Mathematical Algorithms for Two-Dimensional Positioning Based on GPS Pseudorange Technique

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Abstract**—Recently, one has realized that the three-dimensional positioning technique used in GPS can be effectively applied to the modern two-dimensional positioning. Such a technique might has applied to the two-dimensional positioning in fields of the mobile communication, eLORAN and the GPS jamming/ electronic warfare system. In the paper, we have studied on algorithms for two-dimensional positioning based on GPS Pseudorange Technique. The main works and results are summarized below. First, the linearized state equation was mathematically derived based on GPS pseudorange technique. Second, the geometry model with respect to triangles formed using unit-vectors were proposed for investigation of land-based radio positioning. Finally, the corresponding mathematical formulations for DOP values and covariance matrix were designed for two-dimensional positioning.**

Index Terms**—eLORAN, Jamming, GPS, Pseudorange, DOP.**

I. INTRODUCTION

The pseudorange positioning method, which has been initially applied to the GPS satellite navigation, is powerful to deal with the common bias of clock error between the transmitters and the user's receiver. It has the significant advantage to handle the navigation equation under the circumstance incoming signal from all transmitters in view.

Recently, one has also realized that the three-dimensional positioning technique used in GPS (Global Positioning System) can be effectively applied to the modern two-dimensional positioning. Such a technique might has applied to the two-dimensional positioning in fields of the mobile communication, eLORAN (enhanced LOng RAnge Navigation) and the GPS jamming/ electronic warfare system [1],[2].

In addition, it is well known that the positioning accuracy is deeply depended on the geometry influence. The geometry factor such as GDOP (Geometry Dilution of Precision), which affects to the positioning accuracy, has mathematically defined and used for the three-dimensional satellite navigation [3].

Unfortunately, in case of two-dimensional positioning, there are no neat geometrical construction and the useful mathematical formulation from the logical concepts. Most of the previous materials involving research papers might have shown to be the lack of the systematic and integrated formulations for dealing with two-dimensional positioning with regard to the pseudorange technique [4]-[6].

In this paper, analytical algorithms based on mathematical steps are derived for two-dimensional positioning. Mainly, the linearized pseudorange equation and DOP values are investigated and discussed.

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II. CONSIDERATIONS OF PSEUDORANGE POSITIONING AND GEOMETRIC VALUE

A. Pseudorange Multiple Ranging Model

The unknown position $(\boldsymbol{P}_n)$ can be determined by measuring multiple ranges between the unknown position $(\boldsymbol{P}_n)$ and the reference positions $(\boldsymbol{P}_i)$.

In horizontal navigation, if two measurements are used to determine the fixing, then it is called the rho-rho method and rho-rho-rho in case of three measurements [7]. The range between the positions can be mathematically denoted by the length of the vector.

$$\rho_i = |\boldsymbol{X}_i| \tag{1}$$

However, these positioning techniques may not be clear for dealing with measurements having range error due to the erroneous synchronization of the receiver clock and the transmitter clocks.

The pseudorange multiple ranging is more flexible to apply to the erroneous positioning system because it can give the more precise solution in positioning and navigation as well.

That is, the following mathematical model, pseudorange, $\boldsymbol{PR}_i$, can be used to solve the navigation systems which the measured ranges contain the range error $(\delta\rho)$ due to the common asynchronous receiver clock.

$$\boldsymbol{PR}_i = \rho_i + \delta\rho \tag{2}$$

Where, ρ_i is the true range between a receiver and *i-th* transmitter. Fig. 1 shows the concept of the pseudorange positioning fixing.

From the above mathematical formula and Fig. 1, one can realize that the pseudorange equation requires one additional measurement compared to rho-rho/rho-rho-rho method because the range error ($\delta\rho$) is dealt as the unknown parameter to be solved in the pseudorange equation. Therefore, four pseudoranges are required for three-dimensional coordinates of the unknown position. The three range measurements are needed in case of determining the coordinates of the horizontal position.

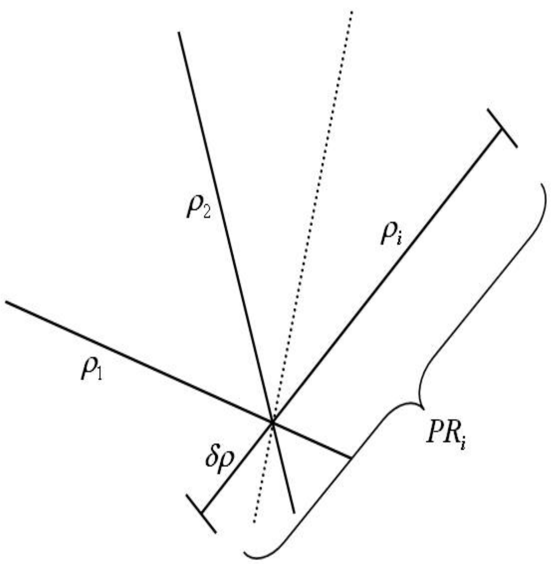

Fig. 1. The Concept of Pseudorange Positioning.

B. Geometric Considerations on Positioning Accuracy

The effect of geometry of satellites on position error is called geometric dilution of precision (GDOP) and it is also interpreted as ratio of position error to the range error. The GDOP generally assumed that the measured pseudorange errors are independent with zero mean and all measurement errors have the same rms value.

In case of three-dimensional satellite positioning, the GDOP value is deeply related to the volume of a tetrahedron formed using four unit-vectors point toward the GPS satellites. As it is the r.s.s (square root of sum of the squares) of the area of the tetrahedron 4 faces, divided by its volume [8].

In our investigation, it has been recognized that the tetrahedron represented by various materials seems to be unclear for ease understanding in proper coordinate. Therefore, we propose the meaningful tetrahedron shown in Fig. 2 with unit vector points toward the assumed broadcasting transmitters in east-north-up local coordinate.

The larger the volume of this body, the smaller the GDOP value. And then the low DOP value gives a better positioning precision because of the wider angular separation between the satellites used to calculate the receiver's position. Therefore, when satellites are far apart, the geometry is strong and DOP value is low. Thus, it gives the better precision. There are several kinds of DOP, HDOP (Horizontal DOP), VDOP (Vertical DOP), PDOP (Position DOP), and TDOP (Time DOP) which can evaluate the position errors for three-dimensional satellite positioning. All the geometry factors are functions of the diagonal elements of the covariance matrix of the parameters, expressed either in a global or local geodetic frame.

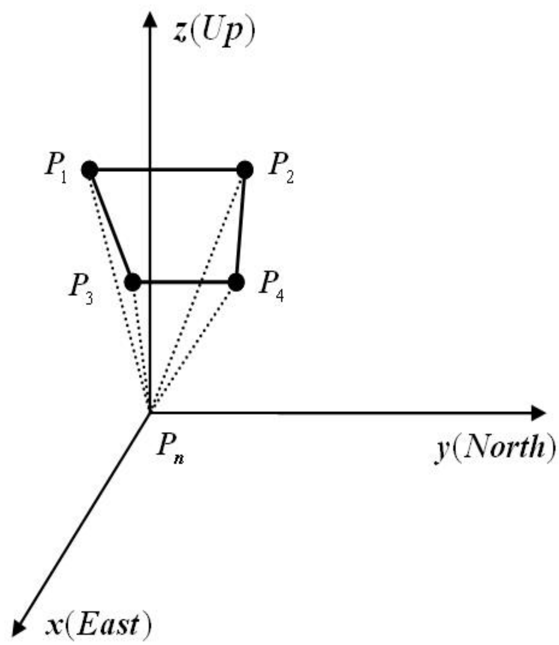

Fig. 2. Tetrahedron with Unit Vectors Points toward the GPS Satellites.

For two-dimensional GDOP, the area of triangles formed using three unit-vectors point toward the transmitters of the broadcasting stations may be considered unlike the tetrahedron. In this case, the receiver lies at the intersection of the circular lines of position that are centered on the transmitters. There is some uncertainty, however, in the receiver's measurements, and the location of the range circles will be inexact and result in an error in computed position. This error depends on the geometry relating the receiver and transmitters.

The two-dimensional DOP values for this would be mathematically derived from the solution of pseudorange equation and covariance matrix.

III. MATHEMATICAL DERIVATION FOR TWO-DIMENSIONAL POSITIONING BASED ON PSEUDORANGE MEASUREMENT

A. Characteristics of Pseudorange Equation for Three-Dimensional Position in GPS Case

The modeling GPS pseudorange equation [9] is investigated to develop the corresponding algorithms for the terrestrial radio navigation system. The following summarized procedures can be applied with some mathematical manipulation and vector analysis as well.

Step 1: Given geometry of satellites (x_i, y_i, z_i) in space and define receiver's true position (x_u, y_u, z_u).
Step 2: Measuring the pseudoranges between the receiver and satellites.
Step 3: Formulating the PR equation for 3D positioning.
Step 4: Linearization of the resultant nonlinear pseudo-range PR equation for solution with iteration.

- Defining the offset $(\delta x_u, \delta y_u, \delta z_u)$ of true position (x_u, y_u, z_u) from known approximate position (x_u', y_u', z_u').
- Expanding the pseudorange equations in Taylor series about known approximate receiver position.
- Completing linearization with respect to unknown position offset $(\delta x_u, \delta y_u, \delta z_u)$ and time offset.

Step5: Expression of the pseudorange offset $\delta\rho$ in the linear equation.
Step6: Constructing the state equation with matrix form.

As a result, the offset pseudorange can be expressed as the linear combination with the direction cosine of the unit vector pointing from the approximate position (x_u', y_u', z_u') to the satellites (x_i, y_i, z_i) and the position offset $(\delta x_u, \delta y_u, \delta z_u)$ and time offset

B. Derivation of Linearization Equation for Two-dimensional Positioning

In the section, we develop the formulations with regard to two-dimensional positioning. As we mentioned previously, the PR method for determining three-dimensional position is very useful to compute the clock bias which is a term of unknown parameter in the linearized equation.

It means that an additional pseudorange measurement compared to the numbers of unknown of position coordinates is needed to construct the simultaneous equation. Therefore, three pseudorange measurements should be existed in order to solve the coordinates of two-dimensional position.

For simple derivation of the state equation, we consider N-transmitting stations on surface of earth or N-available navigation satellites in space.

And then the corresponding pseudoranges are available to formulate the equation.

With referring Fig. 1, let $P_n(x_u, y_u)$ and $P_n(x_i, y_i)$ are the receiver's position and the transmitting station's position, respectively. Then the pseudorange involving time offset, ct_u, can be expressed as the following set of equations.

$$\rho_1 = \sqrt{(x_1 - x_u)^2 + (y_1 - y_u)^2} + ct_u$$

$$\rho_2 = \sqrt{(x_2 - x_u)^2 + (y_2 - y_u)^2} + ct_u$$

$$\rho_3 = \sqrt{(x_3 - x_u)^2 + (y_3 - y_u)^2} + ct_u$$

$$\vdots$$

$$\begin{aligned}\rho_i &= \sqrt{(x_i - x_u)^2 + (y_i - y_u)^2} + ct_u \\ &= f(x_u, y_u, t_u)\end{aligned} \tag{3}$$

The above set of the nonlinear pseudorange equations should be linearized for determining position. According to the shown procedures, the linearization of the above nonlinear equation is taken.

In order to deal with Taylor series about the known approximate position (x_u', y_u'), the corresponding approximate pseudorange might be represented by

$$\begin{aligned}\rho_i' &= \sqrt{(x_i - x_u')^2 + (y_i - y_u')^2} + ct_u' \\ &= f(x_u', y_u', t_u')\end{aligned} \tag{4}$$

And then the function of the true receiver's position can be given with not only the offset but also time offset, δt_u.

$$f(x_u, y_u, t_u) = f(x_u' + \delta x_u,\ y_u' + \delta y_u,\ t_u' + \delta t_u) \tag{5}$$

Therefore, the linearization works using Taylor series-expansion simply can be done with the following.

$$\begin{aligned}&f(x_u' + \delta x_u,\ y_u' + \delta y_u,\ t_u' + \delta t_u) \\ &= f(x_u, y_u, t_u) + \textit{first} - \textit{order derivatives} \\ &\qquad + \textit{higher} - \textit{order derivatives}\end{aligned} \tag{6}$$

In general, the elimination of higher-order derivatives of the above might be cancelled for the linearization [9], [10].

Hence, for two-dimensional problem in the study, the first-order derivatives of the expansion terms are shown below.

$$\frac{\partial f}{\partial x_u'} = -\frac{x_i - x_u'}{r_i'} \qquad (7)$$

$$\frac{\partial f}{\partial y_u'} = -\frac{y_i - y_u'}{r_i'} \qquad (8)$$

$$\frac{\partial f}{\partial t_u'} = c \qquad (9)$$

If we recall the vector analysis about the resulting terms, it is easily realized that the r_i' is the magnitude of vector between transmitting station's position (x_i, y_i) and approximate position (x_y', y_u'). Additionally, the terms $x_i - x_u'$ and $y_i - y_u'$ are directional terms of vector associated with two-dimensional points, (x_i, y_i) and (x_y', y_u').

Then, those terms, $(x_i - x_u')/r_i'$ and $(y_i - y_u')/r_i'$, denote the direction cosines of the unit vector from the receiver's approximate position toward the i transmitter position.

From the previous expression, the offset pseudorange with regard to the offset $(\delta x_u, \delta y_u)$ can be yielded as follow equation.

$$\begin{aligned}\delta \rho_i &= \rho_i' - \rho_i \\ &= \frac{x_i - x_u'}{r_i'}\delta x_u + \frac{y_i - y_u'}{r_i'}\delta y_u - c\,\delta t_u\end{aligned} \qquad (10)$$

Now we simply formulate the matrix forms from the above the linear equation.

$$\delta P = (\delta \rho_1 \; \delta \rho_2 \; \delta \rho_3 \cdots \delta \rho_k)^T \qquad (11)$$

$$\delta X = (\delta x_u \; \delta y_u \; -c\delta t_u)^T \qquad (12)$$

$$G = \begin{bmatrix} \frac{x_1 - x_u'}{r_1'} & \frac{y_1 - y_u'}{y_1'} & 1 \\ \frac{x_2 - x_u'}{r_2'} & \frac{y_2 - y_u'}{y_2'} & 1 \\ \frac{x_3 - x_u'}{r_3'} & \frac{y_3 - y_u'}{y_3'} & 1 \\ \vdots & \vdots & \vdots \\ \frac{x_k - x_u'}{r_k'} & \frac{y_k - x_u'}{r_k'} & 1 \end{bmatrix} \qquad (13)$$

The first two elements of each row of G are the components of unit vector from the receiver to the indicated transmitter. The elements in the third column are the constant with respect to the speed of light.

Finally, the offset (error) of pseudorange value regarding the position offset can be formulated as below.

$$\delta P = G\delta X \qquad (14)$$

From the above pseudorange offset, the position offsets can be obtained with some engineering techniques such as the least square solution and Kalman filtering. The following position offset equations give the corresponding solution for two-dimension.

for $k = 3$, $\delta X = G^{-1}\delta P$ (15)

for $k > 3$, using the pseudoinverse G,

$$\delta X = (G^T G)^{-1} G^T \delta P \qquad (16)$$

G is the geometry matrix due to the direction to the transmitting stations as mentioned previously. The matrix $(G^T G)^{-1} G^T$, called the least-square solution matrix, is a $3 \times k$ matrix. It depends on the relative geometry between the receiver and the transmitting stations in the least square solution computation. It is also noted that three-transmitting stations are required to obtain the two-dimensional positioning problem based on the pseudorange technique.

Practically, G matrix can be obtained using the azimuth θ and elevation ϕ of satellites in case of three-dimensional positioning shown as follows [3]:

$$G = \begin{bmatrix} \cos\phi_1 \sin\theta_1 & \cos\phi_1 \cos\theta_1 & \sin\phi_1 & 1 \\ \cos\phi_2 \sin\theta_2 & \cos\phi_2 \cos\theta_2 & \sin\phi_2 & 1 \\ \cos\phi_3 \sin\theta_3 & \cos\phi_3 \cos\theta_3 & \sin\phi_3 & 1 \\ \vdots & \vdots & \vdots & \vdots \\ \cos\phi_n \sin\theta_n & \cos\phi_n \cos\theta_n & \sin\phi_n & 1 \end{bmatrix} \qquad (17)$$

However, the only azimuth from the receiver position to transmitters is used to formulate directional derivatives of the pseudorange range error. And then the azimuth could be translated into the horizontal plane coordinate frame which is with respect to the east and the north without the up. Therefore, if the third column of elevation term, ϕ, is deleted and the row deleted corresponding to the unused transmitter of the station in above G matrix. With zero degree-elevation, the resulting matrix for N-transmitters can be formulated as follow:

$$G = \begin{bmatrix} \sin\theta_1 & \cos\theta_1 & 1 \\ \sin\theta_2 & \cos\theta_2 & 1 \\ \vdots & \vdots & \vdots \\ \sin\theta_n & \cos\theta_n & 1 \end{bmatrix} \tag{18}$$

C. Mathematical Formulations of GDOP and Covariance Matrix

It is important to evaluate how pseudorange measurements with the error affect the estimate parameters. The pseudorange error is statistically considered to be random variable, and then the covariance matrix could be used to evaluate the magnitude of position error in sense of the accuracy.

The matrix, which has been widely applied to various engineering fields in order to examine the estimation parameters, is sometimes called the propagation error in navigation problem. To promote the GDOP and other values, the geometric model for two-dimensional positioning is proposed in Fig. 3. In the figure, one can imagine that the triangles are formed using three unit vectors points toward the transmitters and the azimuth is measured in 360 degrees with clockwise.

Here the 3×3 covariance matrix would be selected for applying to two-dimensional radio navigation or positioning for the objective of this study.

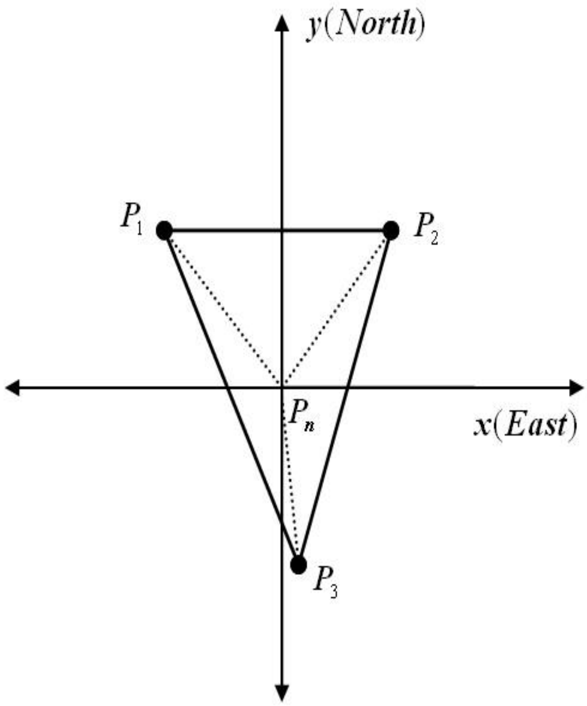

Fig. 3. The Proposed Geometric Model for Two-Dimensional Positioning.

In addition, with recalling the well known the covariance matrix, the expression of the covariance matrix is represented by

$$COV(\delta X) = \begin{bmatrix} \sigma_{x_u}^2 & \sigma_{x_u y_u}^2 & \sigma_{x_u t_u}^2 \\ \sigma_{x_u y_u}^2 & \sigma_{y_u}^2 & \sigma_{y_u t_u}^2 \\ \sigma_{x_u t_u}^2 & \sigma_{y_u t_u}^2 & \sigma_{t_u}^2 \end{bmatrix} \tag{19}$$

$$COV(\delta X) = \sigma_r (G^T G)^{-1} \tag{20}$$

In the above formulas, $(G^T G)^{-1}$ known the GDOP matrix is the matrix of multipliers of ranging variance to give position variance. And σ_r is the pseudorange error factor. Then GDOP can be mathematically computed as trace of the $(G^T G)^{-1}$ matrix as follow:

$$(G^T G)^{-1} = \begin{bmatrix} G_{11} & G_{12} & G_{13} \\ G_{21} & G_{22} & G_{23} \\ G_{31} & G_{32} & G_{33} \end{bmatrix} \tag{21}$$

$$\begin{aligned} GDOP &= \sqrt{trace_3 (G^T G)^{-1}} \\ &= \sqrt{G_{11} + G_{22} + G_{33}} \end{aligned} \tag{22}$$

In case of two-dimensional case, the positional DOP is equal to the horizontal DOP.

$$\begin{aligned} PDOP &= HDOP \\ &= \sqrt{trace_2 (G^T G)^{-1}} \\ &= \sqrt{G_{11} + G_{22}} \end{aligned} \tag{23}$$

Finally, the accuracy of two-dimensional position determined by the receiver can be expressed in terms of DOP values and the pseudorange error factor, σ_r as follow:

$$\begin{aligned} \sqrt{\sigma_x^2 + \sigma_y^2} &= PDOP \times \sigma_r \\ &= HDOP \times \sigma_r \end{aligned} \tag{24}$$

IV. DISCUSSION AND CONCLUSION

The pseudorange positioning technique is powerful to deal with the common bias of clock error between the transmitters and th user's receiver. Therefore, it has been mainly adapted in the GPS positioning with the extremely accurate computing the propagation time of satellite signals.

More recently, even though the land-based radio navigation/positioning has widely developed, there have not been neat algorithms involving the geometrical construction and the useful mathematical formulation. Most of the previous materials involving research papers might have shown to be the lack of the systematic and integrated formulations for dealing with two-dimensional positioning with regard to the pseudorange technique.

In the paper, we have studied on algorithms for two-dimensional positioning based on GPS Pseudorange Technique.

The main works and results are summarized below. First, the linearized state equation was mathematically derived based on GPS pseudorange technique. Second, the geometry model with respect to triangles formed using unit-vectors were proposed for investigation of land-based radio positioning. Finally, the corresponding mathematical formulations for DOP values and covariance matrix were designed for two-dimensional positioning.

In the future works, we would like to continue to apply the resultant algorithms to various practical fields such as GPS jamming, eLORAN and other land-based radio positioning systems.

REFERENCES

[1] K. S. Ko, "A Basic Study on the Jamming Mechanism and Characteristics against GPS/GNSS Based on Navigation Warfare," *Korea Institute of Navigation and Port Research*, vol. 34, no. 2, pp. 97-99, March 2010.

[2] International Loran Association, "Enhanced Loran (eLoran)," Report Version 0.1, January 2007.

[3] B. W. Parkinson, J. J. Spilker Jr., *Global Positioning System: Theory and Applications Volume I*, Washington DC: AIAA, pp. 190-196, pp. 469-477, 1996.

[4] P. H. Dana, "Geogrphic Information System Loran-C Coverage Modeling," *Proc. ILA 22th Annual Technical Symposium*, pp. 280-285, October 1993.

[5] A. H. Phillips, "Geometrical Determination of PDOP," *Journal of Navigation*, vol. 31, no. 4, pp. 329-333, Winter 1984-85.

[6] T. E. Duerr, "Effect of Terrain Masking on GPS Position Dilution of Precision," *Journal of Navigation*., Vol.39, no.3, pp.317-321, Fall 1985.

[7] B. Hofmann-wellenof, K. Legat, M. Wieser, *Navigation Principle of Positioning and Guidance*, New York: Springer Wien, pp. 19-38, 147-162, 2003.

[8] B. A. Stein, "Satellite Selection During Altimeter Aiding of GPS," *Journal of Navigation*, vol. 32, no. 2, pp. 152-155, Summer 1985.

[9] E. D. Kaplan, C. J. Hegarty, *Understanding GPS Principles and Applications*, MA: Artec House, pp. 54-56, 2006.

[10] A. Gelb, *Applied Optimal Estimation*, Massachusetts: The MIT Press, pp. 182-192, July 1979.

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