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A Generalization of the Robust Inventory Problem with Non-Stationary Costs*

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ABSTRACT

This paper considers the robust inventory control problem introduced by Bertsimas and Thiele [4]. In their paper, they have shown that the robust version of the inventory control problem can be solved by solving a nominal inventory problem which is formulated as a mixed integer program. As a proper generalization of the model, we consider the problem with non-stationary cost. In this paper, we show that the generalized version can also be solved by solving a nominal inventory problem. Furthermore, we show that the problem can be solved efficiently.

Keywords: Robust Inventory Theory, Robust Inventory Control Problem, Mixed Integer Program

1. Introduction

A robust optimization approach to inventory control has been proposed in Bertsimas and Thiele [4]. In contrast with the traditional inventory theory where the distribution of the random demand is assumed to be known a priori (for example, see Clark and Scarf [6]), the robust optimization approach assumes that the only known fact is that the demand is a member of a pre-specified set (called *uncertainty set*).

Basically the robust optimization approach is the worst case oriented in that it

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tries to find a solution which works best in the worst-case realization of the uncertain parameters (for example, see Ben-Tal *et al*. [2]). Practically the approach is more suitable than the traditional stochastic models when very limited information on the uncertain parameters is available. However, there is price to pay for the guaranteed robustness, that is, the robust optimal solution can be very conservative. To tackle the problem, a mechanism to control over-conservativeness was proposed by Bertsimas and Sim [5], which was adopted in the robust inventory theory (Bertsimas and Thiele [4]). In the paper, they have shown that the robust optimal policy is of the same form as the case of the stochastic inventory model (the so-called (s, S) policy, see Bertsekas [3]) and the parameters of the optimal policy can be calculated by solving a nominal (deterministic) mixed integer programming problem.

This paper considers a generalization of the robust inventory model where the cost parameters are not stationary. We show that as in the stationary cost case considered in Bertsimas and Thiele [4], the generalized version can also be solved by solving a nominal problem. In addition, we show that the nominal problem can be solved efficiently, which was not addressed in their paper. Hence we can conclude that the generalized robust inventory problem can also be solved efficiently.

In the next section, we briefly review the robust inventory theory presented in Bertsimas and Thiele [4]. Section 3 presents a generalization of the model and polynomial time solvability of the nominal problem. Finally, Section 4 gives concluding remarks.

2. A short Review of the Robust Inventory Theory

This section briefly reviews the main results presented in Bertsimas and Thiele [4].

For each period $k \in \{0, 1, \cdots, T-1\}$, a demand w_k is given. In each period, we can order as much as we want and the cost of ordering the quantity $u \geq 0$ is given by $C(u) = K + cu$ when $u > 0$ and $C(0) = 0$. The unit inventory holding cost and penalty cost due to shortage are given by h and p, respectively. Note that the costs are assumed to be stationary, that is, they are the same for all the periods. For each period k, let x_k be the initial stock level (x_0 is assumed to be given) and u_k the order quantity. Then the following equation holds:

$$x_{k+1} = x_k + u_k - w_k = x_0 + \sum_{i=0}^{k}(u_i - w_i)\,. \tag{1}$$

The robust inventory problem can be formulated as follows.

$$\text{(RIP)} \quad \min \sum_{k=0}^{T-1}(cu_k + Kv_k + y_k) \tag{2}$$

$$y_k \geq h(x_0 + \sum_{i=0}^{k}(u_i - w_i)),\ \ k = 0, 1, \cdots, T-1, \tag{3}$$

$$y_k \geq -p(x_0 + \sum_{i=0}^{k}(u_i - w_i)),\ \ k = 0, 1, \cdots, T-1, \tag{4}$$

$$0 \leq u_k \leq Mv_k,\ \ v_k \in \{0, 1\},\ \ k = 0, 1, \cdots, T-1, \tag{5}$$

where $w_k = \overline{w}_k + \hat{w}_k z_k$ such that $z \in U = \{|z_k| \leq 1, \sum_{i=0}^{k}|z_i| \leq \Gamma_k,\ k = 0, 1, \cdots, T-1\}$ and M is a sufficiently large number.

In the above formulation, the variable y_k denotes either the inventory holding cost or the penalty cost. The demand of each period is given by the nominal value ($\overline{w}_k$) and its possible deviation from the nominal value ($\hat{w}_k z_k$), where z is a member of the uncertainty set *U*. The parameters Γ_k are used to control the conservativeness of the solution, where it is assumed that $\Gamma_k \leq \Gamma_{k+1} \leq \Gamma_k + 1$ (which means the level of uncertainty increases as the number of possible perturbations increases, but with limited increment bounded by the number of perturbations newly added). Note that as the value of Γ_k increases, the number of possible realizations of the demands increases, which results in more conservative solution. When $w_k = \overline{w}_k$ (that is, there is no uncertainty in the demand), the (RIP) is called a *nominal* inventory problem.

Note that a feasible solution to (RIP) should satisfy the constraints (3) and (4) for all possible realizations of *w*, which can be represented as follows:

$$y_k \geq h(x_0 + \sum_{i=0}^{k}(u_i - \overline{w}_i) - \min_{z \in U} \sum_{i=0}^{k} \hat{w}_i z_i),\ \text{for all } k,$$

$$y_k \geq -p(x_0 + \sum_{i=0}^{k}(u_i - \overline{w}_i) - \max_{z \in U} \sum_{i=0}^{k} \hat{w}_i z_i),\ \text{for all } k.$$

By noting that $\max\{\sum_{i=0}^{k} \hat{w}_i z_i \mid z \in U\} = -\min\{\sum_{i=0}^{k} \hat{w}_i z_i \mid z \in U\}$, the problem (RIP) can be reformulated as follows:

$$\min \sum_{k=0}^{T-1} (cu_k + Kv_k + y_k)$$

$$y_k \geq h(x_0 + \sum_{i=0}^{k} (u_i - \overline{w}_i) + A_k), \quad k = 0, 1, \cdots, T-1, \tag{6}$$

$$y_k \geq p(-x_0 - \sum_{i=0}^{k} (u_i - \overline{w}_i) + A_k), \quad k = 0, 1, \cdots, T-1, \tag{7}$$

$$0 \leq u_k \leq Mv_k, \quad v_k \in \{0, 1\}, \quad k = 0, 1, \cdots, T-1,$$

where $A_k = \max\{\sum_{i=0}^{k} \hat{w}_i z_i \mid z \in U\}$, which can be computed easily by a greedy algorithm since $A_k = \max\{\sum_{i=0}^{k} \hat{w}_i z_i \mid \sum_{i=0}^{k} z_i \leq \Gamma_k, 0 \leq z_i \leq 1, \forall i\}$. In Bertsimas and Thiele [4], they propose a different formulation which relies on LP duality.

The main results of the paper are summarized in the following theorem by Bertsimas and Thiele [4].

Theorem 1 [4]

(a) The robust optimal inventory policy is (s, S) policy.

(b) The robust optimal solution can be obtained by solving the nominal inventory problem with the modified demand

$$w_k' = \overline{w}_k + \frac{p-h}{p+h}(A_k - A_{k-1}),$$

where $A_{-1} \equiv 0$.

Hence the robust optimal solution can be computed by solving the nominal inventory problem which is a deterministic mixed integer programming problem.

3. Generalization of the Robust Inventory Problem and its Solution

We consider a proper generalization of the robust inventory problem (GRIP), where the costs are not stationary, that is, for each period *k*, the fixed ordering cost K_k, the unit ordering cost c_k, the unit inventory holding cost h_k, and the unit penalty cost

p_k are given. The following theorem shows that the generalized version (GRIP) can also be solved by solving a nominal deterministic inventory problem. The proof of the theorem closely resembles the corresponding proof in Bertsimas and Thiele [4].

Theorem 2: The problem (GRIP) can be solved by solving the nominal inventory problem with the modified demand

$$w_k^{'} = \overline{w}_k + \frac{p_k - h_k}{p_k + h_k} A_k - \frac{p_{k-1} - h_{k-1}}{p_{k-1} + h_{k-1}} A_{k-1},$$

where $A_{-1} = 0$.

Proof: By noting that the values of the variables v_k and y_k are functions of the variables u_k, the problem (GRIP) can be represented as follows:

$$\min_{u \geq 0} \sum_{k=0}^{T-1} \{c_k u_k + K_k 1_{\{u_k > 0\}} + \max(h_k(\overline{x}_{k+1} + A_k), p_k(-\overline{x}_{k+1} + A_k))\}, \tag{8}$$

where $\overline{x}_{k+1} = \overline{x}_k + u_k - \overline{w}_k = x_0 + \sum_{i=0}^{k} (u_i - \overline{w}_i)$, for all k.

Let us define a modified stock variable $x_k^{'}$ which evolves according to the following equation:

$$x_{k+1}^{'} = x_k^{'} + u_k - w_k^{'},$$

where $x_0^{'} = x_0$. Then we have

$$\max(h_k(\overline{x}_{k+1} + A_k), p_k(-\overline{x}_{k+1} + A_k)) = \max(h_k x_{k+1}^{'}, -p_k x_{k+1}^{'}) + \frac{2 p_k h_k}{p_k + h_k} A_k \tag{9}$$

since the following relation holds:

$$\begin{aligned} x_{k+1}^{'} &= x_k^{'} + u_k - w_k^{'} = x_0 + \sum_{i=0}^{k} (u_i - w_i^{'}) \\ &= x_0 + \sum_{i=0}^{k} (u_i - \overline{w}_i) - \sum_{i=0}^{k} \left(\frac{p_i - h_i}{p_i + h_i} A_i - \frac{p_{i-1} - h_{i-1}}{p_{i-1} + h_{i-1}} A_{i-1}\right) = \overline{x}_{k+1} - \frac{p_k - h_k}{p_k + h_k} A_k. \quad \blacksquare \end{aligned}$$

Note that the modified demand $w_k^{'}$ is independent of the uncertainty. Hence the problem (GRIP) can be solved by solving the nominal problem with the above-defined modified demand (plus fixed cost of $\sum_{k=0}^{T-1}(2p_k h_k/(p_k+h_k))A_k)$).

Next we consider the computational complexity of the nominal problem of (GRIP). First we need the following lemma.

Lemma 1: Let u^* be the optimal solution to the nominal problem of (GRIP). Then if at least one order is made, there exists r ($0 \le r \le T-1$) such that

$$x_0 + \sum_{k=0}^{T-1} u_k^* = \sum_{k=0}^{r} \overline{w}_k .$$

Proof: Assume that at least one order is made in the optimal solution and let F be the last time period when an order is made. Let us define P as the greatest integer such that $x_{F+P} > 0$ (if no such P exists, then it is defined as 0). Then the cost from the period F to T -1 (ignoring the fixed ordering cost K_F) can be written as

$$c_F u_F + \sum_{k=0}^{P-1} h_{F+k}(x_F + u_F - \sum_{i=0}^{k} \overline{w}_{F+i}) + \sum_{k=P}^{T-F-1} p_{F+k}(-x_F - u_F + \sum_{i=0}^{k} \overline{w}_{F+i}),$$

which is linear in u_F with slope $c_F + \sum_{k=0}^{P-1} h_{F+k} - \sum_{k=P}^{T-F-1} p_{F+k}$ subject to the constraint $\sum_{i=0}^{P-1} \overline{w}_{F+i} < u_F + x_F \le \sum_{i=0}^{P} \overline{w}_{F+i}$ from the definition of F.

For a given u_F to be optimal, we should have $c_F + \sum_{k=0}^{P-1} h_{F+k} - \sum_{k=P}^{T-F-1} p_{F+k} \le 0$ since if otherwise, we can decrease u_F to reduce the cost. Hence for the optimal solution, we have

$$u_F + x_F = x_0 + \sum_{k=0}^{F-1}(u_k - \overline{w}_k) + u_F = \sum_{i=0}^{P} \overline{w}_{F+i} .$$

Hence $x_0 + \sum_{k=0}^{T-1} u_k = \sum_{k=0}^{F+P} \overline{w}_k$.

From the above lemma, we can conclude that the optimal solution satisfies one of the following equations:

$$\sum_{k=0}^{T-1} u_k^* = 0 \quad \text{or} \quad x_0 + \sum_{k=0}^{T-1} u_k^* = \sum_{k=0}^{r} \overline{w}_k \,, \quad r = 0, 1, \cdots, T-1 \,.$$

Hence we can decompose the problem into T+1 subproblems each of which with one of the above equations added. Then each subproblem becomes an economic lot sizing problem with backlogging allowed with the demands of the first r periods and all the others set to 0 (with an additional fixed cost representing the penalty cost for the remaining periods). The n-period economic lot sizing problem with backlogging can be solved in $O(n \log n)$ time, see Aggarwal and Park [1]. So the nominal inventory problem can be solved in $O(T^2 \log T)$ time. Note that if the costs are stationary (the model considered in Bertsimas and Thiele [4]), the computational complexity reduces to $O(T^2)$. The results combined with Theorem 2 are summarized in the following theorem.

Theorem 3: The nominal inventory problem as well as the generalized robust inventory problem can be solved in $O(T^2 \log T)$ time (assuming A_k can be computed very easily).

Finally, we want to mention that the above results hold for all the cases when the uncertainty set is symmetric in the sense that $A_k = \max\{\sum_{i=0}^{k} \hat{w}_i z_i \mid z \in U\} = -\min \{\sum_{i=0}^{k} \hat{w}_i z_i \mid z \in U\}$. Such cases include the ellipsoidal uncertainty set with center at the origin. Note that the ellipsoidal uncertainty set arises when we consider the chance constraints where the demand parameters follow multivariate normal distribution [2].

4. Concluding Remarks

This paper considers the generalized robust inventory model where the cost parameters are time-varying. We show that the results presented in [4] also hold in this case. Moreover, we prove that the generalized robust inventory problem can be solved very efficiently. As we mentioned at the end of Section 3, the results remain valid for other uncertainty sets. Hence the results can be used in more general settings in practice.

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