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# Comparison of Numerical Methods on Heat Transfer in a Rod with Second Order-Boundary Value Problem
# 이차 경계문제를 가지는 봉의 열전달에 대한 수치해석적 비교

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주요용어: 수치해석(Numerical Method), 열전달(Heat Transfer), 수정된 오일러 방법(Modified Euler's Method), 이론 해(Analytical Solution)

요 약 : 본 연구는 수정 오일러 법칙을 이용한 봉의 열전달문제를 엄밀해와 수치해를 수치해석적 해법을 이용해 비교한 것이다. 경계조건으로는 열전도 및 대류가 동시에 존재하는 경우의 모델을 가정하여 계산하였고, 봉의 길이가 원주방향에 비해 상당히 길다고 가정하여 1차원으로 지배방정식을 정리하여 2차 상미분방정식을 유도하여 계산을 수행하였다. 계산을 수행한 결과 적절한 초기 추측값인 β값을 정의하면 오일러의 방정식으로도 충분히 만족할만한 결과를 얻을 수 있다는 것을 알았고, 지수함수 형태의 유도 상관식이 엄밀해와 ±1% 범위 내에서 일치한다는 결과를 얻었다.

## Nomenclature

$A$ : Cross sectional area of rod [m$^2$]
$a$ : Coefficient concerned in Eq.(12)
$b$ : Coefficient concerned in Eq.(12)
$c$ : Coefficient concerned in Eq.(12)
$dx$ : Infinitesimal distance [m]
$h$ : Coefficient of convective heat transfer [W/(m$^2$·K)]
$k$ : Thermal conductivity of rod [W/(m·k)]
$P$ : Perimeter of rod [m]
$q$ : Heat transfer rate [W]
$T$ : Temperature [K]
$x$ : Distance of rod [m]
$a$ : Factoring coefficient, $\sqrt{\frac{h \cdot p}{k \cdot A}}$ [$m^{-1}$]
$\beta$ : Derivative of temperature at $x_1$ [K]

## Subscript

$a$ : Ambient
$c$ : Convective heat transfer
$cor$ : Corrected value
$old$ : Old
$pre$ : Present

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## 1. Introduction

In analysis on heat transfer problem, there are many methods to analyze the temperature profile on control volume or control materials.

Recent years, the concerns on numerical analysis are increased[1~5]. So the part of numerical analysis have been expanded in every industrial part, especially in heat transfer[6~9].

Among this part, the Euler's method which is a numerical technique to solve ordinary differential equations has been widely spread in numerical analysis part. However, the Euler's method is adopted only first order ordinary differential equations.

Unfortunately, to solve the heat transfer problems, most of the governing different equations are partial differential equations. Even if, to simplify the PDE(Partial Differential Equation), the governing equation can be derived as second order ODE(Ordinary Differential Equation). So it is impossible to solve these problems with conventional Euler's method.

The goal of this study is to compare the difference between analytic solution and modified Euler's predictor-corrector method in the steady state one dimensional heat transfer problem with boundary values.

## 2. Heat transfer model

In this study, the heat transfer in a rod is used as an example of the development of second order boundary value problem. The heat transfer mechanisms in this rod are combined with heat conduction and convection(shown as Fig. 1).

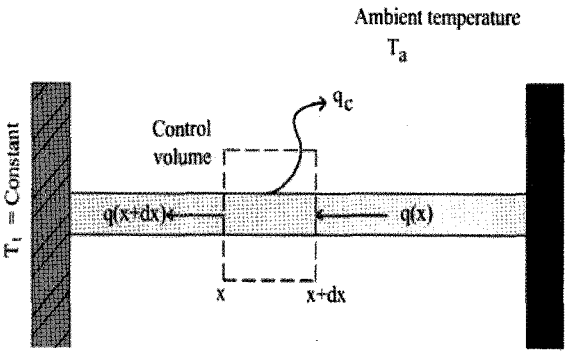


Fig. 1 Considered analysis model

The energy balance of heat transfer from the control volume of Fig. 1 is given by Eq.(1).

$$q(x) = q(x+dx) + q_c \quad (1)$$

The heat balance, composed with heat conduction and convection, written above can also be expressed as Eq.(2).

$$\frac{d}{dx} q(x) \cdot dx + q_c = 0 \quad (2)$$

The heat conduction phenomenon was governed by Fourier's law of conduction. And also the heat transfer by convection is governed by Newton's law of cooling. So the Eq.(2) can be given by Eq.(3)

$$\frac{d}{dx}\left[-k \cdot A \cdot \left(\frac{d}{dx} T\right)\right] \cdot dx + h \cdot (P \cdot dx) \cdot (T - T_a) = 0 \quad (3)$$

To simplify this analysis, the thermal conductivity($k$) and cross sectional area of rod($A$) are assumed as constant values then it can be factored out and rewritten as Eq.(4) and (5).

$$\frac{d^2}{dx^2} T - \left(\frac{h \cdot P}{k \cdot A}\right) \cdot (T - T_a) = 0 \quad (4)$$

where $\alpha^2 = \dfrac{h \cdot P}{k \cdot A}$

$$T'' - \alpha^2 \cdot T = -\alpha^2 \cdot T_a \quad (5)$$

The above Eq.(5) is a second order ordinary differential equation with boundary values. Its solution is the temperature of the rod($T$) as a function of distance along the rod($x$) commonly written as $T(x)$. The solution function is subjected to the constant temperature boundary conditions at each end of the rod.

Fig. 2 is the imaginable solution to the second order ordinary differential equation with known boundary values. The derivative of the temperature ($\beta$) at $x_1$ is a fixed but unknown value.

In this calculation, the $\beta$ is assumed as a proper value and the correct assumption will result in the temperature ($T_2$) at the right end of the rod.

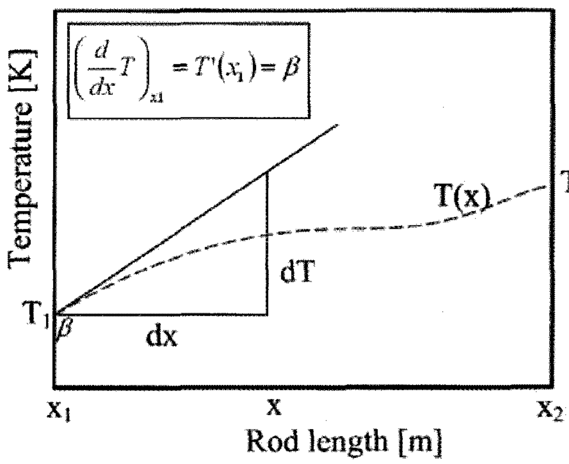


Fig. 2 Imaginable solution of the Eq.(5)

## 3. Solving the problem with modified Euler's method

As mentioned above, to solve the second order ODE is impossible with Euler's method. So it should be rewritten as two first order ODEs.

Using the Eq.(5), let the estimate of first derivative of temperature $T'$ is $\beta$. Therefore, $\beta' = T'' = \alpha^2 T$. So the two first order ODEs are like follows.

$$T' = \beta \qquad T(0) = 0 \tag{6}$$

Eq.(6) can be solved using Euler's method.

$$\frac{d}{dx}T = \beta_{old}$$
$$dT = \beta_{old} \cdot dx \tag{7}$$
$$\text{Tnew} = \text{Told} + \beta\text{old} \cdot \text{dx}$$

The another one first order ODE is also derived as Eq.(8).

$$\beta' = T'' = \alpha^2 \cdot T \qquad \beta(0) = T'(0) \tag{8}$$

Using the same method in Eq.(7), the Eq.(8) can be solved as Eq.(9).

$$\frac{d}{dx}\beta = \alpha^2 \cdot T_{old}$$
$$d\beta = \alpha^2 \cdot T_{old} \cdot dx \tag{9}$$
$$\beta\text{new} = \beta_{old} + \alpha^2 \cdot T_{old} \cdot dx$$

The algorithm of this calculation is that the old value of temperature is replaced with those most recently computed and repeated the procedure at the next point along the rod.

It can be solved these equations using the modified Euler's predictor-corrector method. However, the modified Euler's method needs the initial guess of $\beta$.

To compose the computer program, the above equations can be expressed as Eq.(10). And this calculation was fulfilled with Mathcad(ver. 13).

$$T_{pre} = T_{old} + \triangle x \cdot \beta_{old}$$
$$\beta_{pre} = \beta_{old} + \triangle x \cdot \alpha^2 \cdot T_{old}$$
$$T_{cor} = T_{pre} + \frac{1}{2} \cdot \triangle x \cdot (\beta_{pre} + \beta_{old}) \tag{10}$$
$$\beta_{cor} = \beta_{pre} + \frac{1}{2} \cdot \triangle x \cdot (T_{pre} + T_{old})$$

The analytic and modified Euler's method solutions of temperature profile with rod distance is shown in Fig. 3.

From this graph, if the initial guess value of $\beta$ is well chosen, it is clearly known that the modified Euler's method is well agreed with the analytic solution. In this case, the initial guess value was $\beta$ = 55.

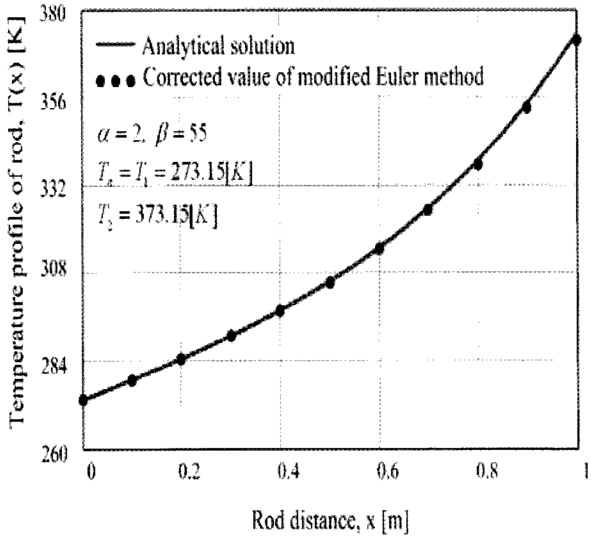


Fig. 3 Comparison of analytic and Euler's method

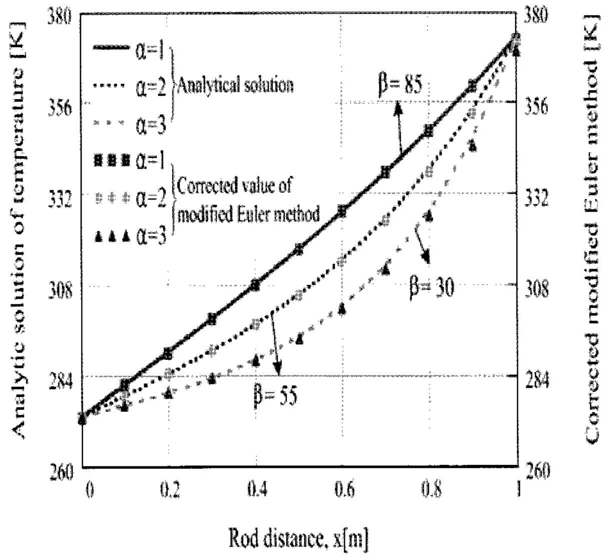


Fig. 4 Comparison of analytic and modified Euler's method with various factoring coefficients

To find the effect of $\alpha$, it is examined in various range of $\alpha$. Fig. 4 shows the calculated results of various $\alpha(\alpha=1\sim10)$. It is known that the initial guess value of $\beta$ is decreased with increasing in $\alpha$.

The increasing of $\alpha$ means that the heat transfer is good. So the temperature profile of rod is increased with exponentially. As the result, the initial value of derivative($\beta$) becomes smaller as the heat transfer goes well.

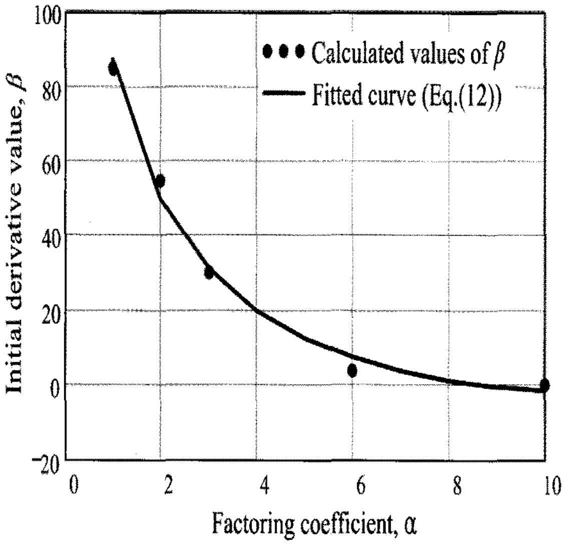


Fig. 5 Relationship between $\alpha$ and $\beta$

From this study, it is clearly known that the factoring coefficient($\alpha$) and initial derivative value($\beta$) has logarithmic relationship. Therefore, the data of $\alpha$ and $\beta$ are fitted with the function of logarithmic equation(Eq. (11)).

$$f(\alpha)=a\cdot\ln(\alpha)+b\cdot\sqrt{\alpha}+c \qquad (11)$$

where, a, b and c are unknown coefficients.

Fig. 5 shows that the logarithmic function with the newly found coefficients values and the original calculated data points reveals a good fit. And the acquired fitting equation is expressed as Eq. (12) within the maximum error of ±1%.

$$f(\alpha)=-81.057\cdot\ln(\alpha)+45.187\cdot\sqrt{\alpha}+42.105 \qquad (12)$$

## 4. Conclusions

In this study, the modified Euler's predictor-corrector method was calculated in the steady state one dimensional heat transfer problem with boundary values. And the following conclusions can be obtained.

(1) If the initial guess value of $\beta$ has been chosen properly, the modified Euler's method is reliable to calculate the heat transfer problem.

(2) It is cleared that the initial value of derivative becomes smaller as factoring coefficient is being larger.

(3) In this calculation range($1\leq\alpha\leq10$), the relationship between factoring coefficient and initial value of derivative was observed with logarithmic function. And the each coefficient of correlation equation was derived within the maximum error of ±1%.

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