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Evaluation of Quantity Discounts for Buyer's Stocking Risk

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ABSTRACT

Quantity discounts provide a practical foundation for supply chain inventory policies, improving the supplier's profit and reducing the buyer's inventory cost simultaneously. Traditional quantity-discount research, which deals with inventory coordination between a buyer and a supplier, is extended to a stationary stochastic environment. This research shows that the magnitude of the optimal discounts scheduled by the deterministic quantity discount models may not be large enough to cover the buyer's additional inventory stocking risks under uncertain conditions. As a result, the buyer's total inventory cost may often increase rather than decrease. In contrast, the proposed model allows the supplier to identify the discount level, which shares the buyer's amplified risk associated with temporary overstocking and ensures that both buyer and supplier benefit economically. The performance of the proposed model was tested in the continuous review environments via numerical experiments. The experimental results support the proposed method as a feasible alternative in coordinating inventory decisions under stochastic demand.

Keywords: Quantity Discounts, Overstocking, Supply Chain Management

1. Introduction

For decades, purchasing quantity discounts have been used in numerous industries

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as an option to coordinate inventory decisions between a supplier and a buyer. The popularity of quantity discounts stems from the fact that the supplier's discount offer as an economic incentive can influence the customers' purchasing behavior. The literature acknowledges that quantity discounts are a practical coordination mechanism for a supply chain in which independent companies cooperate with each other to improve not only the overall performance of the supply chain but also their own profitability [5, 7, 15, 26].

Since Monahan [17], the application of quantity discounts to inventory coordination has resulted in considerable research extensions [2, 4, 6, 10, 11, 13, 14, 20-24]. Recently, the literature has expanded to scheduling quantity discounts for a three-level supply chain [18], coordinating orders of multiple buyers at a time of the supplier's discount offer [12], combining quantity and volume discounts under price-sensitive deterministic demand [25], and deliberating on quantity discount with transportation costs [16]. For a more comprehensive review of the literature, readers may refer to Benton and Park [1], Rubin and Benton [21], Yano and Gilbert [26], and Chan *et al*. [3].

In general, the core philosophy of "inventory coordination" in the supply chain is that the whole system works as if it is controlled by a single decision marker to maximize the whole system profit, and the improved profit should be shared by the supply chain participants. Thus, any inventory coordination mechanism must guarantee that every participant in the supply chain benefits, once the coordination mechanism is implemented. The deterministic quantity discount models, which primarily use quantity discounts as the inventory coordination mechanism, were also developed in order to create a win-win situation for both the supplier and buyer (s).

However, a simulation study by Shin and Benton [22] clearly shows that under stochastic demand conditions they often fail to secure the buying firm's cost savings for two reasons. First, the modeling objective of the deterministic quantity discount models is biased in favor of the supplier. In fact, the optimal policy only maximizes the supplier's profit and offers the buyer (s) zero or marginal cost savings [21]. The maximization of the supplier's profit is achieved with a strong assumption that at all times the buying firm (s) accepts the discount schedule as long as its total cost break-evens under the quantity discount policy. Second, the deterministic quantity discount models never consider the buying firm's amplified risk of temporary overstocking. Note that the buyer's order quantity and cycle stock increase with the use of quantity discounts. The increased cycle stock poses a greater risk of holding larger inventories for a longer period as demand weakens. In other words, if actual demand turns out much lower than the expected demand, the buying firm's inventory carrying cost under the discount policy will escalate significantly due to the increased cycle stock.

Under the circumstances, a fundamental question arises as to why the buying firm should participate in the quantity discount policy given the risk of cost increase? In fact, the modeling convention in favor of the supplier neglects a simple fact that the buying firm as an independent economic entity is not obligated to accept the supplier's discount offer. Once the buying rejects the discount offer, the supplier's attempt for supply chain inventory coordination will not succeed. Therefore, it is necessary to devise a more practical quantity discount schedule in which the supplier must consider the buyer's temporary overstocking risk in order to guarantee the buyer's cost savings under stochastic demand.

In this research, we propose two methods of identifying feasible quantity discounts and corresponding buyer's and supplier's lot sizes. The first proposed model (PM_1 hereafter) is developed for a formal continuous review system, adopting the modeling convention in the literature which maximizes the supplier's profit and minimizes the supply chain's total inventory cost. Next, we propose an alternative model, which will systematically improve the buyer's and the supplier's inventory cost performance simultaneously in the presence of stationary stochastic demand (uncertain demand hereafter). The modeling direction of the second proposed model (PM_2 hereafter) conforms to the philosophy of supply chain management which emphasizes that every participating company should benefit from the governing coordination policy. It should be noted that an earlier version of PM_2 was used in our simulation paper [22]. However, the detailed mathematical representation of PM_2 is original and was never introduced in Shin and Benton [22]. PM_2 in the current form has been much improved from the earlier version by directly incorporating safety stock and shortage costs into the discount schedule.

2. Problem Description And The Extended Model

In typical quantity discount-based inventory coordination models, the primary mechanism to share the improved benefit is the amount of quantity discount itself. In an idealistic condition, the buyer benefits from discounts, and the supplier benefits from the reduced inventory since the buyer has to increase its order quantity in exchange with quantity discounts. Thus, how much discount is offered from the supplier to the buyer is the key decision which eventually determines the buyer's order quantity and the supplier's lot size.

In this context, a typical quantity discount-based inventory coordination (QDIC hereafter) model requires three decision variables. These are 1) the discount amount (d_K), 2) the factor (K) to determine the buyer's increased order quantity (KQ) given the discount, and 3) the supplier's lot size factor (k) as an integer multiple for the buyer's order quantity (KQ). The assumptions of the proposed model are as follows: 1) all-units discounts, 2) non time-phased, price-inelastic, and normally distributed independent demand, and 3) the buyer's inventory cost structure is known to the supplier. In addition, it is further assumed that consumers' demand for the buyer's item is the only source of uncertainty.

The following notation is used throughout the research.

D Expected annual demand

Q Buyer's current order quantity

m_Q Demand per buyer's expected replenishment cycle of Q/D, a random variable

σ_Q Standard deviation of demand per buyer's expected replenishment cycle

ξ Actual demand during the lead time with distribution of $f(\xi)$.

C Supplier's unit cost of producing or acquiring the product

d_K Discount amount per unit

H_1 Buyer's inventory holding cost rate per year expressed as a percentage of P

H_2' Supplier's inventory holding cost rate per year expressed as a percentage of C

H_2 Supplier's inventory holding cost rate per year expressed as a percentage of P

I^O Buyer's overstock per expected replenishment cycle

K A factor to determine the buyer's increased order quantity

k An integer factor to determine the supplier's order quantity as a multiple of the buyer's order quantity

P Unit price paid by the buyer before quantity discounts

p_S Buyer's shortage penalty cost per unit

R Buyer's reorder point

S_1 Buyer's fixed ordering cost per order

S_2 Supplier's fixed set-up cost per order

$ss_{(Q)}$ The optimal safety stock under the no-discount policy

$ss_{(KQ)}$ The near optimal safety stock under the discount policy

2.1 Proposed Model 1 (PM_1): Near Optimal Discounts under Continuous Review System

In this section, we propose an all-units discount model to show how the conventional QDIC modeling approach [14, 17, 20, 23] can be modified within the structure of a two-stage continuous review (Q, R) system (see [19], 475-479). PM_1 follows the conventional modeling objective in favor of the supplier. That is, the supplier is to schedule a single price break (discount) which "ignores the need to provide an incentive to the buyer to switch ([21], 182)" to the QDIC policy and will make the buyer breakeven "but not gain from an optimal discount policy."

The first step to QDIC is to define the buyer's two different inventory cost functions: one under no-discount policy and the other under the proposed QDIC policy. The buyer's total inventory cost function under the continuous review system (with backlogging allowed) is commonly formulated as Eq. (1) (see [8], 205-209; [9], 318-325).

$$TC_E(Q, R_{(Q)}) = PD + \frac{DS_1}{Q} + PH_1\left[\frac{Q}{2} + ss_{(Q)}\right] + \frac{D}{Q}\mathrm{E}[Cs_{(Q)}] \tag{1}$$

where $\mathrm{E}[Cs_{(Q)}]$ *is the expected shortage cost per replenishment cycle under the no-dis-count policy, which is given by* $p_S\left[\int_{R^*}^{\infty}(\xi - R_{(Q)})f(\xi)d\xi\right]$.

The partial derivatives of Eq. (1) with respect to Q and R lead to the optimal solutions of order quantity and reorder point, which are given in (2) and (3). Solving Eqs. (2) and (3) directly to find the closed-form solutions is impossible, yet these solutions can be identified by an iterative search process [9].

$$Q = \left[\{2D(S_1 + \mathrm{E}[Cs_{(Q)}])\}/(PH_1)\right]^{(1/2)} \tag{2}$$

$$\Pr\{D \geq R_{(Q)}\} = (PH_1 Q)/(p_S D) \tag{3}$$

As Zheng [27] uses the term Economic Order Quantity (EOQ) for the optimal order quantity for the continuous review system, we use the term EOQ to represent the buyer's optimal order quantity defined in (2). When a QDIC policy is adopted, the buyer's expected total cost function should be modified to Eq. (4).

$$TC_E(KQ, R_{(KQ)}) = (P-d_K)D + \frac{DS_1}{KQ} + (P-d_K)H_1\left[\frac{KQ}{2} + ss_{(KQ)}\right] + \frac{D}{KQ}\mathrm{E}[Cs_{(KQ)}] \tag{4}$$

where $\mathrm{E}[Cs_{(KQ)}]$ *is the expected shortage cost per replenishment cycle under the no-discount policy, which is given by* $p_S\left[\int_{R^*}^{\infty}(\xi - R_{(KQ)})f(\xi)d(\xi)\right]$.

In general, the optimal service level under the quantity discount policy ($ss_{(KQ)}$) would be smaller than that of a no discount policy ($ss_{(Q)}$) because of the inverse relationship between the optimal order quantity and the optimal safety stock [27]. Due to this difference in the safety stock levels between the two policies (no discount vs. QDIC), the corresponding optimal service levels and expected shortage costs per replenishment cycle will differ as well. The reduction in safety stock is another benefit of quantity discount policies.

Note that in many cases the safety stock should be purchased at the regular price since it's not a part of regular order quantity; that is, $(P-d_K)H_1 * ss \Rightarrow PH_1 * ss$. This modification reduces computational complexity considerably and eliminates the need for the supplier to adjust its discount schedule based on the buyer's safety stock amount. With this modification imposed, Eq. (4) must be rewritten as (5).

$$TC_E(KQ, R_{(KQ)}) = (P-d_K)D + \frac{DS_1}{KQ} + \frac{KQ^*(P-d_K)H_1}{2} + PH_1 ss_{(KQ)} + \frac{D}{KQ}\mathrm{E}[Cs_{(KQ)}] \tag{5}$$

When scheduling quantity discounts for supply chain inventory coordination, the supplier's objective is to identify a minimum level of discounts which will maximize the supplier's own profit and at the same time compensate for the buyer's increased inventory cost caused by a larger order quantity under the quantity discount policy. To achieve this goal (i.e., to make the discount schedule acceptable to the buyer) the supplier has to ensure that the buyer's total cost under the quantity discount policy should be less than equal to the buyer's total cost under no-discount policy, as shown by the inequality condition given in (6). In other words, the buyer must be at least breakeven even to consider accepting the supplier's discount offer.

$$TC_E(Q, R_{(Q)}) - TC_E(KQ, R_{(KQ)}) = 0 \tag{6}$$

By substituting (1) and (5) into (6), we obtain the following equation given in (7).

$$\begin{aligned} & d_K D + \frac{KQH_1 d_K}{2} + \frac{PH_1(1-K)Q}{2} \\ & + \frac{DS_1(K-1)}{KQ} + PH_1(ss_{(Q)} - ss_{(KQ)}) + \frac{D(\mathrm{E}[Cs_{(Q)}]K - \mathrm{E}[Cs_{(KQ)}])}{KQ} = 0 \end{aligned} \tag{7}$$

Solving Eq. (7) with respect to d_K, the minimum discount which will maximize the supplier's profit can be determined as shown in (8).

$$\begin{aligned} & d_K(KQ) = \\ & \frac{2D(1-K)S_1 + 2D(\mathrm{E}[Cs_{(KQ)}] - \mathrm{E}[Cs_{(Q)}]K)}{KQ(2D + KQH_1)} + \frac{PH_1(K-1)Q - 2PH_1(ss_{(Q)} - ss_{(KQ)})}{(2D + KQH_1)} \\ & \textit{for } K \geq 1. \end{aligned} \tag{8}$$

Evaluating the value of (8) is technically complex because of the dynamics among discount amount (d_K), order quantity (KQ) and cycle service level. For instance, an increase in order quantity ($K \uparrow$) will require an increase the discount amount ($d_K \uparrow$) and initially reduces the optimal service level and safety stock. The reduced safety stock increases the expected shortage cost ($\mathrm{E}[Cs_{(KQ)}] \uparrow$), which will in turn require the buyer's order quantity to decrease ($K \downarrow$). Due to this reciprocity of influence between order quantity and safety stock, it usually takes a series of iterations to determine the optimal combinations of solutions, as shown in Hillier and Lieberman (see [9], 323). However, we found that this iterative approach makes the buyer's increased order quantity (KQ) converge around the original EOQ, neglecting the benefits of quantity discounts. Therefore, rather than using the iterative approach directly, we have adopted a sequential process in which 1) the buyer's service level is determined as the best response function to its order quantity (KQ), 2) the estimated service level determines the buyer's safety stock and expected shortage cost, and 3) the safety stock, shortage cost, and order quantity determine the magnitude of discount given in (8).

2.2 Proposed Model 2 (PM_2): Extended Model for Supply Chain Inventory Coordination

The discount schedule proposed in PM_1 extracts the buyer's surplus to a maximum degree since we use the conventional approach which maximizes the supplier's profit and breakevens the buyer's cost at best [21]. Shin and Benton [22] show that this approach is inappropriate for supply chain coordination because it fails to secure the buyer's cost savings in practical conditions where demand fluctuates. If the PM_1 is adopted as a coordination mechanism in practice, it is likely that the buyer's inventory cost may increase (rather than decreases) on occasion, particularly in the short term. Thus, the quantity discount amount given in Eq. (8) should be improved systematically. In the following section, we show how to incorporate the buyer's expected overstocking cost in order to make the discount schedule more appealing to the buyer.

2.2.1 Buyer's expected (temporary) overstocking cost

Let demand (m_Q) per buyer's expected replenishment cycle (Q/D) be a normally distributed independent random variable. If the buying firm increases its order quantity to KQ under a QDIC policy, the buyer's expected inventory cycle extends to [KQ/D]. The expected demand during the cycle of [KQ/D] becomes KQ because $K(\bar{m}_Q) =$ KQ. The notation "m_{kQ}" will be used to denote the real demand during the extended cycle of [KQ/D]. Then, the buyer's overstock per expected replenishment cycle is given by Eq. (9).

$$I^O = \begin{cases} (KQ - m_{KQ}) & if \quad m_{KQ} < KQ \\ 0 & if \quad m_{KQ} \geq KQ \end{cases} \tag{9}$$

The buyer's overstock given in (9) represents the amount of unsold items (inventory remnants) due to the weak demand during the buyer's expected replenishment cycle. I^O is a linear combination of KQ and m_{kQ} with a possible range of [0, KQ]. If the buyer's overstock were estimated in the range of (– ∞, ∞), the distribution of I^O would have a mean of zero and a standard deviation of $\sigma_{KQ} \approx \sigma_Q \sqrt{K}$. The probability function $f(I^O)$ would follow a half of the normal distribution with a possible range of [0, KQ].

Using Eq. (9), the estimation of the buyer's expected overstock per replenishment cycle is straightforward as shown in Eq. (10). Eq. (10) can be standardized with a unit normal variable (u), and the result is Eq. (11).

$$\mathrm{E}[I^O] = \int_{-\infty}^{KQ} (KQ - m_{KQ}) f(m_{KQ}) d(m_{KQ})\,. \tag{10}$$

$$\mathrm{E}[I^O] = \sigma_{KQ} \int_{-\infty}^{0} \left[-(u) f(u)\right] d(u)\,, \tag{11}$$

$$where \quad u = \frac{m_{KQ} - KQ}{\sigma_{KQ}}, \quad u \sim N(0,\ 1)\,.$$

An integral function is defined as Eq. (12) to represent the expectation of (u). Using (12), Eq. (11) can be simplified into Eq. (13).

$$G(u) = \int_{-\infty}^{0} \left[-(u) f(u)\right] d(u) \tag{12}$$

$$\mathrm{E}[I^O] = \sigma_{KQ} G(u) \tag{13}$$

Thus, the expected overstocking cost ($C_E[I^O]$) per expected replenishment cycle (KQ/D) can be formulated as Eq. (14).

$$C_E[I^O] = \sigma_{KQ} G(u)(P - d_K) H_1 (KQ / D) \tag{14}$$

The value of $G(u)$ depends on the supplier's willingness to cover a certain magnitude of demand variation. If the supplier intends to fully cover the buyer's risk with stochastic demand as shown in (11), i.e. $u \to \left(-\infty,\ 0\right]$, the value of $G(u)$ becomes about 0.4 at the maximum.

2.2.2 Buyer's Expected Total Cost and Estimation of Quantity discounts

In the prior section, we estimated the buyer's expected overstocking cost per replenishment cycle in order to accommodate the buyer's overstocking risk into the discount schedule. If the buyer's increased order quantity and $C_E[I^O]$ are considered, the buyer's cost function should be reorganized to Eq. (15).

$$\begin{aligned} TC_E(KQ, R_{(KQ)}) &= (P - d_K)D + \frac{DS_1}{KQ^*} + (P - d_K)H_1\left[\frac{KQ^*}{2}\right] \\ &+ PH_1 ss_{(KQ)} + \frac{D}{KQ^*}\mathrm{E}[Cs_{(KQ)}] + C_E[I^O](D / KQ) \end{aligned} \tag{15}$$

Unlike Eqs. (4) and (5) used in PM_1, Eq. (15) is the buyer's actual cost function. Rather, as discussed at the beginning of this section, Eq. (15) must be understood as a sum of buyer's cost components that the supplier must consider to devise a discount schedule appealing to the buyer. Specifically, the current modeling approach using (15) is illustrated in Figure 1. As shown in Figure 1, the triangular area **A** represents the buyer's expected cycle inventory under the no-discount policy. If a conventional QDIC policy is adopted, the buyer's order quantity escalates up to *KQ*, increasing the amount of cycle inventory (area **B**). The increased portion of cycle inventory carrying cost for area **B** is covered by the supplier's proposed discounts. Under PM_2, we acknowledge that the buyer has a risk of holding temporary overstock ($\mathrm{E}[I^O]$) possibly due to the realized demand less than the expected demand. The use of (15) forces the supplier to provide additional coverage for a possible increase in buyer's inventory carrying cost (area **C**). The triangular area **C** is measured as [$\sigma_{kQ}\, G(u)(KQ/D)$], and the discounted inventory holding cost per unit is [$(P\text{-}d_K)H_1$]. Consequently, additional inventory cost from the area **C** becomes [$\sigma_{kQ}\, G(u)(KQ/D)(P\text{-}d_K)H_1$], which is $C_E[I^O]$ given in (14).

Eq. (16) is a necessary condition that must be met to at least secure the buyer's cost savings under the discount policy.

$$TC_E(Q, R_{(Q)}) - TC_E(KQ, R_{(KQ)}) = 0 \tag{16}$$

By substituting (1) and (15) into (16), we obtain the following equation given in (17).

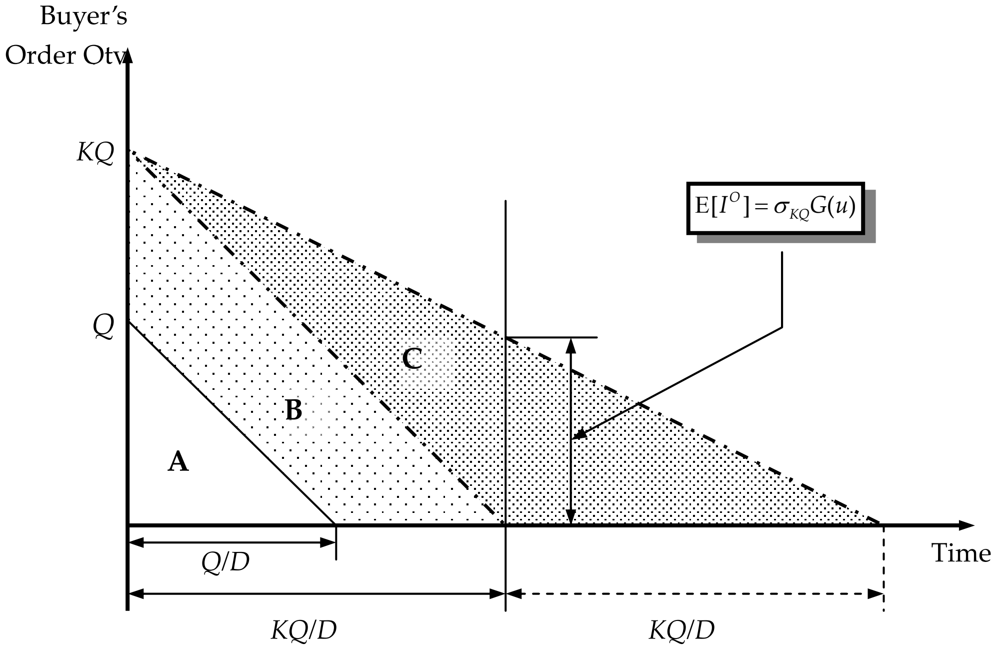

Figure 1. Schematic Illustration of All−Units Discounts' Coverage of Buyer's Inventory Cost

$$d_K D + \frac{KQH_1 d_K}{2} + \frac{PH_1(1-K)Q}{2} + \frac{DS_1(K-1)}{KQ} + PH_1(ss_{(Q)} - ss_{(KQ)}) + \frac{D(\mathrm{E}[Cs_{(Q)}]K - \mathrm{E}[Cs_{(KQ)}])}{KQ} - \sigma_{KQ}G(u)(P-d_K)H_1 = 0 \quad (17)$$

Eq. (17) has a structure similar to that of (7). The only difference is that the expected overstocking cost per year, $[\sigma_{KQ}G(u)(P-d_K)H_1]$, is added to reduce the possible occasions where the buyer's temporary overstocking cost exceed its cost savings from the reduced purchase cost. Solving Eq. (17) with respect to d_K, the revised discount schedule is determined as shown in (18).

$$d_K(KQ) = \frac{2D(1-K)S_1 + 2D(\mathrm{E}[Cs_{(KQ)}] - \mathrm{E}[Cs_{(Q)}]K)}{KQ[2D + KQH_1 + 2H_1\sigma_{KQ}G(u)]} + \frac{PH_1(K-1)Q - 2PH_1(ss_{(Q)} - ss_{(KQ)})}{[2D + KQH_1 + 2H_1\sigma_{KQ}G(u)]} \quad \text{for } K > 1. \quad (18)$$

The discount schedule given in (18) is an increasing function of K when $K > 1$.

The proposed discounts compensate for the buyer's additional inventory costs caused by both increased lot size and possible overstocking. Given that the value of $G(u)$ is positive, the value of d_K from (18) is always positive even if $K = 1$ (meaning the buyer adheres to EOQ without increasing his order quantity). Thus, when $K = 1$, d_K is set to zero since the supplier is not obligated to offer quantity discounts. The increased discount in (18) functions as a buffer against stochastic demand and provides the buyer with additional cost savings if demand intensifies.

A more precise way to schedule the minimum discount is to compare (15) with a buyer's total cost function under the EOQ policy with the expected overstocking cost incorporated. However, we choose to compare (1) and (15) because our objective is to lower the probability of increase in the buyer's inventory cost (caused by temporary overstocking) when PM_2 is adopted.

2.2.3 The Supplier's Expected Annual Profit Function

For both proposed models, the objective function is the supplier's profit function. In the supplier's profit function, the supplier's inventory level is a critical element because the supplier's inventory level is directly influenced by the buyer's ordering behavior. As the buyer orders a fixed quantity under constant demand, the supplier's inventory follows a stepwise reduction and the average inventory level is estimated as $[(k\text{-}1)KQ]/2$ [20]. If the buyer orders a fixed quantity under stochastic demand, the buyer's time-between-orders (TBO) varies, creating a lead-time uncertainty to the supplier. However, the supplier's expected inventory level remains the same as $[(k - 1)KQ]/2$ as described in Lemma 1. In fact, this is one of the special cases where the buyer's varying time-between-orders has essentially no effect on formulation of the supplier's profit function [28].

Lemma 1: *The supplier's expected inventory level is* $[(k\text{-}1)KQ]/2$, *if the expected annual demand rate is D on a continuous time horizon*-the corresponding proof is available in the Appendix.

With the supplier's average inventory of $[(k\text{-}1)KQ]/2$ described in Lemma 1, the supplier's expected annual profit function is determined as Eq. (19).

$$AP_E = D(P-C) - Dd_K - \frac{DS_2}{kKQ} - \frac{(k-1)KQPH_2}{2} \tag{19}$$

$$\textit{where} \quad H_2 = (H_2^{'}C/P)$$

By substituting (8) or (18) into (19), we can finalize the supplier's expected profit function for each proposed model. We omit the representation of the supplier's profit function in the final form for the simplicity of paper.

3. The Solution Search Procedure

The solution process is to find the supplier's and the buyer's optimal lot size factors (k^* and K^*) that maximize the value of (19). Once the optimal combination (k^* and K^*) is identified, the discount amounts (d_K) for the proposed models can be computed, using (8) and (18). In the deterministic QDIC models, the buyer's and the supplier's lot size factors are often found as closed-form solutions. However, as Eq. (8) or (18) is incorporated into the supplier's profit function, the solution in a closed-form becomes far more complex. Therefore, we search the solutions by identifying the upper bounds for both k and K. The lower bounds for k and K are to equal one, indicating the buyer maintains its economic lot size (EOQ) and the supplier uses a lot-for-lot policy. The following Lemmas 2 and 3 describe the procedure of obtaining the upper bounds for k and K.

Lemma 2: *The upper bound for the supplier's lot size factor* (k^*) *is* $[1/Q][(2DS_2)/(PH_2)]^{(1/2)}$, *given* $K \geq 1$-the corresponding proof is available in the Appendix.

Lemma 3: *The upper bound for the buyer's optimal order quantity factor* (K^*) *is found to be* $[1/Q][(2DS_2)/(k(k-1)PH_2)]^{(1/2)}$ *if* $k > 1$ – the corresponding proof is available in the Appendix.

If k is one, the buyer's optimal order quantity factor (K^*) can be searched by imposing a constraint that the supplier's profit estimated from Eq. (19) should be positive. Using the two upper bounds (k^+ *and* K^+), the optimal combination of (k^* and K^*) is obtained by the following search algorithm.

[a] Start with $k(j) = 1$.

[b] Search the range (a vector) of $K_k(i)$, starting from $K(i) = 1$ to $K(i) = K^+(i)$ with an increment of 0.01.

[c] Given $K_k(i)s$, estimate buyer's corresponding order quantities ($K_k(i)Q$), op-

timal safety stocks ($ss_{(KQ)}$), and expected shortage costs ($E[Cs_{(KQ)}]$) to determine the discount per unit ($d_{K(i)}$).

[d] Identify the supplier's corresponding profits, $AP_E(K_k(i))$, and $k(i) = k(i) + 1$.

[e] Repeat steps [b] through [d] while $k(i) \le k^+$.

[f] Find the maximum $AP_E(i)$ and the corresponding [$K_k^*(i)$, $k^*(i)$].

[g] Stop.

The proposed algorithm is a search process that supplier's lot size factors (*k*) as the bases to find the local optimal solution from the objective function (19). In Steps [a] through [d], the algorithm first sets *k* as one and searches $K_k(i)$ that maximizes $AP_E(K_k(i))$, given *k* of one. The combination of $K^*(i)$ and *k* of one is the local optimum for $AP_E(k=1)$. The search for the local optimal solutions continues until *k* reaches its upper bound. In Step [f], all the local optimal solutions will be compared to find the global optimum solution. The algorithm stops at Step [e] as the global optimum is identified.

4. Computational Experiments

The primary objective of the computational experiments is to understand whe-ther or not implementing the QDIC policies **for a limited period** can actually improve each party's performance under practical conditions in which stochastic demand presents. For simplicity, the following abbreviation is adopted for the various policies examined in this research. "EOQ" represents a no discount policy. In this case, the buyer adheres to the original economic lot size given in Eq. (2), and the supplier's lot size is found when the objective function (19) is maximized with no discount. The EOQ policy is the base case which the performance of other QDIC policies are compared with. The abbreviation "W&W" represents the deterministic optimal model developed by Weng and Wong [23].[1] Finally, PM_1 and PM_2 are used consistently to represent the

[1] Previous researchers (e.g. see [17, 18, 21, 23]) have recognized a potential shortcoming of QDIC policies that the supply chains global optimal is found when the buyer gains nothing (i.e. the buyer is at its breakeven switching from the no-discount EOQ policy to a QDIC policy). Recognizing this problem, Weng and Wong [23] introduces a kind of shared saving pa-

two proposed models.

For the experiments, three example problems are adopted from Shin and Benton [22] as shown in Table 1, and their solutions are summarized in Table 2. The environmental characteristics of these example problems are well defined in Shin and Benton [22]. As shown in Table 2, order quantities under W&W are almost identical regardless of the magnitude of demand variation since its quantity discount schedules are not influenced by the demand variation. In contrast, the quantity discounts scheduled by PM_2 tend to increase as the level of buyer's risk with temporary overstocking risk (represented by Cv) intensifies.

Table 1. Input Parameter Setting for the Example Problems

Controlled Parameters			
Unit price (P)	100		
Unit cost (C)	70		
Number of periods per year (n)	50		
Supplier's holding cost rate (H_2')	0.250		
Supplier's holding cost rate (H_2)	0.175		
Buyer's order cost (S_1)	1,000		
Lead time	1		
Shortage Penalty Cost	P-C = 30		
*Standard Deviation of Demand	*0.1, 0.2, or 0.3 of the mean demand ($\bar{d}$)		
Experimental Variable			
Example No.	01	02	03
Average demand per period ($\bar{d}$)	40	512	96
Expected annual demand (D)	2,000	25,600	4,800
Buyer's holding cost rate (H_1)	0.16	0.20	0.26
Supplier's setup cost (S_2)	10,000	15,000	20,000
Deterministic EOQ	500	1,600	608

Note) *The ratio of standard deviation to its mean (0.1, 0.2, or 0.3) is the Standard Coefficient for Variation (Cv).

rameter (α) to one of the constraints in scheduling the discounts (Eq. (9) in [23]) to secure some economic benefits for the buyer. However, a positive value of α increase the discount amount, lowers supplier's profit, and eventually increases the supply chain system's inventory cost. In other words, W&W is not true optimal if the value of α is greater than zero. Therefore, we set the value of α to zero in our experimentation.

Table 2. Summary of the Solutions

Ex. No.	Policies	EOQ			W&W			PM_1			PM_2		
	Coefficient Variation	0.1	0.2	0.3	0.1	0.2	0.3	0.1	0.2	0.3	0.1	0.2	0.3
01	***Buyer's* K**	1.00	1.00	1.00	*3.46	*3.45	*3.43	3.31	3.28	3.29	3.30	3.30	3.28
	Buyer's order quantity	502	504	506	1,739	1,739	1,738	1,662	1,653	1,665	1,657	1,663	1,660
	Supplier's *k*	3.00	3.00	3.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
	Supplier's lot size	1,506	1,512	1,518	1,739	1,739	1,738	1,662	1,653	1,665	1,657	1,663	1,660
	Discount per unit ($)	0.00	0.00	0.00	3.30	3.30	3.30	3.08	3.07	3.09	3.08	3.14	3.19
02	***Buyer's* K**	1.00	1.00	1.00	*4.02	*3.97	*3.92	2.09	2.07	2.04	2.09	2.07	2.04
	Buyer's order quantity	1,621	1,642	1,664	6,522	6,521	6,521	3,388	3,399	3,395	3,388	3,399	3,395
	Supplier's *k*	4.00	4.00	4.00	1.00	1.00	1.00	2.00	2.00	2.00	2.00	2.00	2.00
	Supplier's lot size	6,484	6,568	6,656	6,522	6,521	6,521	6,776	6,798	6,790	6,776	6,798	6,790
	Discount per unit ($)	0.00	0.00	0.00	1.41	1.41	1.41	0.37	0.38	0.38	0.39	0.44	0.47
03	***Buyer's* K**	1.00	1.00	1.00	*1.81	*1.79	*1.78	1.78	1.80	1.79	1.78	1.80	1.76
	Buyer's order quantity	612	617	622	1,105	1,105	1,105	1,089	1,111	1,113	1,089	1,111	1,095
	Supplier's *k*	5.00	5.00	5.00	3.00	3.00	3.00	3.00	3.00	3.00	3.00	3.00	3.00
	Supplier's lot size	3,060	3,085	3,110	3,381	3,381	3,381	3,268	3,332	3,340	3,282	3,300	3,246
	Discount per unit ($)	0.00	0.00	0.00	0.58	0.58	0.58	0.56	0.60	0.61	0.57	0.63	0.63

Note) * Weng and Wong (1993) algorithm does not provide the buyer's K. Rather it gives direct solution for the buyer's order quantity (Q). Therefore, K is reverse-calculated from Q for comparison purposes.

As addressed in the modeling section, the buyer's optimal service level was estimated as the best response function of its order quantity, using the formula modified from Eq. (3). Based upon the cycle service levels, the buyer's optimal safety stocks, and reorder points are estimated as provided in Table 3. The optimal cycle service levels under the QDIC policies are lower than those in EOQ policy because the buyer's increased order quantity in exchange for quantity discounts reduces the number of order placements, limiting the number of possible stock-outs.

In the simulation, the lead time for the buyer is fixed to one unit period. The supplier's inventory is assumed to be replenished on time within the buyer's lead time (i.e, the supplier's lead time is less than or equal to the buyer's lead time) so that no safety stock or lead-time buffer is imposed at the supplier's procurement side. A total of twelve experiments (four models times three Cvs) for each example is conducted by discrete-event terminating-simulation. A total of 200 replications were run for each experimental setting. Each simulation was run for 2,500 periods, which represents a fifty-year period.

Table 3. Buyer's Safety Stock, Reorder Point, and Cycle Service Level in Each Policy

Ex.	Policies	**EOQ**			**W&W**			**PM1**			**PM2**		
No.	Coefficient Variation	0.1	0.2	0.3	0.1	0.2	0.3	0.1	0.2	0.3	0.1	0.2	0.3
01	Service Level	0.866	0.866	0.865	0.552	0.552	0.552	0.557	0.559	0.556	0.558	0.557	0.557
	Safety Stock	4	9	13	1	1	2	1	1	2	1	1	2
	Reorder Point	44	49	53	41	41	42	41	41	42	41	41	42
02	Service Level	0.958	0.958	0.957	0.833	0.833	0.833	0.912	0.911	0.912	0.912	0.911	0.912
	Safety Stock	88	176	263	49	99	148	69	138	207	69	138	207
	Reorder Point	600	688	775	561	611	660	581	650	719	581	650	719
03	Service Level	0.889	0.889	0.888	0.802	0.802	0.802	0.803	0.799	0.799	0.803	0.799	0.802
	Safety Stock	12	23	35	8	16	24	8	16	24	8	16	24
	Reorder Point	108	119	131	104	112	120	104	112	120	104	112	120

Four performance criteria were modified from Shin and Benton [22]-Cost Reduction Ratio (CRR) for the buyer, Profit Improvement Ratio (PIR) for the supplier, and Supply Chain Improvement Ratio (SIR) and Failure Rate (FR).[2] Among them the most important performance criterion is Failure Rate (FR). An FR represents a percentage that a QDIC policy fails to provide cost savings to the buyer or profit improvement to the supplier. Since the objective of QDIC modeling is to improve all the participants' performance within the notion of supply chain coordination, any policy that produces a high FR cannot be qualified as a practical inventory coordination mechanism. A high FR implies that the supplier's or the buyer's risk of facing cost increases (rather than savings) is high under a certain QDIC policy.

As illustrated by Eqs. (21) through (25), the buyer's total inventory cost (B_TC) and the supplier's total profit (S_TP) and inventory cost (S_TC) are the basic components of CRR, PIR, and SIR. CRR and PIR calibrate respectively the buyer's relative cost savings and the supplier's relative profit improvement with QDIC policies. For

[2] A fundamental difference of the proposed measures from those in Shin and Benton [22] is the incorporation of buyer's shortage cost component in case of a stock-out. When a stock-out occurs, the shortage cost were measured as a loss of profit (P-C), and the number of items in short was backordered.

example, a CRR measures what percentage of cost is saved under a QDIC policy compared with the EOQ policy. SIR measures if the supply chain's system inventory cost (B_TC plus S_TC) has been reduced under the QDIC policies.

$$CRR = \frac{B\_TC_{(EOQ)} - B\_TC_{(QDIC)}}{B\_TC_{(EOQ)}} \tag{20}$$

$$B\_TC(\cdot) = S_1 \sum O_1 - d_K KQ \sum_{i=1}^{N} O_1 + [H_1(P - d_K)/n] \sum_{i=1}^{N} [(I_{1(i)}^{B} - I_{1(i)}^{+})/2] + p_S \left| I_{1(i)}^{-} \right| \tag{21}$$

$$PIR = \frac{S\_TP_{(EOQ)} - S\_TP_{(QDIC)}}{S\_TP_{(EOQ)}} \tag{22}$$

$$S\_TP(\cdot) = \left[P - C - d_K\right] KQ \sum_{i=1}^{N} O_1 - S_2 \sum_{i=1}^{N} O_2 - [H_2^{'} C/n] \sum_{i=1}^{N} I_{2(i)}^{E} \tag{23}$$

$$SIR = \frac{[B\_TC_{(EOQ)} - B\_TC_{(QDIC)}] + [S\_TC_{(EOQ)} - S\_TC_{(QDIC)}]}{B\_TC_{(EOQ)} + S\_TC_{(EOQ)}} \tag{24}$$

$$S\_TC(\cdot) = S_2 \sum_{i=1}^{N} O_2 + [H_2^{'} C/n] \sum_{i=1}^{N} I_{2(i)}^{E} + d_K KQ \sum_{i=1}^{N} O_1 \tag{25}$$

where:

O_1: a binary variable to count the number of buyer's order placements.

O_2: a binary variable to count the number of supplier's order placements.

$I_{1(i)}^{B}$: the buyer's beginning inventory at period *i*.

$I_{1(i)}^{+}$: the buyer's positive ending inventory at period *i*.

$I_{1(i)}^{-}$: the buyer's negative ending inventory at period *i* in case of stock-out.

$I_{2(i)}^{E}$: the supplier's ending inventory at period *i*.

N: total number of periods in a simulation run.

n: number of periods in a year.

5. Results and Discussion

The experimental results are summarized in Table 4.[3] Although the experimental

[3] No QDIC policy failed in improving the supplier's and the supply chain system's performance, thus we omit the representation of the corresponding Failure Rates (FRs) in PIR and SIR.

method has been modified in this research, the simulation results in general are consistent with those in Shin and Benton [22]. Therefore, we focus on unique findings which demonstrate that our modeling objective is achieved.

Table 4. Performance Comparison among the QDIC policies

CRR: Buyer's Cost Reduction Ratio (%)										
Policies		**W&W**			**PM_1**			**PM_2**		
CV		0.1	0.2	0.3	0.1	0.2	0.3	0.1	0.2	0.3
[01]	Mean	0.16	0.22	0.36	0.57	0.66	1.07	0.71	2.03	2.72
	Min	–0.65	–1.06	–2.17	–0.55	–0.23	–0.48	–0.28	–0.19	–0.35
	FR*	**34.00**	**32.50**	**28.50**	**10.00**	**9.00**	**11.00**	**1.50**	**1.50**	**3.00**
[02]	Mean	0.00	0.00	0.00	0.15	0.16	0.15	0.28	0.39	0.46
	Min	–0.16	–0.23	–0.22	–0.06	–0.03	–0.03	0.02	0.02	0.07
	FR	**48.00**	**50.50**	**52.00**	**23.50**	**17.00**	**18.50**	**0.00**	**0.00**	**0.00**
[03]	Mean	0.00	0.00	0.00	0.02	0.04	0.03	0.04	0.06	0.07
	Min	–0.07	–0.08	–0.11	–0.06	–0.07	–0.08	0.02	–0.01	0.01
	FR	**47.50**	**46.50**	**42.00**	**17.00**	**13.00**	**11.00**	**0.00**	**0.05**	**0.00**
PIR: Supplier's Profit Improvement Ratio (%)										
[01]	Mean	12.21	12.22	12.22	12.20	12.12	11.94	12.10	12.06	11.79
[02]	Mean	1.50	1.49	1.51	1.31	1.30	1.28	1.27	1.22	1.12
[03]	Mean	0.03	0.03	0.03	0.03	0.03	0.03	0.02	0.02	0.02
SIR: Supply Chain System Improvement Ratio (%)										
[01]	Mean	13.17	13.18	13.21	12.74	12.59	12.22	12.57	12.50	12.23
[02]	Mean	2.71	2.66	2.69	2.76	2.73	2.69	2.64	2.63	2.61
[03]	Mean	0.02	0.02	0.02	0.03	0.04	0.03	0.02	0.02	0.03

Note) * FR represents Failure Rates in CRR.

1. The experiments confirm that the conventional QDIC modeling approach (W&W and PM_1) to maximize the supplier's profit is inappropriate for inventory coordination because of its failure in protecting the buyer's cost savings. As shown in Table 4, the minimum CRRs under W&W and PM_1 range from -2.17 to -0.07% and from -0.55 to -0.03% respectively. The negative CRRs imply that the buyer's relative inventory cost increases on occasion under W&W and PM_1 (compared with the EOQ policy). In particular, Weng and Wong [23]'s FRs in CRR increased up to 52.0% in Example [02]. The results imply that the buyer has more than a 50/50 chance to face a

cost increase after running Weng and Wong's QDIC policy for about 50 years. In contrast, PM_2 has shown zero FRs in a majority of cases, but marginally failed to secure the buyer's cost saving especially in Example [01]; its highest FR was 3.0%. In Example [01], the buyer's overstock risk is relatively low because of the small magnitude of overall demand. Thus, the benefits of buyer's risk adjustment approach proposed by PM_2 on CRR seem to be tenuous.

2. In Figure 2, we present three-year moving averages of CRRs for both W&W and PM_2. in order to highlight the buyer's risk of accepting quantity discounts. The graph in Figure 2 was obtained from one of the simulation runs, using the parameters values in Example [01] with a Cv of 0.2. Figure 2 clearly demonstrates that under W&W the buyer's inventory cost often increases (up to 8.00% compared with the EOQ policy) due to temporary overstocking problems and that this risk aggravates as the evaluation period shortens to three years. This happens because the modeling objective of W&W is to provide the buyer with least cost savings, extracting the buyer's surplus the most. In contrast, when PM_2 is used as inventory coordination mechanism, the buyer's inventory cost under PM_2 appears constantly lower than that of no discount policy, satisfying our modeling objective.

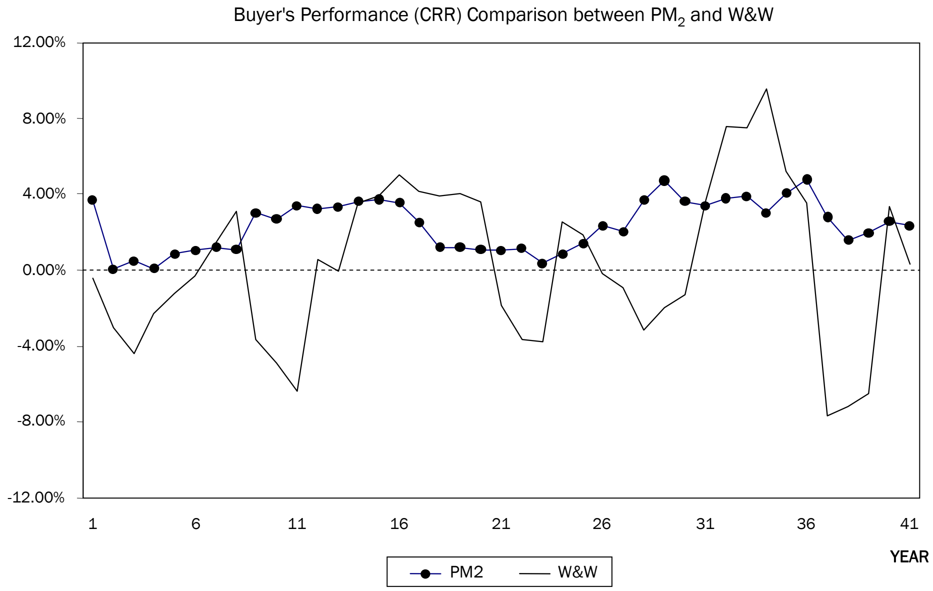

Figure 2. Buyer's Inventory Cost Behavior under the QDIC Policies

3. As indicated in Table 4, the comparison of supply chain's overall performance poses an interesting result. The PM_2 is the worst performing out of the three models compared. This was expected, given that the supply chain's global optimal is found when the buyer is at breakeven switching from the no discount policy to a quantity discount policy [21]. In terms of SIR, W&W is better than PM_1 in Example [1] whereas PM_1 produces better results in Examples [2] and [3]. One of the critical differences between Example [1] and Examples [2] and [3] can be found in Table 2. As indicated in the table, in Example [1] the average weekly demand is only 40, and the required optimal service level for the QDIC policies ranges from 0.552 to 0.559, leading to the safety stock amount of 1 or 2 units. Thus, the adjustments made by PM_1 based on the buyer's safety stock and expected shortage cost did not appear to play a critical role. In Examples [2] and [3], however, the relatively greater amount of safety stocks and shortage costs seem to create conditions in favor of PM_1

In summary, PM_2 secures cost savings for the buyer and profit improvement to the supplier in most cases because it schedules quantity discounts large enough to compensate for the buyer's overstocking risk with uncertain demand. Thus, PM_2 may be the most appropriate for supply chain inventory coordination despite the fact that it was worst in achieving the system's optimal. If quantity discounts are to be adopted for inventory coordination under stochastic demand, the supplier must share the buyer's amplified risk with temporary overstocking. When the supplier's objective is to maximize its own profit, PM_1 might be better than W&W particularly when the size of demand is large, requiring a larger amount of safety stock.

6. Concluding Remarks

In this paper, we extended the traditional QDIC research and showed how the deterministic optimal discounts should be improved to cope with the buyer's increased overstocking risk under stochastic demand. The critical distinction between the traditional QDIC models and the second proposed model is that in the proposed model the supplier's discount offer becomes flexible, depending on the magnitute of demand variablity. In other words, the supplier is forced to share the buyer's expected overstocking costs by offering larger quantity discounts in order to protect the buyer's cost savings. Given that the discount amount is subject to the magnitute of demand variablity in the proposed modeling, availability of information on stochastic demand pattern is essential for the supplier to use the proposed approach. Indeed, the proposed modeling approach initiates an operational way of coordinating inventory decisions, capitalizing on information availability

Managerial implications from the experimental results can be summarized as follows. First, throughout the modeling and experiments, we were able to show that suppliers can identify an appropriate discount schedule even under stochastic demand to improve both parties' inventory performance. If suppliers can offer the right schedules of quantity discounts to buyers, fierce price negotiations may be avoided. Second, not every quantity discount policy should be treated as a panacea for supply chain inventory coordination. The conventional QDIC models may not work effectively as an inventory coordination mechanism, particularly when the buyer seeks for cost savings in the short term.

In order to understand the impact of demand uncertainty on the effectiveness QDIC policies in detail, it may be necessary to evaluate different magnitudes of buyer's shortage cost, setup cost, and holding cost in conjunction with demand uncertainty [27]. However, in the proposed modeling and experimentation, we only focused the changes in buyer's order quantity under various QDIC policies. In other words, the numerical study was conducted with an assumption that the supplier's and the buyer's inventory cost parameters do not change under the quantity discount policies. If one can examine the dynamics of these parameter changes in the future, it may provide meaningful insights to researchers and managers.

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APPENDIX

A. Proof of Lemma 1.

Consider Figure 3, which is prepared to illustrate the core idea behind this proof. As illustrated in Figure 3, if the buyer's orders arrive at T_1, T_2, ⋯, and T_i, on average, the supplier's average inventory is determined as $[(k-1)KQ]/2$. Thus, the proof must show that each T_i is the expected moment for the buyer's order arrival within the supplier's replenishment cycle. In fact, T_i is a random variable, and the distribution of T_i can be determined if and only if demand is defined as a stochastic process. However, it is possible to obtain the first moment of T_i without defining the distribution of T_i. In Figure 3, $[(KQ\text{-}\,m_{KQ})/D]$ represents the absolute deviation from T_i with respect to the buyer's varying TBO's. If the realized demand (m_{KQ}) during the expected inventory cycle is smaller (or larger) than the order quantity (KQ), the expected order arrival moment, $E[M_{(T_i)}]$, will be extended to $E[T_i + (KQ - m_{KQ})/D]$.

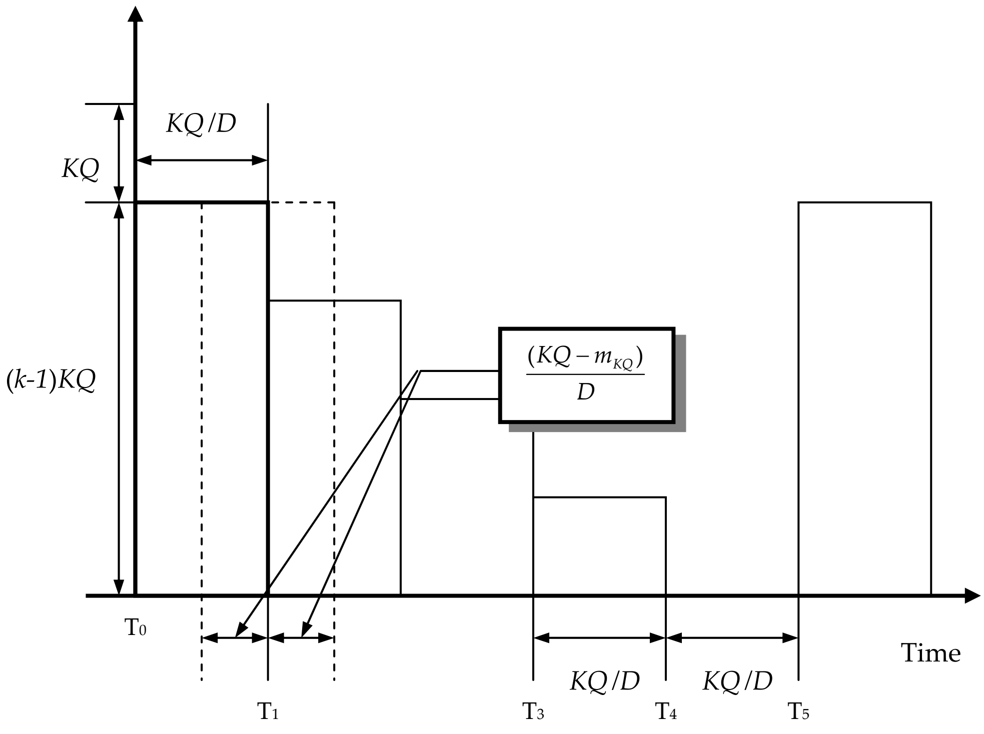

Figure 3. Expected Inventory Level at the Supplier

Therefore;

$$\mathrm{E}[M_{(T_i)}] = \mathrm{E}\,[T_i + (KQ - m_{KQ})/D] = T_i + \mathrm{E}[(KQ - m_{KQ})/D]$$
$$\mathrm{E}[(KQ - m_{KQ})/D] = (1/D)\mathrm{E}[KQ - m_{KQ}]$$
$$\mathrm{E}[KQ - m_{KQ}] = \int_{-\infty}^{\infty}(KQ - m_{KQ})f(m_{KQ})d(m_{KQ}) = 0\,, \quad \because \quad \mathrm{E}[M_{(T_i)}] = T_i\,.$$

■

B. Proof of Lemma 2.

In order to find the upper bound for the supplier's lot size factor (k), we assume that k is a real number of a continuous function. The first derivative of (19) is given by Eq. (B1). By rearranging (B1) with respect to k, Eq. (B2) is obtained.

$$\frac{\Delta AP_E}{\Delta k} = \frac{DS_2}{k^2 KQ} - \frac{KQPH_2}{2} = 0 \qquad \text{(B1)}$$

$$k^* = [1/(KQ)][(2DS_2)/(PH_2)]^{(1/2)} \quad where \quad K \geq 1. \qquad \text{(B2)}$$

In (B2), k will be maximized when K equals one. Therefore, the upper bound of k is the smallest integer which satisfies the condition given in (B3).

$$k^+ \geq (1/Q)[(2DS_2)/(PH_2)]^{(1/2)} \qquad \text{(B3)}$$

■

C. Proof of Lemma 3.

To find the upper bound for K^*, we modify Eq. (19) into Eq. (C1) for computational simplicity.

$$AP_E(KQ) = D(P - C) - \frac{DS_2}{kKQ} - \frac{(k-1)KQPH_2}{2} \qquad \text{(C1)}$$

Eq. (C1) is formulated by subtracting the total discount amount (Dd_K) from (19). From Eq. (C1), we can derive the buyer's optimal order quantity factor (K^*) which not only maximizes the value of (C1) but also minimizes the total inventory cost (setup costs and holding costs) in (C1). Suppose that any value of K greater than the optimal K^* is used in (19). Then, the supplier's profit estimated by (19) will always decrease (19) since the total discount amount (Dd_K) increases because of the positive association between K and d_K, and the total inventory cost also increases. In other words, any K greater than K^* leads to lowering the profit of (19). Consequently, we can use the optimal K^* from (C1) as the upper bound for the buyer's optimal order quantity factor for (19). The first derivative of (C1) with respect to K is given by Eq. (C2). By solving (C2) with respect to K, Eq. (C3) is obtained.

$$\frac{\Delta AP_E}{\Delta K} = \frac{DS_2}{K^2 kQ} - \frac{(k-1)QPH_2}{2} = 0 \tag{C2}$$

$$K^+ \;\geq\; [1/Q]\left[(2DS_2)/(k(k-1)PH_2)\right]^{(1/2)} \quad where \quad k > 1 \tag{C3}$$

■