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REMARKS ON THE GAP SET OF $R = \mathcal{K} + C$

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Abstract. $\tilde{G}(P,Q)$, a new generalization of the set of gap numbers of a pair of points, was described in [1]. Here we study gap numbers of local subring $R = \mathcal{K} + C$ of algebraic function field over a finite field and we give a formula for the number of elements of $\tilde{G}(P,Q)$ depending on pure gaps and R.

1. Introduction

The concept of Weierstrass point is a well known notion, and it has many generalizations and applications. One of them, generalization to the semilocal subring of an algebraic function field, was given by Karakaş [6]. This allows a function field $\mathcal{F}$ may have subrings with Weierstrass points even if genus $g \in \{0, 1\}$. Now, we give a brief review of the theory.

Let $\mathcal{F}$ be an algebraic function field over a finite field $\mathcal{K}$, and $\mathcal{F}^{'} = \mathcal{K}^{'}\mathcal{F}$ be a constant field extension of $\mathcal{F}/\mathcal{K}$. Let R be a semilocal subring of $\mathcal{F}$ with $\mathcal{K} \subset R \subseteq \mathcal{F}$ and let the quotient field of R be $\mathcal{F}$. Our notations will be as follows:

$\mathbf{P}_{\mathcal{F}}$: the set of prime divisors (or points) of $\mathcal{F}/\mathcal{K}$.

O_P, v_P : for any $P \in \mathbf{P}_{\mathcal{F}}$, the valuation ring of P, and valuation at P.

(x): divisor of $x \in F$; (w): divisor of w Weil differential.

$S(R)$: the set of prime divisors whose valuation rings contain R.

It is known that $|S(R)| < \infty$. If $R \neq \mathcal{F}$, then $S(R) \neq \emptyset$.

$\mathbf{P}_R := \mathbf{P}_{\mathcal{F}} \setminus S(R)$.

$\mathbf{D}_{\mathcal{F}}$ (resp., $\mathbf{D}_R$): the free abelian group generated by $\mathbf{P}_{\mathcal{F}}$ (resp., $\mathbf{P}_R$).

If $R^{'} := \mathcal{K}^{'}R$, then $R^{'}$ is a semilocal subring of $\mathcal{F}^{'}$ with

$$S(R^{'}) = \{P^{'} \in \mathbf{P}_{\mathcal{F}^{'}} : P^{'}|P \text{ for some } P \in S(R)\}.$$

For any $D \in \mathbf{D}_{\mathcal{F}}$, we define $L(D) = \{x \in F : (x) \geq -D\} \cup \{0\}$ and $\mathcal{L}(D) = L(D) \cap R$. It is well known that $\dim_{\mathcal{K}} L(D)$, $\dim_{\mathcal{K}} \mathcal{L}(D)$, $\delta = \dim_{\mathcal{K}} \overline{R}/R$ and $\gamma = \dim_{\mathcal{K}} \overline{R}/C$ are finite, where $\overline{R}$ is the integral closure of R in $\mathcal{F}$ and C is the conductor of R in $\overline{R}$. For $P \in S(R)$, there is uniquely determined non-negative integer c_P such that $C = \{y \in \mathcal{F} : v_P(y) \geq c_P,\ P \in S(R)\}$. The

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divisor $\sum_{P\in S(R)} c_P P$ is also denoted by C, and $\deg C = \gamma$. For any $D \in \mathbf{D}_R$, Riemann-Roch-Rosenlicht's theorem can be formulated as

$$\dim_{\mathcal{K}} \mathcal{L}(D) = \deg D + 1 - g_R + \dim_{\mathcal{K}} \widetilde{\mathcal{L}}(D),$$

where $g_R = g + \delta$, and g is the genus of $\mathcal{F}$. It can be shown that $\delta \leq \gamma \leq 2\delta$. $\gamma = 2\delta$ if and only if, for any $D \in D_R$ and ω Weil differential such that $(\omega)+C \in D_R$, $\dim_{\mathcal{K}} \widetilde{\mathcal{L}}(D) = \dim_{\mathcal{K}} \mathcal{L}((\omega) + C - D)$. As a result of this formulation, we immediately see that $\dim_{\mathcal{K}} \mathcal{L}((\omega) + C - D) = 0$ if $\deg D \geq 2g + \gamma - 1$. For any $P \in P_R$ of degree one, we have $\dim_{\mathcal{K}} \mathcal{L}((2g+\gamma-1)P) = g+\gamma-\delta$ and therefore there are g_R positive integers $\lambda \leq 2g + \gamma - 1$ for which $\dim_{\mathcal{K}} \mathcal{L}(\lambda P) = \dim_{\mathcal{K}} \mathcal{L}((\lambda - 1)P)$. We call these integers *gap numbers* (otherwise, *pole numbers*) of R at P. We call P a *Weierstrass point* of R, if the number of points of R having the same gap numbers as P is finite. Otherwise, P is called an *ordinary point* of R. If $R = \mathcal{F}$, then we have classical Weierstrass points theory. During the last decades the set of pole numbers $H(P)$, called Weierstrass semigroup, is also generalized to the r-tuple semigroup $H(P_1, \dots, P_r)$, see [3], [4], [7]. Its many applications take place in the coding theory, for example in [8] and [9].

For any r-tuple $P_1, \dots, P_r$ of distinct rational points we denote by $R_{P_1,\dots,P_r}$ the ring of functions that are regular outside the points $P_1, \dots, P_r$. Throughout this paper, we will assume that $\#\mathcal{K} > r$. For notational simplicity we define $v_{P_1,\dots,P_r}(f) := (v_{P_1}(f), \dots, v_{P_r}(f))$ for any non-zero function f. In [1], authors gave a new generalization of the concept r-tuple Weierstrass semigroup

$$\widetilde{H}(P_1, \dots, P_r) := \widehat{H}(P_1, \dots, P_r)/v_{P_1,\dots,P_r}(R^*_{P_1,\dots,P_r}),$$

and the complement of $\widetilde{H}(P_1, \dots, P_r)$ is called the set of gaps, and denoted by $\widetilde{G}(P_1, \dots, P_r)$. For $\widehat{H}(P_1, \dots, P_r)$ and $\widehat{G}(P_1, \dots, P_r)$ see [1]. $\#\widetilde{G}(P_1, \dots, P_r) < \infty$ and their properties were investigated extensively in [1]. Now, we specialize to the case $r = 2$. Let P and Q be two rational points of $\mathcal{F}$. There exists a positive integer m, called the *period* of the semigroup $\widetilde{H}(P, Q)$, such that $\{\lambda \cdot (m, -m) \,|\, 0 \leq \lambda \leq 1\}$ is a fundamental region. We have $\sum_{i=c}^{m+c-1}(i + \sigma_{P,Q}(i)) = mg$, where $c \in \mathbb{Z}$ and $\sigma_{P,Q}$ is the map whose definition and basic properties can be found in [1]. In particular, we are able to calculate the number of gaps of the $\widetilde{H}(P, Q)$ in terms of $\sigma_{P,Q}$. By Theorem 18 in [1], for any integer c, we have

$$\#\widetilde{G}(P, Q) = mg + \sum_{i=c}^{m+c-1} \#\{j > i \,|\, \sigma_{P,Q}(j) > \sigma_{P,Q}(i)\}.$$

In this work we study gap set of the subrings of the form $R = \mathcal{K} + C$, where C is the conductor of R, and give a formula for the number of $\widetilde{G}(P, Q)$ via the set of pure gap numbers $\widetilde{G}_0(P, Q)$ and gap numbers of R. Additionally, it contains some examples illustrating our theorems.

2. Gap numbers of local subring $R = \mathcal{K} + C$

Definition 1. Let $\mathcal{F}$ be an algebraic function field over $\mathcal{K}$ and $P_1, P_2, \ldots, P_r$ be rational points of $\mathcal{F}$. We define $\sigma_j : \mathbb{Z}^{r-1} \to \mathbb{Z}$ by

$$\begin{aligned}&\sigma_j(i_1, \ldots, i_{j-1}, i_{j+1}, \ldots, i_r)\\ :=\ &\min\{k \,|\, (i_1, \ldots, i_{j-1}, k, i_{j+1}, \ldots, i_r) \in \widehat{H}(P_1, P_2, \ldots, P_r)\}.\end{aligned}$$

If $r = 2$, then the map σ_2 will be the map $\sigma_{P,Q}$, introduced in [1]. Now we study gap set of R.

Lemma 2. *If D is any integral divisor of the ring $R = \mathcal{K} + C$, then*

$$\mathcal{L}(D) = \mathcal{K} + L(D - C).$$

Proof. One can easily prove this lemma. □

Here note that the set of gap numbers of $R_n = \mathcal{K} + Q^n$ at P will be denoted by $G_n(P)$.

Lemma 3. (1) *Let $R_n = \mathcal{K} + Q^n$ be a local subring of $\mathcal{F}$, where $Q \in P_{\mathcal{F}}$ of degree one, $n \geq 2$ and $P \in P_{\mathcal{F}}$ with $P \neq Q$. Then we have*

(a) *If j is a gap number of R_n at P, then $(j, -n) \in \widehat{G}(P, Q)$.*
(b) *If $j \in G(P)$, then $j \in G_n(P)$.*
(c) *Let $j \in H(P)$. $j \in G_n(P)$ if and only if $\sigma_2(j) > -n$.*

(2) *Let $R_{n_1,n_2,\ldots,n_r} = \mathcal{K} + Q_1^{n_1} Q_2^{n_2} \cdots Q_r^{n_r}$ be a local subring of $\mathcal{F}$, where $n_1 + \cdots + n_r \geq 2$, $\deg Q_i = 1$, $i = 1, 2, \ldots, r$, and Q_i and Q_j be distinct points for $i \neq j$. If $i \in H(P)$, $(i, -n_1, \ldots, -n_r) \in \widehat{G}(P, Q_1, \ldots, Q_r)$ such that $\sigma_1(-n_1, \ldots, -n_r) < i$, then $L(iP - \sum_{t=1}^{r} n_t Q_t) = L((i-1)P - \sum_{t=1}^{r} n_t Q_t)$.*

Proof. (1) To prove (a), let us assume the opposite. If $(j, -n) \notin \widehat{G}(P, Q)$, then $(j, -n) \in \widehat{H}(P, Q)$. There exists a function $f \in R_{P,Q} \backslash \{0\}$ such that $(j, -n) = (-v_P(f), -v_Q(f))$. Hence, $L(jP - nQ) \neq L((j-1)P - nQ)$ and $j \notin G_n(P)$. (b) is clear from definitions. (c) Let $j \in H(P)$ and $-n \geq \sigma_2(j)$. Since $(j, \sigma_2(j)) \in \widehat{H}(P, Q)$, there is a function $f \in R_{P,Q} \backslash \{0\}$ such that $v_P(f) = -j$, $v_Q(f) = -\sigma_2(j) \geq n$. Therefore, $L(jP - nQ) \neq L((j-1)P - nQ)$. For converse, assume that $j \notin G_n(P)$. By Lemma 2, $L(jP - nQ) \neq L((j-1)P - nQ)$. There exists a function f such that $v_P(f) = -j$, $v_Q(f) \geq n$. On the other hand, since $(j, \sigma_2(j)) \in \widehat{H}(P, Q)$, there exists a function g such that $v_P(g) = -j$, $v_Q(g) = -\sigma_2(j) < n$. Let take $h := f + g$, we see that for some $t < j$, $v_P(h) = -t$, $v_Q(h) = -\sigma_2(j)$. Hence, $(t, \sigma_2(j)) \in \widehat{H}(P, Q)$ and $(t, \sigma_2(j)) < (j, \sigma_2(j))$, but this contradicts with the definition of σ_2. Assuming opposite of the assertion, we can prove (2) by similar reasoning. □

The following lemma explains action of the constant field extensions to the semilocal subrings, and it can be proven by using constant field extension properties, see [12]. Now on, $\mathcal{F}'/\mathcal{K}_0'$ will denote the costant field extension of $\mathcal{F}/\mathcal{K}$.

Lemma 4. *Let R' be a semilocal subring of $\mathcal{F}'$, $\overline{R'}$ be denote the integral closure of $R^{'}$ in $\mathcal{F}'$ and $C^{'}$ be denote the conductor of $R^{'}$ in $\overline{R'}$. Then the followings hold.*

(a) $\dim_{\mathcal{K}'} \overline{R'}/R^{'} = \delta$, $\dim_{\mathcal{K}'} \overline{R'}/C^{'} = \gamma$.

(b) $C^{'} = \mathrm{Con}_{\mathcal{F}'/\mathcal{F}}(C)$.

(c) *For any $D \in \mathbf{D}_R$, we have $\mathrm{Con}_{\mathcal{F}'/\mathcal{F}}(D) \in \mathbf{D}_{R'}$, $\deg D = \deg \mathrm{Con}_{\mathcal{F}'/\mathcal{F}}(D)$, and $\dim_{\mathcal{K}'} \mathcal{L}(\mathrm{Con}_{\mathcal{F}'/\mathcal{F}}(D)) = \dim_{\mathcal{K}} \mathcal{L}(D)$. Thus for any $D \in \mathbf{D}_R$,*

$$\dim_{\mathcal{K}'} \tilde{\mathcal{L}}(\mathrm{Con}_{\mathcal{F}'/\mathcal{F}}(D)) = \dim_{\mathcal{K}} \tilde{\mathcal{L}}(D).$$

(d) *Let $\mathcal{F}^{'}/\mathcal{F}$ be a constant field extension and $R^{'}$ be as defined above. Let $Q^{'} \in \mathbf{P}_{R'}$ be an extension of $Q \in \mathbf{P}_R$ of degree one. Then, any gap number of Q is also a gap number of Q'. In particular if $\deg Q = 1$, then a positive integer n is a gap number of Q if and only if it is a gap number of Q'.*

Theorem 5. *Let $\mathcal{F}$ be an algebraic function field over a finite field $\mathcal{K}$, $\#\mathcal{K}>2$. Let $R_n = \mathcal{K} + Q^n$ be a local subring of $\mathcal{F}$ and $Q \in P_{\mathcal{F}}$, $\deg Q = 1$, $n \geq 2$.*

(a) *If $P \in P_{R_n}$ and $\deg P = 1$, then the gap set of R_n at P is given by*

$$G(P) \cup \{j \in H(P) : \sigma_2(j) > -n\}.$$

(b) *If $P \in P_{R_n}$ and $\deg P = r$, then the gap set of R_n at P is given by*

$$G(P) \cup \{i \in H(P) : (i, \ldots, i, -n) \in \widehat{G}(P_1, P_2, ...P_r, Q'), \sigma_{r+1}(i, \ldots, i) > -n\},$$

where $\mathcal{F}' = \mathcal{F}\mathcal{K}'/\mathcal{K}'$ is a constant field extension of degree r of $\mathcal{F}/\mathcal{K}$, and $P_k|P$, $Q'|Q$, $k = 1, 2, 3, \ldots, r$.

Proof. (a) Let $R_n = \mathcal{K} + Q^n$, $\deg Q = 1$, $P \neq Q$ and $\deg P = 1$, $n \geq 2$. Now, $C = nQ$, and we consider the spaces

$$L(P - nQ) \subset L(2P - nQ) \subset \cdots \subset L(nP - nQ) \subset \cdots \subset L((2g + n - 1)P - nQ).$$

It follows that the first $n - 1$ gap values of P consist of $1, 2, \ldots, n - 1$. The remaining g gap values range between $\delta + 1 = n$ and $2g + n - 1$. By Lemma 3 and Lemma 2, the gap set of R_n is $\{1, 2, \ldots, n - 1\} \cup \{i \in G(P) : i \geq n\} \cup \{j \in H(P) : \sigma_2(j) > -n\}$, and this set is also written of the following form

$$G(P) \cup \{j \in H(P) : \sigma_2(j) > -n\}.$$

(b) Let $P \in P_{R_n}$ and $\deg P = r$. P has at most $\left[\frac{g_R}{r}\right]$ gap numbers, and for any gap number $j \leq \left[\frac{2g_R - 2}{r}\right] + 1$. If $\mathcal{F}' = \mathcal{F}\mathcal{K}'$ is a constant field extension of degree r, then P splits completely in $\mathcal{F}'/\mathcal{F}$. Hence, there are r distinct places $P_1, P_2, \ldots, P_r$ of degree one lying over the place P. By Lemma 4, $R'_n = \mathcal{K}' + Q'^n$ is a local subring of $\mathcal{F}'$ and an extension of R_n, $C' = nQ'$and $\mathrm{Con}_{\mathcal{F}'/\mathcal{F}}(iP) = \sum_{t=1}^{r} iP_t$. We can easily see that $i \in G_n(P)$ if and only if

$$\dim_{\mathcal{K}} \mathcal{L}(iP) - \dim_{\mathcal{K}} \mathcal{L}((i-1)P) = \dim_{\mathcal{K}'} \mathcal{L}(\sum_{t=1}^{r} iP_t) - \dim_{\mathcal{K}'} \mathcal{L}(\sum_{t=1}^{r} (i-1)P_t).$$

Moreover, we have $\mathcal{L}(\sum_{t=1}^r iP_t) = \mathcal{K}' + L(\sum_{t=1}^r iP_t - nQ')$ and $L(\sum_{t=1}^r iP_t - nQ') = L(\sum_{t=1}^r (i-1)P_t - nQ')$. On the other hand, if $i \in G(P)$, then $i \in G_n(P)$. Now, we take $i \in H(P)$. It is easy to see that $i \in H(P)$ if and only if $(i, i, \ldots, i) \in H(P_1, \ldots, P_r)$, i.e., $(i, i, \ldots, i, 0) \in \widehat{H}(P_1, \ldots, P_r, Q')$. If $(i, \ldots, i, -n) \in \widehat{G}(P_1, \ldots, P_r, Q')$ and $\sigma_{r+1}(i, \ldots, i) > -n$, using similar idea which can be seen in the proof of Lemma 3, we have $L(\sum_{t=1}^r iP_t - nQ') = L(\sum_{t=1}^r (i-1)P_t - nQ')$. $\square$

Corollary 6. (a) $\#\{j \in H(P) : \sigma_2(j) > -n\} = n - 1$.

(b) *If $P \in P_{R_n}$ is a rational and Weierstrass point of $\mathcal{F}$, then P is also Weierstrass point of R_n.*

(c) *Let $R_n = \mathcal{K} + Q^n$ be a local subring of $\mathcal{F}$ over a finite field $\mathcal{K}$,* $\deg Q = 1$ *and $n \geq 2$, $\#\mathcal{K}>2$. If $n > m$ and j is a gap number of R_m at a point P of* $\deg P = 1$, *then j is also gap number of R_n at P.*

Proof. Using the fact that the number of gaps of R_n is $g_R = g + \delta = g + n - 1$ and Theorem 5, we have (a). We get (b) from the definition of Weierstrass point and Theorem 5. If $n > m$ and $\sigma_2(j) > -m$, then $\sigma_2(j) > -n$. Hence, (c) is clear by Theorem 5. $\square$

Definition 7. Let $(a, b) \in \widehat{G}(P, Q)$. If $\dim L(aP+bQ) = \dim L(aP+(b-1)Q) = \dim L((a-1)P + bQ)$, then (a, b) is called a pure gap. The set of pure gaps is denoted by $\widehat{G}_0(P, Q)$, and the pure gaps set in $\widetilde{G}(P, Q)$ is denoted by $\widetilde{G}_0(P, Q)$.

Lemma 8. $(a, b) \in \widetilde{G}_0(P, Q)$ *if and only if* $(k, b), (a, t) \in \widetilde{G}(P, Q)$ *for all* $k \leq a$ *and* $t \leq b$.

Proof. Let $(a, b) \in \widehat{G}(P, Q)$. If $\dim L(aP+bQ) = \dim L((a-1)P+bQ)+1$, then there exists j such that $j \leq a$, $(j, b) \in \widehat{H}(P, Q)$. In this case $(a, b') \in \widehat{H}(P, Q)$ for $b' \leq b$, then $(a, b) \in \widehat{H}(P, Q)$, but this is a contradiction. Similarly, we obtain $\dim L(aP+bQ) \neq \dim L(aP+(b-1)Q)+1$. Hence we have lemma. $\square$

Now we denote $\Gamma := \{(a, \sigma_2(a)) : 0 \leq a \leq 2g + n - 1\}$ and we use natural partial order defined as

$$(a_1, b_1) \leq (a_2, b_2) \Longleftrightarrow a_1 \leq a_2, b_1 \leq b_2,$$

and for any pair (a, b) define

$$\Gamma_{(a,b)} = \{(a', b') \in \Gamma : a' \leq a, b' \leq b\}.$$

Lemma 9. $\#\Gamma_{(a,\sigma_2(a))} = \dim L(aP + \sigma_2(a)Q)$.

Proof. Since $(a, \sigma_2(a)) \in \widehat{H}(P, Q)$, we have $\dim L((a-1)P + \sigma_2(a)Q) + 1 = \dim L(aP + \sigma_2(a)Q)$. Then $(j, \sigma_2(a)) \in \widehat{H}(P, Q)$ for some $j \leq a$. Since $(j, \sigma_2(j)) \in \widehat{H}(P, Q)$, we get $\sigma_2(j) < \sigma_2(a)$. Hence, $(j, \sigma_2(j)) \in \Gamma_{(a,\sigma_2(a))}$ and we have lemma. $\square$

Now, we can calculate the number of gaps at the fundamental period via $\widetilde{G}_0(P,Q)$.

Theorem 10. *Denote by m the period of the semigroup $\widehat{H}(P,Q)$. The number of gaps of the semigroup $\widetilde{H}(P,Q)$ is given by*

$$\#\widetilde{G}(P,Q) = \#\widetilde{G}_0(P,Q) + 2\sum_{n=2}^{m+1} \dim \widetilde{\mathcal{L}}(\sigma_2^{-1}(-n)P).$$

Proof. Let take $-n = \sigma_2(i)$ and $i \in H(P)$. In this case we have

$$\dim L(iP + \sigma_2(i)Q) = \#\{(j,\sigma_2(j)) : (j,\sigma_2(j)) \leq (i,\sigma_2(i))\}$$

by Lemma 9. It is easy to see that

$$i-1 = (n-1) + \#\{j \in G_n(P) : n \leq j < i\} + \#\{(j,\sigma_2(j)) : (j,\sigma_2(j)) < (i,-n)\}$$

therefore,

$$i-1 = (n-1) + \#\{j \in G_n(P) : n \leq j < i\} + \dim L(iP + \sigma_2(i)Q) - 1,$$

and using Riemann-Roch-Rosenlicht theorem, we have

$$\#\{j \in G_n(P) : n \leq j < \sigma_2^{-1}(-n)\} = g - \dim \widetilde{\mathcal{L}}(\sigma_2^{-1}(-n)P),$$

$$\sum_{n=2}^{m+1} \#\{j \in G_n(P) : n \leq j < \sigma_2^{-1}(-n)\} = mg - \sum_{n=2}^{m+1} \dim \widetilde{\mathcal{L}}(\sigma_2^{-1}(-n)P).$$

Hence

$$mg = \#\text{ Pure Gaps } + \sum_{n=2}^{m+1} \dim \widetilde{\mathcal{L}}(\sigma_2^{-1}(-n)P).$$

On the other hand,

$$\#\{j > \sigma_2^{-1}(-n) \,|\, \sigma_2(j) > -n)\} = g + n - i + \dim L(\sigma_2^{-1}(-n)P - nQ) - 1$$

and

$$\sum_{n=2}^{m+1} \#\{j > \sigma_2^{-1}(-n) \,|\, \sigma_2(j) > -n)\} = \sum_{n=2}^{m+1} \dim \widetilde{\mathcal{L}}(\sigma_2^{-1}(-n)P).$$

From Theorem 18 in [1], we obtain

$$\#\widetilde{G}(P,Q) = \#\text{ Pure Gaps} + 2\sum_{n=2}^{m+1} \dim \widetilde{\mathcal{L}}(\sigma_2^{-1}(-n)P).$$

□

Example 1. Let $\mathcal{F} = \mathcal{K}$(x,y) be a function field given by $y^2 = x^3 - x$, where $\mathcal{K}$ is a finite field, $\operatorname{char} \mathcal{K} = 5$. Then $(x) = 2P - 2Q$. Now we consider the subring $R_2 = \mathcal{K} + Q^2$. Since 2 is period for the semigroup $\widehat{H}(P,Q)$, 1 and 3 are gap numbers of R_2 at P.

Example 2. Let q be a prime power and m a positive integer $(m,\, q) = 1$. Now, we will investigate the algebraic function field $\mathcal{C}_m$ defined over $\mathbb{F}_{q^2}$ by the equation $y^q + y = x^m$. If $m = q+1$, C_m is the Hermitian function field [2]. Let $P = P_{00}$ and $Q = P_\infty$, where P_∞ denotes the point at infinity and P_{ab} denotes the common zero of $x - a$ and $y - b$. The divisors of x and y are given by

$$(x) = \sum_{\beta^q+\beta=0} P_{0\beta} - qP_\infty \quad \text{and} \quad (y) = m(P_{00} - P_\infty).$$

In fact m is the period of the semigroup $\widehat{H}(P,Q)$. We have $\sigma_2(i) = -iq$ for $-m < i \leq 0$, and $\sigma_2(-i \cdot m) = i \cdot m$, $\sigma_2(-1 - im) = q + im$ for any $i \in \mathbb{Z}$.

Theorem 11. *Let $\mathcal{C}_m$ be the algebraic function field over $\mathbb{F}_{q^2}$ given by the equation $y^q + y = x^m$. We denote by P the zero of y and by Q the pole of y. Let $R_n = \mathbb{F}_{q^2} + Q^n$ be a local subring of $\mathcal{C}_m$, where $n \geq 2$, and $m = q+1$ or $m|q+1$. Then the set of gap numbers of R_n at P is one of the following sets*

(i) $\{\ aq : 0 < a < n = \alpha\} \cup G(P)$.
(ii) $\{1, 2, \ldots, n-1,\ j+n\ ; j \in G_\alpha(P),\ n = m\beta + \alpha,\ 0 \leq \alpha < m\}$.

Proof. Now, we give a proof for $m = q+1$, the other case can be proven similarly. Let $R_n = \mathbb{F}_{q^2} + Q^n$ be a local subring of $\mathcal{C}_{q+1}$, $n \geq 2$. If $j \in G(P)$, then $j \in G_n(P)$. If $j \in H(P)$ and $j \leq 2g + n - 1$, then $j = aq + bm$, where a, b are non-negative integers and $\sigma_2(j) = -(a + bm)$.

case 1. If $n = \alpha < m$, then $\sigma_2(j) = -a < \alpha$ and $j = aq \in G_n(P)$.

case 2. If $n = \alpha + m\beta$ and $\alpha < m$, then $\sigma_2^{-1}(-n) = \alpha q + m\beta$. Using periodicity, we have $(j, -n) = (aq + mb, -(\beta m + \alpha)) = (aq + m(b - \beta), -\alpha)$. If $(aq + m(b-\beta), -\alpha) \in \widehat{G}(P,Q)$, then $(j, -n) \in \widehat{G}(P,Q)$ and $j \in G_n(P)$. Hence, we have $G_n(P) = \{1, 2, \ldots, n-1,\ j+n\ ;\ j \in G_\alpha(P)\}$.

case 3. Let $n = \beta m$. If j is a gap number of the $\mathcal{C}_{q+1}$, then $j + \beta m \in G_n(P)$. On the other hand, $(\beta m, -\beta m)$ is always in $\widehat{H}(P,Q)$. Hence,

$$\{1, 2, \ldots, m\beta - 1, j + m\beta\ ; j \in G(P)\}.$$

is gap set of $R_{\beta m} = \mathbb{F}_{q^2} + Q^{\beta m}$ at P. □

It is well known that any rational point of Hermitian curve is also a Weierstrass point. By Corollary 6, one sees that if $P \in P_{R_n}$ is a rational point, then P is a Weierstrass point of R_n. Now, we consider the curve $y^4 + y = x^5$ over the finite field with 16 elements. Let $R_4 = \mathcal{K} + Q^4$, where Q is the point at infinity, and let take $(w) = (xy^2dx)$, $(dx) = 10Q$. Consider divisor $W_R := (W_x) + (\sum_{i=1}^{g_R} \epsilon_i)(dx) + g_R(w) = ((xy^2)^9(x^{16} + x)^4) + 390Q + 9(xy^2dx)$, where W_x is the Wronskian determinant. Any point $P \in P_R$ is a Weierstrass points of R if and only if $v_P(W_R) > 0$. Therefore, Weierstrass points of the local subring $R_4 = \mathcal{K} + Q^4$ are exactly rational points of the curve different from Q.

Here we consider subring $R_{n_1,n_2} = \mathcal{K} + Q_1^{n_1}Q_2^{n_2}$, $n_1 + n_2 \geq 2$, $\deg Q_k = 1$, $k = 1, 2$. For any $P \in P_R$ of degree one if $i \in G(P)$, then we have $i \in G_{n_1,n_2}(P)$.

Let define the map $\sigma_1(-n_1,-n_2)=\min\{j:(j,-n_1,-n_2)\in \widehat{H}(P,Q_1,Q_2)\}$. If $i\in H(P)$ and if there exists a pair (a,b) such that $(a,b)>(-n_1,-n_2)$ and $\sigma_1(a,b)=i>\sigma_1(-n_1,-n_2)$, then $i\in G_{n_1,n_2}(P)$. Thus the gap set of R_{n_1,n_2} is $G(P)\cup\{i\in H(P):(a,b)>(-n_1,-n_2),\sigma_1(a,b)=i>\sigma_1(-n_1,-n_2)\}$. Now, we take the Hermitian curve C_5 and the local subring $R_{1,1}=F_{q^2}+Q_1Q_2$ and points $Q_1=P_{00}$, $Q_2=P_{0b}$, where $b^q+b=0$. Let $P=P_\infty$, $\sigma_1(-1,-1)=4$, $(4,-1,-1)\in\widehat{H}(P,Q_1,Q_2)$. Since $5\in H(P)$, we have $(5,-1,0)$, $(5,0,-1)$, $(5,0,0)$, $(5,-5,0)$, $(5,0,-5)\in\widehat{H}(P,Q_1,Q_2)$. Hence, 5 is a gap number of $R_{1,1}$ at P. Similarly, $G_{1,1}(P)=\{1,2,3,5,6,7,11\}$. For another example, we take the subring $R_{3,2}=F_{q^2}+Q_1^3Q_2^2$. Then $\sigma_1(-3,-2)=12$, $(12,-3,-2)\in\widehat{H}(P,Q_1,Q_2)$, and $G_{3,2}(P)=\{1,2,3,\dots,11\}$.

Corollary 12. *The number of pure gaps of $\mathcal{C}_m$ in the fundamental region is given by*

(a) $\#\widetilde{G}_0(P,Q)=\frac{1}{3}q(q^2-1)$, *where* $m=q+1$

(b) $\#\widetilde{G}_0(P,Q)=\frac{1}{6}(m-1)(2mq-q-m-1)$, *where* $m|q+1$.

Proof. It follows from Theorem 10 and the following formulas

$$\#\widetilde{G}(P,Q)=(m+1)g+\binom{q}{3},\quad m=q+1,$$

$$\#\widetilde{G}(P,Q)=(m+1)g+\frac{1}{6}(m-2)(m-1)(q-2),\quad m|q+1$$

which are given in [1]. □

For an example, the curve $y^{11}+y=x^4$ has 36 pure gap numbers.

Example 3. The Suzuki curve over the field $\mathbb{F}_q$ is defined by the equation $y^q-y=x^{q_0}(x^q-x)$, where $q_0=2^s$, $q=2^{2s+1}$ and s is a positive integer. We denote by $P=P_{00}$ the zero of both x and y and by $O=P_\infty$ the pole of x. It is also known that the divisor $(q+2q_0+1)(P_{00}-P_\infty)$ is principal (see [9]). Here, $m:=q+2q_0+1$ and $H(P_\infty)=\langle q,q+q_0,q+2q_0,q+2q_0+1\rangle$. One easily sees that $\sigma_2(q)=-1$, $\sigma_2(q+q_0)=-1-q_0$, $\sigma_2(q+2q_0)=-1-2q_0$, and $\sigma_2(m)=-m$. This determines the involution σ_2 completely. For $j\in H(P)$ and $j=aq+b(q+q_0)+c(q+2q_0)+d(q+2q_0+1)$, then we have $\sigma_2(j)=-(a+b(1+q_0)+c(1+2q_0)+d(q+2q_0+1))$. If $n<m$,

$$n=(1+2q_0)\alpha+l,\ 0\le l<1+2q_0,\qquad l=\beta(q_0+1)+\tau,\quad \tau<q_0+1$$

and we get $\sigma_2^{-1}(-n)=\sigma_1(-n)=\tau q+\beta(q+q_0)+\alpha(q+2q_0)$.

Theorem 13. *Let $\mathcal{F}$ be the Suzuki function field over $\mathbb{F}_q$ and $R_n=\mathcal{K}+Q^n$ be a local subring of $\mathcal{F}$, $\deg Q=1$. Assume that $P\in P_{R_n}$ of degree one, and $m>n\ge 2$. If $j\in G_n(P)\backslash G(P)$ and n is in the form above, then j can be written one of the following forms*:

(I) (i) $\tau q+b(q+q_0)+\alpha(q+2q_0)$, $b\le\beta$.

(ii) $a+b(q+q_0)+\alpha(q+2q_0)$, $b\le\beta$, $a<q_0+1$.

(iii) $aq + b(q + q_0) + \alpha(q + 2q_0)$, $b < \beta$, $\tau < a < q_0 + 1$.

(II) $c < \alpha$, $\alpha = c + u$, *where u is an integer*,

(i) $\tau + b(q + q_0) + c(q + 2q_0)$, $b \le \beta + u \left[\frac{uq_0}{1+q_0}\right]$.

(ii) $a + b(q + q_0) + c(q + 2q_0)$, $a < \tau$, $b \le \beta + u + \left[\frac{uq_0+\tau}{1+q_0}\right]$.

(iii) $a + b(q + q_0) + c(q + 2q_0)$, $\tau < a$, $b \le \beta + u \left[\frac{uq_0}{1+q_0}\right]$.

Proof. Let $n < m$. If $j \in H(P)$, then $j = aq + b(q + q_0) + c(q + 2q_0)$. Here we note that $a + b(1 + q_0) < 1 + 2q_0$ and if $j = km$, then j is not a gap number at P. Using Theorem 5, we have the theorem. □

If $n = m$, we have $G_n(P) = \{1, 2, \ldots, m - 1, j + m \ ; \ j \in G(P)\}$. Generally, if $n = km + l$ with $0 \le l < m$, using periodicity we obtain

$$G_n(P) = \{1, 2, \ldots, n - 1, j + n \ ; \ j \in G_l(P)\}.$$

Lemma 14. *For the Suzuki function field defined by $y^q - y = x^{q_0}(x^q - x)$ over $\mathbb{F}_q$*

$$\#\widetilde{G}_0(P, Q) = \frac{1}{15} q_0 \left(5q_0^2 - 25q_0 - 40q_0^3 + 46q_0^6 - 6\right).$$

Proof. It follows from [1] and Theorem 10. □

If $q_0 = 2$, the number of pure gaps is 136 for Suzuki curve.

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