h_i + t^{-1} - 1$ and $h_i = \pi(\sigma_i)$ for all i. Therefore, it suffices to show that the images of monomials in the σ_i's belong to V_n. Use induction on the word length of monomials in the σ_i's. By Proposition 4, it is easy to get the desired result. □

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