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퍼지 준-위상 공간에 관한 연구

Fuzzy Quasi Topological Spaces

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요 약

본 논문에서는 smooth topology 와 Chang's fuzzy topology의 개념을 일반화시킨 fuzzy quasi topology의 개념을 소개한다. 그리고 퍼지 준-위상의 일반적인 위상적 성질들을 조사한다.

Abstract

In this paper, we introduce the concept of fuzzy quasi topologies which are generalizations of smooth topologies and Chang's fuzzy topologies, and obtain some basic properties of their structures.

Key Words : fuzzy generalized topological spaces, smooth topology, fuzzy quasi continuous, weakly fuzzy quasi continuous.

1. Introduction and Preliminaries

Let X be a set and $I=[0,1]$. Let I^X denote the set of all mapping $A: X\to I$. A member of I^X is called a *fuzzy subset* [4] of X. 0_X and 1_X will denote the characteristic functions of $\varnothing$ and X, respectively. Let X be a set and $I=[0,1]$. Let I^X denote the set of all mapping $A: X\to I$. A member of I^X is called a fuzzy subset of X. And unions and intersections of fuzzy sets are denoted by $\cup$ and $\cap$, respectively, and defined by

$$\cup A_i = \sup\{A_i(x) \mid i\in J \text{ and } x\in X\}$$

$$\cap A_i = \inf\{A_i(x) \mid i\in J \text{ and } x\in X\}$$

A Chang's fuzzy topological space [2] is an ordered pair (X, τ) is a non-empty set X and $\tau\subseteq I^X$ satisfying the following conditions:

(O1) $0_X, 1_X \in \tau$.

(O2) If $A, B \in \tau$, then $A\cap B \in \tau$.

(O3) If $A_i \in \tau$, for all $i\in J$, then $\cup A_i \in \tau$.

(X, τ) is called a *fuzzy topological space*. Members of τ are called fuzzy open sets in (X,τ) and complement of a *fuzzy open set* is called a *fuzzy closed* set.

A *smooth topological space* [1,3] is an ordered pair (X, τ), where X is a non-empty set and $\tau: I^X \to I$ is a mapping satisfying the following conditions:

(O1) $\tau(0_X)=\tau(1_X)=1$.

(O2) $\tau(A_1\cap A_2)\geq \tau(A_1)\wedge \tau(A_2)$ for $A_1, A_2\in I^X$.

(O3) $\tau(\cup A_i)\geq \wedge\tau(A_i)$ for $A_i\in I^X$.

Then $\tau: I^X\to I$ is called a *smooth topology* on X. The number $\tau(A)$ is called the *degree of openness* of A.

A mapping $\tau^*: I^X\to I$ is called a *smooth cotopology* [3] iff the following three conditions are satisfied:

(C1) $\tau^*(0_X)=\tau^*(1_X)=1$.

(C2) $\tau^*(A_1\cup A_2)\geq\tau^*(A_1)\wedge\tau^*(A_2)$ for $A_1, A_2\in I^X$.

(C3) $\tau^*(\cap A_i)\geq\wedge\tau^*(A_i)$ for $A_i\in I^X$.

Let f be a mapping from a set X into a set Y. Let A and B be respectively the fuzzy sets of X and Y. Then $f(A)$ is a fuzzy set in Y, defined by

$$f(A)(y) = \begin{cases} \sup A(z)_{z\in f^{-1}(y)}, & \text{if } f^{-1}(y)\neq 0,\ y\in Y \\ 0, & \text{otherwise}, \end{cases}$$

and $f^{-1}(B)$ is a fuzzy set in X, defined by $f^{-1}(B)(x)=B(f(x)),\ x\in X$.

2. Fuzzy quasi topological spaces

Definition 2.1. A *fuzzy quasid topological space* (simply, FQTS) is an ordered pair (X, τ), where X is a non-empty set and $\tau: I^X\to I$ is a mapping satisfying the following conditions:

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(QO1) $\tau(0_X)=1$.
(QO2) $\tau(A_1 \cap A_2) \geq \tau(A_1) \wedge \tau(A_2)$ for $A_1, A_2 \in I^X$.
(QO3) $\tau(\cup A_i) \geq \wedge\, \tau(A_i)$ for $A_i \in I^X$.

Then the mapping $\tau : I^X \to I$ is called a *fuzzy quasi topology* on X. The number $\tau(A)$ is called the *degree of quasi openness* of A.

Chang's fuzzy topology ⇒ smooth topology ⇒ fuzzy quasi topology

Definition 2.2. A mapping $T^*: I^X \to I$ is called a *fuzzy quasi cotopology* if the following three conditions are satisfied:
(QC1) $T^*(1_X)=1$.
(QC2) $T^*(A_1 \cup A_2) \geq T^*(A_1) \wedge T^*(A_2)$ for $A_1, A_2 \in I^X$.
(QC3) $T^*(\cap A_i) \geq \wedge\, T^*(A_i)$ for $A_i \in I^X$.

Then $T^*(A)$ is called the *degree of quasi closedness* of A.

Theorem 2.3. If T is a fuzzy quasi topology on X, then the mapping $T^*: I^X \to I$ defined by $T^*(A)= T(A^c)$ is a fuzzy quasi cotopology on X.

Proof. From $T^*(1_X)= T(1_X{}^c)= T(0_X)=1$, we have the condition (QC1).

For $A, B \in I^X$,
$$\begin{aligned} T^*(A \cup B) &= T((A \cup B)^c) \\ &= T(A^c \cap B^c) \\ &\geq T(A^c) \wedge T(B^c) \\ &= T^*(A) \wedge T^*(B). \end{aligned}$$
So (QC2) is obtained.

For every subfamily $\{A_i : i \in J\} \subseteq I^X$,
$$\begin{aligned} T^*(\cap A_i) &= T((\cap A_i)^c) \\ &= T(\cup (A_i^c)) \\ &\geq \wedge\, T(A_i^c) \\ &= \wedge\, T^*(A_i). \end{aligned}$$
Therefore, (QC3) is obtained.

Theorem 2.4. If T^* is a fuzzy quasi cotopology on a nonempty set X, then the mapping $T: I^X \to I$ defined by $T(A)= T^*(A^c)$ is a fuzzy quasi topology on X.

Theorem 2.5. Let (X, T) be a FQTS. Set
$$T_\alpha = \{A \in I^X : T(A) \geq \alpha\}.$$
Then
(1) $0_X \in T_\alpha$.
(2) If $A, B \in T_\alpha$, then $A, B \in T_\alpha$.
(3) If $A_i \in T_\alpha$ for each $i \in J$, then $\cup A_i \in T_\alpha$.

Proof. (1), (2) Obvious.

(3) Let $A_i \in T_\alpha$ for each $i \in J$; then
$$T(\cup A_i) \geq \wedge\, T(A_i) \geq \alpha.$$
This implies that $\cup A_i \in T_\alpha$.

Definition 2.6. Let $f:(X, T_1) \to (Y, T_2)$ be a mapping on FQTS's. Then
(1) f is said to be *fuzzy quasi continuous* if for every $A \in I^Y$, $T_1(f^{-1}(A)) \geq T_2(A)$.
(2) f is said to be *weakly fuzzy quasi continuous* if for every $A \in I^Y$, we have
$$T_2(A)>0 \Rightarrow T_1(f^{-1}(A))>0.$$

fuzzy quasi continuous ⇒ weakly fuzzy quasi continuous

Theorem 2.7. Let $f:(X, T_1) \to (Y, T_2)$ be a mapping on FQTS's. Then
(1) f is fuzzy quasi continuous if and only if $T_2^*(\text{A}) \leq T_1^*(f^{-1}(\text{A}))$.
(2) f is said to be weakly fuzzy quasi continuous if and only if
$$T_2^*(A)>0 \Rightarrow T_1^*(f^{-1}(A))>0.$$

Proof. (1) From the definition of fuzzy quasi continuous function, it follows
$$\begin{aligned} T_2^*(A) &= T_2(A^c) \\ &\leq T_1(f^{-1}(A^c)) \\ &= T_1((f^{-1}(A))^c) \\ &= T_1^*(f^{-1}(A)). \end{aligned}$$
The other implication is obvious.

(2) See the proof of (1).

3. 0-Closure operator ond 0-Interior operator in fuzzy quasi topological spaces

Definition 3.1. Let (X,T) be a FQTS and $A \in I^X$. Then
(1) The *0-closure* of A, denoted by A_-, is defined by
$$A_- = \cap \{K \in I^X : T^*(K) > 0,\ A \subseteq K\},$$
where $T^*(K)=T(K^c)$.
(2) The *0-interior* of A, denoted by A_o, is defined by
$$A_o = \cup \{K \in I^X : T(K)>0,\ K \subseteq A\}.$$

Theorem 3.2. Let (X,T) be a FQTS and $A, B \in I^X$. Then
(1) $(0_X)_o = 0_X$,
(2) $A_o \subseteq A$,

(3) $A \subseteq B \Rightarrow A_o \subseteq B_o$.

Proof. Obvious.

Theorem 3.3. Let (X, T) be a FQTS and $A, B \in I^X$. Then
(1) $(1_X)_- = 1_X$,
(2) $A \subseteq A_-$,
(3) $A \subseteq B \Rightarrow A_- \subseteq B_-$,

Proof. Obvious.

Theorem 3.4. Let (X, T) be a FQTS and $A \in I^X$. Then
(1) $(A_-)^c = (A^c)_o$.
(2) $(A_o)^c = (A^c)_-$.

Proof. (1) For $A \in I^X$,

$$\begin{aligned}(A_-)^c &= (\cap\{K \in I^X : T^*(K) > 0,\ A \subseteq K\})^c \\ &= \cup\{K^c : K \in I^X,\ T(K^c) = T^*(K) > 0, \\ &\qquad K^c \subseteq A^c\} \\ &= \cup\{U \in I^X : T(U) > 0,\ U \subseteq A^c\} \\ &= (A^c)_o.\end{aligned}$$

(2) Obvious.

Theorem 3.5. Let (X, T) be a FQTS and $A \in I^X$. Then
(1) If $T(A) > 0$, then $A = A_o$.
(2) If $T^*(A) > 0$, then $A = A_-$.

Proof. Obvious

Example 3.6. Let $X = [0,1]$ and N the set of all natural numbers. Let us consider for each $n \in N$, a fuzzy set A_n as the following: $A_n(x) = \frac{n}{n+1}x$, for $x \in I$ and a fuzzy set $A(x) = x$, for $x \in I$.

Consider a fuzzy quasi topology $T : I^X \to I$ defined as follows

$$T(\mu) = \begin{cases} 1, & \text{if } \mu = 0_X, \\ \frac{1}{n+1}, & \text{if } \mu = A_n, \\ 0, & otherwise. \end{cases}$$

Then $A = A_o$ and $T(A) = 0$. Thus in the above Theorem 3.5, we know that the converses may not be true in general.

Theorem 3.7. Let (X, T) be a FQTS and $A, B \in I^X$. Then
(1) $(A_o)_o = A_o$.
(2) $(A_-)_- = A_-$.

Proof. (1) In $T(A_o) > 0$, it is obvious from Theorem 3.5.

For $T(A_o) = 0$, let $A_o = \cup\{K \in I^X : T(K) > 0,\ K \subseteq A\}$. Then for $K \subseteq A_o$ satisfying $T(K) > 0$, $K = K_o \subseteq (A_o)_o$. Hence we have $(A_o)_o = A_o$.

(2) It follows from (1).

Theorem 3.8. Let (X, T_1) and (Y, T_2) be FQTS's. If $f : X \to Y$ is weakly fuzzy quasi continuous, then we have
(1) $T_2^*(B) > 0 \Rightarrow T_1^*(f^{-1}(B)) > 0$ for $B \in I^Y$,
(2) $f(A_-) \subseteq f(A)_-$ for $A \in I^X$,
(3) $f^{-1}(B)_- \subseteq f^{-1}(B_-)$ for $B \in I^Y$,
(4) $f^{-1}(B_o) \subseteq f^{-1}(B)_o$ for $B \in I^Y$.

Proof. (1) Let $T_2^*(B) > 0$ for $B \in I^Y$. Then $T_2(B^c) > 0$, from hypothesis, it follows $T_1(f^{-1}(B^c)) > 0$, and hence $T_1^*(f^{-1}(B)) > 0$.
(2) Let $A \in I^X$. Then

$$\begin{aligned} f^{-1}(f(A)_-) &= f^{-1}[\cap\{U \in I^Y : T_2^*(U) > 0 \text{ and } f(A) \subseteq U\}] \\ &= \cap\{f^{-1}(U) \in I^X : T_1^*(f^{-1}(U)) > 0 \text{ and } A \subseteq f^{-1}(U)\}.\end{aligned}$$

By $T_1^*(f^{-1}(U)) > 0$, $A_- \subseteq f^{-1}(U)_- = f^{-1}(U)$, and so $A_- \subseteq \cap\{f^{-1}(U) \in I^X : T_1^*(f^{-1}(U)) > 0,\ A \subseteq f^{-1}(U)\}$. This implies $f(A_-) \subseteq f(A)_-$.
(3) It is similar to (2).
(4) It follows from (3).

From Theorem 3.8, the next theorem is easily obtained:

Theorem 3.9. Let (X, T_1) and (Y, T_2) be FQTS's. If $f : X \to Y$ is fuzzy quasi continuous, then we have
(1) $f(A_-) \subseteq f(A)_-$ for $A \in I^X$,
(2) $f^{-1}(B)_- \subseteq f^{-1}(B_-)$ for $B \in I^Y$,
(3) $f^{-1}(B_o) \subseteq f^{-1}(B)_o$ for $B \in I^Y$.

References

[1] R. Badard, “Smooth axiomatics”, *First IFSA Congress, Palma de Mallorca,* July 1986.
[2] C. L. Chang, "Fuzzy topological spaces", *J. Math. Anal. Appl.*, Vol. 24, pp. 182-190, 1968.
[3] A. A. Ramadan, "Smooth topological spaces", *Fuzzy Sets and Systems*, Vol. 48, pp. 371-375, 1992.
[4] L. A. Zadeh, "Fuzzy sets", *Inform. and Control*, Vol. 8, pp. 338-353, 1965.

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