대한수학교육학회지 <학교수학> 제 12 권 제 3 호
Journal of Korea Society of Educational Studies in Mathematics
School Mathematics Vol.12, No.3, 439-454. Sep 2010

Construction of a Student-Generated Algorithm for Fraction Measurement Division[1)]

Shin, Jae Hong*

This study presents how two eighth grade students generated their own algorithms in the context of fraction measurement division situations by modifications of unit-segmenting schemes. Teaching experiment was adopted as a research methodology and part of data from a year-long teaching experiment were used for this report. The present study indicates that the two participating students' construction of reciprocal relationship between the referent whole [one] and the divisor by using their unit- segmenting schemes and its strategic use finally led the students to establish an algorithm for fraction measurement division problems, which was on par with the traditional invert-and-multi- ply algorithm for fraction division. The results of the study imply that teachers' instruction based on understanding student-generated algorithms needs to be accounted as one of the crucial characteristics of good mathematics teaching.

I. Background

An algorithm has been defined as "a precise, systematic method for solving a class of problems." (Maurer, 1998, p. 21) or "a step-by-step process that guarantees the correct solution to a given problem, provided the steps are executed correctly." (Barnett, 1998, p. 69). That is, mathematical algorithms, as a part of mathematics, are powerful tools that contribute to effective problem solving.

However, in spite of such efficiency and correctness of mathematical algorithms, Plunkett (1979) argued that formal written algorithms do not necessarily correspond to the ways in which people tend to think about numbers even though they have the advantage of providing a standard routine that will work for any numbers. Therefore, algorithms might become harmful in that algorithms encourage students to give up their wn thinking, and prevent students from development of number sense (Kamii & Dominick, 1998). Thus, the traditional algorithms that all students must memorize should not define elementary school mathematics (Campbell, Rowan, and Suarez, 1998). Burns (1994) also cautioned against the risk of teaching standard algorithms in classroom.

Imposing the standard arithmetic algorithms on children is pedagogically risky. It interferes with their learning, and it can give students the idea that mathematics is a collection of mysterious and often magical rules and procedures that must be memorized and practiced. Teaching children sequences of prescribed steps for computing focuses their attention on following the steps, rather than on making sense

* Korea National University of Education (jhbasket@gmail.com)
1) The research reported in this manuscript was part of the author's doctoral dissertation completed at the University of Georgia.

of numerical situations. (p. 472)

Especially, it is hard to negate that an algorithm for fraction division, so called, 'invert-and-multiply' algorithm has been taught in school mathematics without students' relational understanding (Skemp, 1987) and little has been done in mathematics education research to find how the invert-and-multiply algorithm can be meaningfully constructed by learners. Actually, students' construction of unit-segmenting schemes has been studied in a whole number measurement division situation. For a situation to be established as divisional, it is always necessary to establish at least two composite units, one composite unit to be segmented and the other composite unit to be used in segmenting. The goal is to find how many times one can use the measuring unit [the unit to be used in segmenting] with a given unit to be segmented. Steffe (1992) reported that Johanna [a participating student in his teaching experiment] was able to establish her unit-segmenting scheme as anticipatory and the units to be used in segmenting were available to her as iterating units prior to operating. However, when the composite unit to be segmented is not completely measured out by the unit used in segmenting [when producing a remainder], the divisional situation might be assimilated as novel and cause perturbation in a student's use of her unit-segmenting scheme because the result of a unit-segmenting scheme produces a fractional quantity in terms of the segmenting unit, which seems an unexpected result for a student who has no fractional knowledge. Therefore, it remains to be investigated how students modify their whole number division scheme to solve measurement division problems involving fractions.

Paper-and-pencil algorithms, nevertheless, are important tools that equip students for computational fluency. They should not be disregarded only as obstacles in students' mathematical learning. For algorithms to be beneficial to students, Ashlock (2006) argued, the students' use of the algorithms should involve conceptual knowledge as well as procedural skill before the algorithms become mechanical procedures for the students. In Principle and Standards for School Mathematics, the National Council of Teachers of Mathematics (2000) suggested that "when students compute with strategies they invent or choose because they are meaningful, their learning tends to be robust and they are able to remember and apply their knowledge" (p. 86). Mathematical reasoning and justification are inherent in the invention of procedures (Kilpatrick, Swafford, & Findell, 2001) and thus students reveal their own construct of understanding with the procedures that they create (Baek, 1998). Especially, Huinker (1998) reported several advantages of allowing students to invent their own algorithms in teaching fractions: 1) interest in solving and posing word problems with fractions, 2) flexibility in the choice of strategy for solving fraction word problems and computation exercises, 3) proficiency in translating among real-world, concrete, pictorial, oral language, and symbolic representations, and 4) acquaintance with communicating and justifying their thinking and reasoning.

II. Configuration of the Study & Research Questions

Providing opportunities for students to develop, use and discuss invented algorithms helps to enhance number and operations sense (Kamii, Lewis, & Livingston, 1993). Then how can we, as teachers, create learning environments so that students can understand where and when to use an algorithm? McClain, Cobb and Bowers (1998) suggested that relational understanding (Skemp, 1987) occurs as teachers systematically support students' construction of personally meaningful algorithms and these algorithms could emerge when students engage in sequences of problem-solving activities designed to provide opportunities for them to make sense of their mathematical activity. They also argued that such approach can not only value students' construction of non-standard algorithms, but also avoid two extremes: to encourage students to invent their own algorithms with minimal guidance or to teach students to perform traditional algorithms.

The present study is a report about two eighth grader's constructive processes of their own algorithms for fraction measurement division as a part of a year-long constructivist' teaching experiment. Teaching experiment environment is supportive for students' own mathematical development in that 1) a primary goal of the teacher in a teaching experiment is to establish living models of students' mathematics that students can possibly develop and 2) teaching experiment consists of a sequence of teaching episodes that contain problem-solving activities depending on students' progress and 3) the important duty of the teacher-researcher in the teaching experiment is to attempt to put aside his or her own concepts and operations and not to insist that the students learn what he or she knows (Steffe & Thompson, 2000).

Therefore, research questions in my teaching experiment for this study are as following;

What mathematical operations did the two participating students develop in the context of fraction measurement division situations?

What records of the students' mathematical operations did emerge in symbolic notation as a student-generated algorithm?

III. Method of Inquiry

The data that I analyzed in the present study was part of a year teaching experiment whose broad purpose was to understand middle school students' mathematical reasoning. The teaching experiment began in October of 2008 and finished in May of 2009 at a rural middle school in northern Georgia. Rosa, one of the two participants, was chosen after individual selection interview conducted in October of 2008. Carol, the other participant, had been chosen in October of 2007 and was paired with Rosa for her second year of the teaching experiment. The criterion for selection of the two students was the ability to use composite units as iterable units, which was an indicator of their multiplicative reasoning. During the teaching experiment, we met once or twice a week in about 40-minute teaching episodes where I participated in mostly as a teacher-researcher, or sometimes as a witness-researcher. All teaching episodes were videotaped with two cameras for on-going and retrospective analysis. One camera usually captured the whole picture of interactions among the pair of students and the teacher-researcher, and the other camera followed the students' written or computer work with the aid of two witness-researchers. The role of the witness-researcher was not only assisting in video recording but also providing other perspectives during all three phases of the experiment: the actual teaching episodes, the on-going analysis between episodes during the experiment, and the retrospective analysis of the videotapes. Among the collected data during a year of the teaching experiment of 2008, eight teaching episodes were retrospectively analyzed and part of them were transcribed for the present study.

In terms of data analysis, the first type of analysis was ongoing analysis that occurred by watching videos of the teaching episodes and discussing and planning future episodes. Then a sequence of summaries for the teaching episodes were created, each of which provided not only a written description of students' mathematical activities and interactions with the teacher-researcher, but also emerging key points in students' thinking and learning that were taken into account for the next teaching episode. The second type of analysis was a retrospective analysis. The purpose of the retrospective analysis of the sequence of teaching episodes was to make models of students' ways of operating mathematically through conceptual analysis of students' mathematical activities. I, first of all, attempted to understand what the students' behaviors were and hypothesize why the students behaved in such ways. Then the attribution of the researchers' construction of a scheme[2] to the students was made at this stage.

Ⅳ. Analysis

1. Modifications of Unit-Segmenting Schemes for Fraction Measurement Division - Context of Two Students' Construction of Algorithms

When a whole number division problem that produced a remainder [finding how many times 3 meters is contained in 5 meters] was posed, Carol and Rosa assimilated the problem as a divisional situation, which led them to use a conventional division calculation method. However, Rosa could not convert her decimal answer to a fraction form that I had requested. Even though she later re-assimilated the problem as a situation for her unit-segmenting scheme, as indicated in her comments "It's gonna be one and then something fraction," the divisional situation, where a composite unit to be segmented was not completely measured out by the other composite unit used in segmenting, was a novel situation, which produced an unexpected quantity, which was very difficult (for Rosa) to measure. It was Carol who eliminated this perturbation in using her unit-segmenting scheme. With perceptual materials [a 3-part bar and a 5-part bar on paper], she was able to regard 5 as one and two-thirds units of 3 as well as five units of 1 and one unit of 5. I conjecture that this was possible by her association of the result of her unit-segmenting scheme [the leftover 2-part bar] as a situation for her partitive fraction scheme[3]. Therefore, if a student constructed a new unit-segmenting scheme through a modification whereby her partitive fraction scheme was embedded as a subscheme in the assimilating part of her unit-segmenting scheme, I would attribute to the construction of a unit-segmenting scheme with a remainder, which can be considered as a modified unit-segmenting scheme of whole number measurement division.

In the divisional situations with a fraction divisor

2) A scheme consists of three parts: an experiential situation which is activated or recognized by a student, the specific activity associated with the conceived situation, and a certain result of the activity engendered by the student's prediction (cf. von Glasersfeld, 1995; Olive & Steffe, 2002)

3) A partitive fraction scheme is the first scheme to be a genuine fractional scheme (Steffe, 2002). It enables a child to establish a substantial but limited understanding of fractions as parts of a specific partitioned whole (Tzur, 1999).

and a whole number dividend, when the fraction divisor evenly divided the whole dividend, Carol showed her generalizing assimilation[4] of her unit-segmenting scheme, which resulted in inclusion of fractional quantities as segmenting units in the assimilating part of the scheme. Similarly Rosa's numeric calculation of division was connected to her unit-segmenting scheme, which meant that her division algorithm stood in for her unit-segmenting scheme. However, when the fractional divisor did not evenly divide the whole number dividend, Rosa seemed to fail to associate her division calculation result with her unit-segmenting scheme in order to deal with the entailed remainder and it indicated that she was yet to construct a unit-segmenting scheme with a remainder. However, in the teaching episode held on December, 5 of 2008, Rosa finally constructed a unit-segmenting scheme with a remainder when measuring 1-meter with 3/5-meter, but it was a construction by retrospective accommodation[5] in the sense that her construction was possible through communications with Carol, not by herself.

2. Finding What Part of 7/5-Meter is Contained in 1 Meter

Following the previous fraction division problem (measuring 1-meter with 3/5-meter), my concern was to further investigate how the participating students modified their unit-segmenting scheme (with a remainder) when a fraction divisor did not evenly divide a whole number dividend, and when a fraction divisor larger than a whole number dividend was used as a measuring unit. Originally, the problem in the present protocol was "How many times is 7/5-meters contained in 1-meter" However, the two students were totally at a loss with the problem. To construct a smaller quantity by multiplying [times] a number to a larger quantity seemed to them a sort of an unimaginable situation because the word 'times' had always been used for increasing a quantity. Thus, I decided to replace 'how many times' with 'how much part' in the problem hoping that the students could attend to the larger unit as a referent unit and compare two different quantities.

For the following protocols, R stands for Rosa, C for Carol, T for the teacher-researcher (myself), and W for a witness-researcher. Comments enclosed in parentheses describe students' nonverbal actions or interactions from the teacher-researcher's perspective. Ellipses (...) indicate a sentence or an idea that seems to trail off. Four periods (....) denote omitted dialogue or interaction.

Protocol I on 12/05/08:

T: Yeah, then let's go to this part. How much part is seven-fifths meters contained in one meter? (Carol draws a 5-part bar and two more parts separately and shades them all. Rosa also draws a 5-part bar and adds two more parts on the right end of the 5-part bar using dotted lines. However, she turns to numeric calculation. She writes '$7/5 \times 5/5 = 35/25$' and '7/5' but crosses a line through the numeric expressions as she feels something wrong with her calculation. See Figure Ⅳ-1a and Figure Ⅳ-1b)

4) An assimilation is generalizing if the scheme involved in assimilation is used in a situation that contains sensory material or conceptual items that are novel for the scheme but the scheme does not recognize it (Steffe & Olive, 2010).

5) A retrospective accommodation involves selecting and using conceptual elements already constructed. From the student's perspective, a retrospective accommodation is self-initiated in that it is the student who must select and use the concept. From an observer's perspective, the conceptual elements may be selected as result of interactive communication (Steffe & Wiegel, 1994)

C: (Carol counts five parts and seven parts several times in turn and makes a face.) (Without a confidence,) would it be five-sevenths?

....

R: How did you get it, Carol?

1)C: Um... I'm not really sure but I was looking at it if you have one, two, three, four, five, that's the whole.

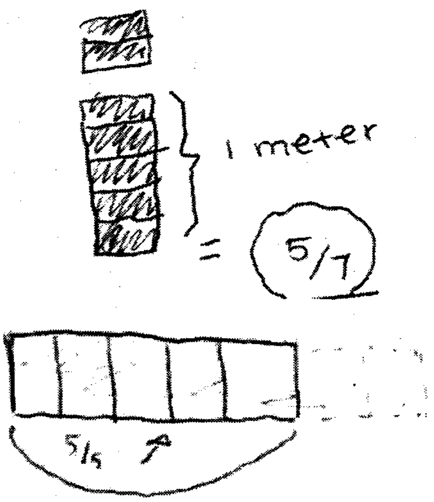
R: (At the same time with Carol) that's the one.

2)C: And you have the extra two and then I would make it seven but then you have the extra one meter which is what you have.

R: Oh~ I see how she did it. I see. Okay. I totally see how she got it.

T: Okay. So, (Teacher turns to look at Rosa' work.)

R: Five out of the seven pieces. This (her five parts drawn in solid lines) is the one and here is the extra two for the seven. So this (five parts drawn in solid lines) is one. So five over total seven pieces because it's seven. I see, I see it now.



<Figure Ⅳ-1a & 1b> Carol's (Above) & Rosa's (Below) drawings for 7/5-meter and 1-meter

When the students assimilated this problem as a situation for a unit-segmenting scheme, they struggled with it because the segmenting unit [7/5-meter] was larger that the unit to be segmented [1-meter], which was a novel situation for them to use their unit-segmenting schemes. Rosa also drew a 5-part bar and added two more parts on the right end of the 5-part bar using dotted lines but suddenly she turned to numeric calculation again. She wrote down '7/5×5/5 = 35/25' and '7/5' and then crossed a line over the numeric expression as she felt that something went wrong. On the other hand, it was Carol who found the answer first. She drew a 5-part bar and two more parts separately and shaded them all. After counting five parts and seven parts several times in turn, she provided an answer, "five-sevenths." Her comment 1) and 2) indicated that her answer came from the part-whole relation based on the visual sensory-motor information through comparing the sizes of her five parts with her seven parts. At this time, what I would conjecture was that a situation where the unit used in segmenting was larger than the unit to be segmented, might be another epistemological obstacle for the students and further a reason to inhibit them from expanding their measuring-out activity using a unit-segmenting scheme. Carol and Rosa needed to expand the range of assimilating situations of their unit-segmenting scheme so that it could include a situation where a smaller quantity was to be measured with a larger quantity. I conjecture that until the students realize that the result can be obtained in the same way as they deal with a remainder in the use of a unit-segmenting scheme with a remainder, which leads that the range of assimilated situations of their unit-segmenting scheme is generalized to include a situation where a smaller quantity is to be measured with a larger quantity, the epistemological obstacle observed in this protocol will remain as a main cause of their perturbation when using their unit-segmenting scheme.

3. Finding How Much Part of 7/5-Meters is Contained in 2 Meters

On the line of exploring the participating students' use of their unit-segmenting scheme with a fraction divisor, I immediately posed a similar problem to what was asked in the previous protocol with an expectation of their flexible use of the scheme in a similar but different problem situation.

Protocol II on 12/05/08:

W: How many times is seven-fifths contained in two meters? Seven-fifths meters contained in two meters.

C: Would it be twice?

T: Twice of what?

C: Twice seven-fifths.

3)R: So it's seven-fifths in ten-fifths, right?

C: Okay. Two meters. (Carol draws two 5-part bars and shades the whole parts of one 5-part bar and two parts of the other 5-part bar. Rosa draws a 7-part bar vertically and adds three more parts on top of the bar using dotted lines. See Figure Ⅳ-2a and 2b. Carol writes down '1 2/7' on the paper.) Is that the answer? (Witness-researcher shakes his head.) Wait. (Carol writes '2/7+3/7=5/7' and '1 5/7') Right? (Witness-researcher shakes his head again.)

R: Is that five-sevenths?

C: One and five-sevenths?

R: It has to be more than itself.

T: (To Carol) can you explain it to me?

4)C: Um, I think I did it wrong. I think I added too many, but if you have it one time (pointing out the shaded seven parts) it goes into the second one and you have one and two-sevenths. But then you still have the extra three-sevenths. Then it can go into... even if you don't have the whole one.

T: Sorry, it's hard to see. Can you, yeah, seven-fifths is

C: Wait! No, no, no, no.

R: (At the same time with Carol) is it one and three-sevenths? One and three-sevenths, one and three-sevenths. I'm sorry.

C: Yeah, it's one and three-sevenths. Because I added those two (the shaded 5-part bar and the shaded two parts of the other 5-part bar) cause that's one right there. (Carol draws a circle holding the shaded seven parts.) This equals one.

R: (At the same time with Carol) I miscalculated. It's one and three-sevenths.

....

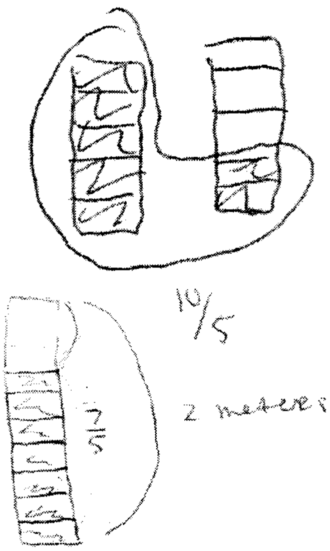
R: It's kind of obvious, like there's three, three-sevenths left because this (seven parts of the 10-part bar) is the whole.

C: Yes, that kind of that is two-sevenths, one that should've been one whole.

....

R: Ten over five which is the two meters times five over seven and I got fifty over thirty five which is one and three-sevenths because this three-sevenths (the unshaded three parts of her 10-part bar) which is the ten-fifths of you want is out of here (the shaded seven parts of her 10-part bar.) So, that's gonna be three-sevenths, right? Yes...yes, yes, yes.

T: Three-sevenths. Okay.



<Figure Ⅳ-2a & 2b> Carol's (Above) & Rosa's (Below) drawings for 7/5-meter in 2-meters

If Rosa's comment 3) stood in for her conception that seven-fifths consists of seven one-fifths and ten-fifths consists of ten one-fifths, any one-fifth of which can be iterated seven times and ten times to make seven-fifths and ten-fifths, she might have immediately solved this problem as in the same way as to measure 10 meters out with a unit of 7 meters. In other words, the construction of a fractional connected number[6], say, 10/7, as a composite unit containing one composite unit consisting of seven one-fifths and another composite unit consisting of three one-fifths can be crucial to solve this problem. Although Rosa and Carol constructed bars for 7/5 meters and two [10/5] meters on paper, there was no evidence that they assimilated the 7/5-meter bar and the 10/5-meter bar based on the construction of interiorized fractional connected numbers. Once again, for this problem Rosa showed her reliance on a division algorithm, which was a typical behavior of Rosa whenever she got stuck with her drawing to figure out the answer. However, in that she independently corrected her answer and established a relation between her division algorithm with her drawing in terms of measuring out with 7/5-meter, she retrospectively accommodated her concept of fraction measurement division using conceptual elements of unit-segmenting scheme and partitive fraction scheme.

On the other hand, Carol's writing of '2/7+3/7=5/7' and '1 5/7' revealed an interesting aspect of her fractional knowledge because it showed her conflation of units when measuring out a quantity with an (improper) fractional quantity. Somehow she felt the necessity of measuring out the whole two meters each meter of which was partitioned into five parts. She could measure out the remaining three parts [3/5-meter] using seven parts [7/5-meter] as a segmenting unit and get three-sevenths of the segmenting unit for the remainder parts. However, she conflated two units [a given unit of a meter and a segmenting unit, 7/5-meter] when she measured seven parts consisting of one 5-part bar and two parts of the other 5-part bar. Her confusion of the two measuring units was indicated by her comments 4), and such confusion of units caused her to indentify seven parts as one and two-sevenths, which led her to get one and five-sevenths as a final answer. My conjecture is that, as in the case of Rosa, if Carol had assimilated the problem situation based on the interiorized iterability of a unit fraction [1/5], she might have found the answer more easily. In other words, in the previous problems such as "measuring 5 meters with 3 meters" or "measuring 1 meter with 3/5 meter" the students did not show any evidence that they used the multiplicative relationship of a unit fraction with a whole unit. Although the former might require to construct a three-levels-of-units structure in order to deal with the remainder [2 meters], where 5 meters was measured by 3 meters and thus 3-meter should emerge as another level of unit in relation to the given two units [1 meter and 5 meters,] the construction of such a three-levels-of-units structure was inherited from the students' construction of an iterability of one, rather than a unit fraction. Likely, in the latter problem although it involved a fractional quantity [3/5-meter] and required the students' construction of three levels of units to see the remainder [2/5-meter] as two-thirds of the newly constructed unit of 3/5-meter, the relationship of a unit fraction [one-fifth] with the whole one was implicit just as the unit fraction in a partitive

6) Fractional connected numbers are connected numbers in which unit fractions are the units of the connected numbers. The connected numbers are numbers whose countable items are the elements of a connected but segmented continuous unit (cf. Steffe & Olive, 2010)

fraction scheme has an implicit iterability not transcending the one whole. However, in order to construct a three-levels-of-units structure involving 7/5-meters in the present problem, I would argue that the construction of a multiplicative relationship of one-fifth with one whole was essential because the students should have conceived seven-fifths as a unit of seven units of one-fifth, any of which could be iterated five times to make a referent whole, five-fifths and also seven times for seven-fifths. As described above, both students could not establish such relationships with the given 7/5 meters and 2 meters based on a unit fraction of 1/5 for a while, even with two drawings [seven parts for 7/5-meter and ten parts for 2-meter]. Also, the fact that they were able to reflect on their mathematical operations and self-corrected the answer by themselves with two drawings implied that perceptual information for the given quantities was still one of the critical factors for them to conduct their mathematical [unit-segmenting] operations.

4. Finding How Much Part of 5/3 Meters is Contained in 176 Meters - Carol's Construction of a Student-Generated Algorithm for Fraction Measurement Division

Since the two students got the answers for the two problems, "How much of or how many times 7/5 meters is contained in 1 meter or 2 meters?" as a result of their unit-segmenting scheme with a remainder, I wanted to know whether the students could strategically use those results as material for the other measuring-out situations, especially when a relatively large quantity should be measured, that is, when to make drawings to get perceptual sensory-motor information is actually not allowed. When the students were asked about how many times 7/5 meters was contained in 10 meters, both of them easily found the answer, fifty-sevenths by multiplying five to the previous result, ten-sevenths, which was the answer for the problem "How many times is 7/5 meters contained in 2 meters?" Further, when I increased the length to be measured to 31 meters, they easily wrote down '155/7' on their own paper by calculating '5/7×31/1' with an aid of the witness-researcher's reminding question, "How many times is 7/5 meters contained in one meter?" and encouraging the students to use the result for their solution. Therefore, the concern for the next teaching experiment was whether such a strategic use of the result of their unit-segmenting schemes with a remainder remains permanent so that it can be used in a different but similar problem.

When they were asked to find how many times 5/3 meters was contained in 176 meters, it was Rosa that first multiplied one hundred seventy six by three-fifths. However, it was not a strategic use of her unit-segmenting scheme with a remainder. Rather, it was reemergence of the memorized invert-and-multiply algorithm for a fraction division problem whenever Rosa assimilated a problem as a fraction divisional situation. Such lack of her mathematical reasoning was indicated by the uncertainty of her answer right after finishing calculation of her fraction multiplication as "Is that right? It doesn't look right." Although I eagerly attempted to induce her to use the other way related to what she did before, Rosa could not use her unit-segmenting scheme strategically for this problem. I had to directly ask her how much part of five-thirds was contained in one, but she could not figure out three-fifths of five-thirds was one. Her answer was two-thirds emphasizing the difference between five-thirds and one whole. Upon my request of verification for her answer, she wrote down '5/3×3/3.' She seemed to use an invert-and-multiply algorithm for the division of '5/3÷3/3', rather than '3/3÷5/3.' This kind of lacuna in her mathematical reasoning coincided with what she showed in the solving process for the previous measuring-out problem when the measuring quantity was larger than the measured quantity (cf. Protocol I). On the other hand, Carol was catching up with the present problem with an aid of the witness-researcher because she spent more time in converting her answer into a decimal form for the previous problem.

Protocol III[7] on 12/05/08:

W: Remember, Carol. The problem is how many times is five-thirds contained in

C: One hundred seventy six.

W: One hundred seventy six. (When Carol writes down '5/3×176/1' for calculation, the witness-researcher intervenes.) Wait, you missed a step.

5)C: Oh, wait. I have to find how many times it goes into the one. (Carol draws two 3-part bars and shades all three parts of the first bar and two parts of the second bar. Then she writes down '5/6') Five-sixths? Is that how many times it goes into one? No?

T: No.

W: Make it five-thirds.

C: Oh, it goes in three-fifths times, wouldn't it?

T: Yes.

W: Right, three-fifths.

C: Three-fifths. Then it would times... (Carol writes down '3/5×176/1' and accidentally got '628/5' by miscalculation. On the other hand, Rosa struggles to find how much part of five-thirds is contained in one.)

C: I got it!

T: Okay, hold a second.

(Rosa insists that her answer should be two-thirds rather than three-fifths. Upon the teacher's request of verifying her answer, she writes down '5/3×3/3' and gets '5/3.' She looks bewildered by the unexpected answer.)

C: Have you seen the pattern? Like, um

R: Sometimes it works, sometimes it doesn't.

6)C: Rosa, on these (pointing her drawings for the problem in the Protocol I. See Figure IV-1a) if it was seven-fifths, it turned into five-sevenths. There is kind of a pattern. And then on the other one, it was five-sevenths and that ended up being seven-fifths. And if you look at this one (back to the current problem), it's five-thirds and it ends up, if you recognize the pattern, it would be

R: One and two-thirds?

C: No, three-fifths.

T: How much part of... Carol, can I ask a question? How much part of five-thirds contained in one meter?

C: One? Three-fifths.

R: If the three is the total, okay, I think I get it.

T: (To Rosa) can you see the three-fifths?

R: Okay, so this is, okay. I have to draw it over here. So here, (Rosa draws a 5-part bar.) There is three or five. Okay, so this (three parts of the 5-part bar) is one right here and this (the whole 5-part bar) is five. And if you want to know how many times is this (three parts) in the total right?

T: Um-hm.

R: Okay, um... (Rosa writes down '5/3') and it's three-fifths because it's three out of the total five. (The teacher nods his head.) Okay, like I see how she is doing it but then I forget what I do, what I do.

C: Isn't there also the pattern? Like I said five-thirds ends up being three-fifths? And the other one was

7) Note that a part of Rosa's verbal expressions and communications with the teacher (myself) are omitted because the intention of this protocol is to show Carol's construction of an algorithm for fraction measurement division and almost all parts of activities were done individually. However, brief descriptions of Rosa's work are also described to help understand the context.

seven-fifths ends up being five-sevenths.

R: So it's just the reciprocal of it. If you want to find it in one, it's the reciprocal.

C: Yeah.

T: So can you see why the reciprocal works together?

R: Yeah, I see how it works together.

....

T: Okay. Carol, can you explain from the start so that Rosa and I share?

7)C: Okay. I did from the five-thirds in a hundred seventy six. And then I found out the three-fifths for one like one whole, it was three-fifths of one whole. Then I times that by a hundred seventy six because that was the number in the problem. I got six hundred twenty eight over five. Denominator multiplication. And then I simplified it.

Unlike Rosa, Carol's numeric notations of '3/5×176/1' did not come from the conventional invert-and-multiply algorithm for fraction division. If so, the order of writing for fraction multiplication should have been reversed like '176/1×3/5' for the division problem of '176/1÷5/3' as Rosa usually did. Carol apparently formed the goal for activating her unit-segmenting scheme as in her comment 5). Further, Carol's comment 7) to reflect her solving processes indicated that her unit-segmenting operation was fundamental in her further mathematical operations for the present problem. Actually, her mathematical operations also involved units-coordinating operations. That is, she distributed three-fifths over each of one hundred seventy six. However, the difference from her units-coordinating operation in her whole number multiplication[8], was that a fraction was distributed over a whole number. She constructed the fraction, three-fifths as an iterating unit to get one hundred seventy six of three-fifths like getting fifteen by iterating a unit of three five times. It means the assimilating situations of Carol's units-coordinating scheme were expanded and started to include a fraction. However, such a generalizing assimilation of her units-coordinating scheme was still to be investigated.

More importantly, she abstracted her unit-segmenting operations from the previous two problems, "How much of seven-fifths is contained in one or two meters?" and found a pattern for fraction measurement division problems as in her comment 6). Therefore, for Carol to find the answer for how many times five-thirds was contained in one hundred seventy six, she could just flip five-thirds to make three-fifths and multiply it by one hundred seventy six, which exactly coincided with the conventional invert-and-multiply algorithm for fraction division. I would call Carol's construction as a student-generated algorithm for fraction measurement division. "Child-generated algorithms as they are manifest in notation are nothing but records of operation, and these records serve the function of constructive generalization" (Steffe & Ulrich, 2010, p. 274). The flipping pattern was an abstracted record of her unit-segmenting operations, especially when measuring a unit whole with a fractional quantity more than the whole. On the other hand, even with Carol's explanation, Rosa did not seem to understand what Carol was trying to say. Rosa's using the invert-and-multiply algorithm was a procedure. A procedure is a scheme in which the activity is only connected to rather than contained in the first part of the scheme (Olive & Steffe, 2010, p. 214). In Rosa's case, the first part of her procedure was constituted by the words "How many times, contained." Her activity

8) To find the product of five and three, if a child mentally inserts the unit of three into each unit of five to produce five threes prior to actual activity, the involved scheme is referred to as a units-coordination scheme (Steffe, 1991).

of dividing using the invert-and-multiply algorithm can be regarded as her meaning for the words.

5. Finding a Fraction of 3/4-Meter that Amounts to 31 Meters - Rosa's Construction of a Unit-Segmenting Scheme with a Remainder and its Strategic Use

Almost three months later, a similar fraction measurement division problem was posed to the participating students again partially due to the winter break between the teaching protocols. At that time, I focused my attention mainly on Rosa because she had not indicated construction of her self-generated algorithm for fraction measurement division as Carol did in the previous protocol.

Protocol IV on 02/27/09:

T: What fraction of five-sevenths should be one-meter? Five-sevenths.
(Rosa makes a 5-part bar and pulls out two parts from the bar to arrange them with the bar in a row.)
C: Sevenths. One and two-fifths?
R: (Rosa does not seem to listen to Carol's answer.) I should've set those together. And you have one, two, three, four, five and you need to make one meter?
T: Yeah. What fraction of...
R: Okay. One... and two-fifths.
C: That's what I said.
T: One and two-fifths.
R: Yeah. Because five is your whole. We were looking at four instead of three[9].
T: Let's listen to your, Rosa's explanation.
8)R: Okay. Instead making seven, um... I took, this (5-part bar) is five-sevenths. And here is two other ones to make the whole one meter. But... so this (5-part bar) is one and you need two more to make seventh-sevenths. But two more of this (5-part bar), because this is your whole. So if there is five in total and you need two, it's gonna be two-fifths plus one original bar that you have right there.
....
T: Okay, let's go back to the second problem. How could we solve this problem? Some fraction of three-fourths meter amounts to thirty-one meters.
....
T: What did you multiply?
R: Oh, what did I multiply. Four over three times thirty-one over one.
T: Why?
R: Why. Okay, I know that four over three is one meter. And you're trying to see how many, um... I'm trying to get thirty-one meters. So you just multiply it times thirty-one.

Immediately before the problem in this teaching protocol, a similar problem [what fraction of 3/4-meter amounts to 1-meter?] was posed to both students and they struggled to solve it for about ten minutes. Although they constructed a 4-part bar for one meter and a 3/4-meter bar by pulling out three parts from the one meter bar partitioned into four parts, the problem that began with "What fraction of..." did not seem to provoke any fraction scheme available to them in order to cope with the problem situation. Until the witness-researcher changed the problem into "What times of 3/4-meter is contained in 1-meter," they could not assimilate the problem as a situation for their unit-segmenting scheme with a remainder that they had constructed before (cf. Protocol I and II). Once they assimilated the problem as a situation for their unit-segmenting scheme, this

9) Both students had struggled with the problem to find what fraction of 3/4-meter amounts to one meter right before this teaching episode.

teaching protocol revealed the first indication that Rosa had constructed a unit-segmenting scheme with a remainder by independently solving the posed problem. Actually, in the previous teaching episodes involving fraction measurement division, it was Carol who provided explanations first and who led communications among us. Rosa, in contrast, mostly assimilated Carol's activities and explanations to find solutions, rather than by herself, although she was very quick to assimilate Carol's explications and find the pattern of Carol's work. However, in this protocol although Carol suggested her answer first, Rosa did not seem to attend to Carol's answer and instead found her answer by herself. Her comments 8) before Carol's explanation corroborated that Rosa had constructed a unit-segmenting scheme with a remainder embedding her partitive fraction scheme as a subscheme in the first part of the unit-segmenting scheme. That is, she was explicitly aware of her segmenting unit [5/7-meter] and conceived the leftover [2/7-meter] in terms of the unit used in segmenting operations [two-fifths of the 5/7-meter]. Further, Rosa indicated a similar strategic use of her unit-segmenting scheme to solve a more complex problem [What fraction of 3/4-meter does amount to 31 meters?] in a way similar to Carol in Protocol III [How many times is 5/3 meters contained in 176 meters?]. Rosa knew that four-thirds of 3/4-meter was contained in one meter and used it to find a fraction of 3/4-meter to get 31 meters by iterating four-thirds 31 times. Obviously that was a big mathematical progress for Rosa, when compared with her struggles in Protocol III, because at that time she could not use the result of her unit-segmenting scheme with a remainder for other mathematical situations, even with Carol's very detailed explanations. Now Rosa can be attributed with construction of a pattern for fraction measurement division calculation, which is on par with the traditional invert-and-multiply algorithm for fraction division.

V. Discussion

According to Campbell et al. (1998), a student-generated algorithm can be acceptable under the conditions that 1) the procedure is efficient enough to be used regularly without considerable loss of time and without frustration due to the number of recorded steps required, 2) the algorithm is mathematically valid and 3) generalizable. In that sense, the construction of a pattern that the two participating students indicated in the context of fraction measurement division can be acceptable as a student-generated algorithm because 1) the procedure was efficient enough to be applied to fraction measurement division situations just as the invert-and-multiply algorithm could, 2) the pattern was mathematically sound based on the reciprocal relationship between the divisor and one whole and 3) it was generalizable enough to involve fractional numbers as well as whole numbers in both divisor and dividend.

Thus, this analysis provides a possible constructive itinerary that teachers could use as a reference when they encourage their students to engage in meaningful mathematical activities with an invert-and-multiply algorithm. In other words, I was able to observe the two students in my teaching experiment not only know the algorithm, but also when and how to apply the algorithm correctly to novel situations through generating their own algorithms on the basis of their fractional knowledge, This implies that students needs to be encouraged and allowed to explore such knowledge-construction processes in classroom which possibly leads the students to feel a logical necessity to develop better notation and language to be more efficient or to communicate their ideas to the other students. Then the understanding demonstrated by students can be guided into the conventional algorithms, resulting in both conceptual and procedural understanding.

In addition, the results of the present study imply that instruction based on understanding student-generated algorithms needs to be accounted as one of the crucial characteristics of good mathematics teachers' in that no pedagogical decision can be made until students' mathematical actions and operations are comprehended by the teachers. In other words, mathematical knowledge for teaching should entail the ability to appraise students' inventing work, otherwise teachers are less able to provide meaningful instructions from learners' [students'] point of view.

References

Ashlock, R. B. (2006). *Error patterns in computation: Using error patterns to improve instruction.* Upper Saddle River, NJ: Pearson Merrill Prentice Hall.

Baek, J. (1998). Children's invented algorithms for multidigit multiplication problems. In L. J. Morrow & M. J. Kenney (Eds.), *The teaching and learning of algorithms in school mathematics, 1998 Yearbook of the National Council of Teachers of Mathematics* (pp. 151-160). Reston, VA: National Council of Teachers of Mathematics.

Barnett, J. H. (1998). A brief history of algorithms in mathematics. In L. J. Morrow & M. J. Kenney (Eds.), *The teaching and learning of algorithms in school mathematics, 1998 Yearbook of the National Council of Teachers of Mathematics* (pp. 69-77). Reston, VA: National Council of Teachers of Mathematics.

Burns, M. (1994). Arithmetic: The last holdout. *Phi Delta Kappan,* 75, 471-476.

Campbell, P. F., Rowan, T. E., & Suarez, A. R. (1998). What criteria for student-invented algorithms? In L. J. Morrow & M. J. Kenney (Eds.), *The teaching and learning of algorithms in school mathematics, 1998 Yearbook of the National Council of Teachers of Mathematics* (pp. 49-55). Reston, VA: National Council of Teachers of Mathematics.

Huinker, D. (1998). Letting fraction algorithms emerge through problem solving. In L. J. Morrow & M. J. Kenney (Eds.), *The teaching and learning of algorithms in school mathematics, 1998 Yearbook of the National Council of Teachers of Mathematics* (pp. 170-182). Reston, VA: National Council of Teachers of Mathematics.

Kamii, C., & Dominick, A. (1998). The harmful effects of algorithms in grades 1-4. In L. J. Morrow & M. J. Kenney (Eds.), *The teaching and learning of algorithms in school mathematics, 1998 Yearbook of the National Council of Teachers of Mathematics* (pp. 130-140). Reston, VA: National Council of Teachers of Mathematics.

Kamii, C., Lewis, B. A., & Livingston, S. J. (1993). Primary arithmetic: Children inventing their own procedures. *Arithmetic Teacher, 41,* 200-203.

Kilpatrick, J., Swafford, J., & Findell, B. (2001). *Adding it up: Helping children learn mathematics*. Washington, DC: National Academy Press.

Maurer, S. B. (1998). What is an algorithm? What is an answer? In L. J. Morrow & M. J. Kenney (Eds.), *The teaching and learning of algorithms in school mathematics, 1998 Yearbook of the National Council of Teachers of Mathematics* (pp. 21-31). Reston, VA: National Council of Teachers of Mathematics.

McClain, K., Cobb, P., & Bowers, J. (1998). A contextual investigation of three-digit addition and subtraction. In L. J. Morrow & M. J. Kenney (Eds.), *The teaching and learning of algorithms in school mathematics, 1998 Yearbook of the National Council of Teachers of Mathematics* (pp. 141-150). Reston, VA: National Council of Teachers of Mathematics.

National Council of Teachers of Mathematics. (2000). *Principles and Standards for School Mathematics*. Reston, VA: Author

Olive, J., & Steffe, L. (2002). Schemes, schemas and director systems- an integration of Piagetian scheme theory with Skemp's model of intelligent learning. In D. Tall & M. Thomas (Eds.), *Intelligence, learning and understanding in mathematics*. Flaxton, Australia.

Olive, J., & Steffe, L. P. (2010). The partitive, the iterative, and the unit composition schemes. In L. P. Steffe & J. Olive (Eds.), *Children's fractional knowledge*. New York: Springer.

Plunkett, S. (1979). Decomposition and all that rot. Mathematics in School, 8, 2-5.

Skemp, R. R. (1987). *The psychology of learning mathematics*. Hillsdale, NJ: Lawrence Erlbaum Associates.

Steffe, L. P. (1991). Operations that generate quantity. *Learning and Individual Differences, 3*(1), 61-82.

Steffe, L. P. (1992). Schemes of action and operation involving composite units. *Learning and Individual Differences, 4*(3), 259-309.

Steffe, L. P. (2002) A new hypothesis concerning children's fractional knowledge. *Journal of Mathematical Behavior, 102*, 1-41

Steffe, L. P., & Olive, J. (2010). *Children's fractional knowledge*. New York: Springer.

Steffe, L. P., & Thompson, P. W. (2000). Teaching experiment methodology: Underlying principles and essential elements. In A. E. Kelly & R. A. Lesh (Eds.), *Handbook of research design in mathematics and science education* (pp. 267-306). Mahwah, NJ: Erlbaum.

Steffe, L. P., & Ulrich, C. (2010). Equipartitioning operations for connected numbers: Their use and interiorization. In L. P. Steffe & J. Olive (Eds.), *Children's fractional knowledge*. New York: Springer.

Steffe, L. P., & Wiegel, H. G. (1994). Cognitive play and mathematical learning in computer microworlds. *Journal of Research in Childhood Education, 8*(2), 117-131.

Tzur, R. (1999). An integrated study of children's construction of improper fractions and the teacher's role in promoting that learning. *Journal for Research in Mathematics Education, 30*(4), 390-416.

von Glasersfeld, E. (1995). *Radical constructivism: A way of knowing and learning*. New York, NY: Routledge Falmer.

분수나눗셈을 해결하기 위한 학생들의 자기-생성 알고리듬 구성에 관한 연구

신 재 홍 (한국교원대학교)

본 연구는 두 명의 중학교 2학년 학생들이 어떻게 단위 분할 도식의 수정, 변경을 통하여 분수나눗셈 상황에서 그들 자신만의 자기-생성 알고리듬을 만들어 나가는지 보여주고 있다. 교육실험이 연구방법으로 사용되었고, 일년간 행해진 교육실험 중 일부분의 자료가 본 연구를 위해 분석되었다. 두 명의 참여 학생들은 기준단위와 제수사이의 상호 관계 구성과 활용으로 분수나눗셈을 위해 전통적으로 학습되어 왔던 '뒤집어서 곱하기'와 같은 역할을 하는 그들 자신의 자기-생성 알고리듬을 구성할 수 있었다. 본 연구결과는 또한 학생들이 만들어낸 알고리듬을 이해할 수 있는 것이 훌륭한 수학 교사로서의 질을 결정하는 하나의 요소로 고려되어야 함을 보여주고 있다.

* **key words** : student-generated algorithm (자기-생성 알고리듬), unit-segmenting scheme (단위 분할 도식), fraction measurement division (분수 포함제 나눗셈)

논문접수 : 2010. 6. 21
논문수정 : 2010. 9. 4
심사완료 : 2010. 9. 11