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ASYMPTOTIC BEHAVIOR OF $\mathcal{A}$-HARMONIC FUNCTIONS AND p-EXTREMAL LENGTH

Seok Woo Kim, Sang Moon Lee, and Yong Hah Lee

Abstract. We describe the asymptotic behavior of functions of the Royden p-algebra in terms of p-extremal length. We also prove that each bounded $\mathcal{A}$-harmonic function with finite energy on a complete Riemannian manifold is uniquely determined by the behavior of the function along p-almost every curve.

1. Introduction

Let Ω be an open subset of an n-dimensional complete Riemannian manifold M and $W^{1,p}(\Omega)$ be the Sobolev space of all functions u in $L^p(\Omega)$ whose distributional gradient ∇u also belongs to $L^p(\Omega)$, where p is a constant such that $1 < p < \infty$. We equip $W^{1,p}(\Omega)$ with the norm $||u||_{1,p} = ||u||_p + ||\nabla u||_p$. We denote by $W_0^{1,p}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{1,p}(\Omega)$. Let $\mathbf{F} : \bigcup_{x\in\Omega} T_xM \to \mathbb{R}$ be a variational kernel satisfying following conditions:

(A1) the mapping $\mathbf{F}_x = \mathbf{F}|_{T_xM} : T_xM \to \mathbb{R}$ is strictly convex and differentiable for all x in Ω, and the mapping $x \mapsto \mathbf{F}_x(\xi)$ is measurable whenever ξ is;

(A2) for a constant $1 < p < \infty$, there exist constants $0 < C_1 \leq C_2 < \infty$ such that

$$C_1|\xi|^p \leq \mathbf{F}_x(\xi) \leq C_2|\xi|^p$$

for all x in Ω and ξ in T_xM.

It is instructive to note that if $\mathcal{A}_x(\xi) = (\mathcal{A}_x^1(\xi), \mathcal{A}_x^2(\xi), \dots, \mathcal{A}_x^n(\xi))$, where $\mathcal{A}_x^i(\xi) = \frac{\partial}{\partial \xi^i} F_x(\xi)$ for each $i = 1, 2, \dots, n$, then $\mathcal{A}$ satisfies the following properties: (See [1] and [7])

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($A3$) the mapping $\mathcal{A}_x = \mathcal{A}|_{T_xM} : T_xM \to T_xM$ is continuous for a.e. x in Ω, and the mapping $x \mapsto \mathcal{A}_x(\xi)$ is a measurable vector field whenever ξ is;

for a.e. x in Ω and for all ξ, ξ' in T_xM,

($A4$) $\langle \mathcal{A}_x(\xi), \xi\rangle \geq C_1|\xi|^p$;
($A5$) $|\mathcal{A}_x(\xi)| \leq C_2|\xi|^{p-1}$;
($A6$) $\langle \mathcal{A}_x(\xi) - \mathcal{A}_x(\xi'), \xi - \xi'\rangle > 0$ whenever $\xi \neq \xi'$.

We say that a function u in $W^{1,p}_{\mathrm{loc}}(\Omega)$ is a solution (supersolution, respectively) of the equation

$$-\mathrm{div}\mathcal{A}_x(\nabla u) = 0 \ (\geq 0, \text{respectively}) \tag{1.1}$$

in Ω if

$$\int_\Omega \langle \mathcal{A}_x(\nabla u), \nabla \phi\rangle = 0 \ (\geq 0, \text{respectively})$$

for any (nonnegative, respectively) function ϕ in $C_0^\infty(\Omega)$. A function v in $W^{1,p}_{\mathrm{loc}}(\Omega)$ is called a subsolution of (1.1) in Ω if $-v$ is a supersolution of (1.1) in Ω. We say that a function u is $\mathcal{A}$-harmonic (of type p) if u is a continuous solution of (1.1). In a typical case $\mathcal{A}_x(\xi) = \xi|\xi|^{p-2}$, $\mathcal{A}$-harmonic functions are called p-harmonic and, in particular, if $p = 2$, then we obtain harmonic functions. Suppose that E is a measurable set and that $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ for an open neighborhood Ω of E. Then the variational integral

$$\mathbf{J}(u, E) = \int_E \mathbf{F}_x(\nabla u)$$

is well defined. If $\mathbf{J}(u, M) < \infty$, then we say that u has finite energy. In fact, given $f \in W^{1,p}(\Omega)$, each $\mathcal{A}$-harmonic function u with $u - f \in W^{1,p}_0(\Omega)$ minimizes the energy functional $J(v, \Omega)$ on the set $\{v \in W^{1,p}(\Omega) : v - f \in W^{1,p}_0(\Omega)\}$ (See [1]). A Green's function $G = G(o, \cdot)$ for $\mathcal{A}$ on M denotes (if exists) a positive solution of the equation

$$-\mathrm{div}\mathcal{A}(\nabla G) = \delta_o \tag{1.2}$$

for each o in M, in the sense of distributions, i.e.,

$$\int_M \langle \mathcal{A}(\nabla G), \nabla \phi\rangle = \phi(o)$$

for any function ϕ in $C_0^\infty(M)$. In fact, there exists a Green's function G satisfying (1.2) if and only if M has positive p-capacity at infinity, i.e., there exists a compact subset K of M such that

$$\mathrm{Cap}_p(K, \infty, M) = \inf_u \int_M |\nabla u|^p > 0,$$

where the infimum is taken over all functions u in $C_0^\infty(M)$ with $u = 1$ on K. In particular, we say that a complete Riemannian manifold M is p-parabolic if $\mathrm{Cap}_p(K, \infty, M) = 0$ for every compact subset K of M. Otherwise, M is called p-nonparabolic. It is well known that a complete Riemannian manifold M is

p-nonparabolic if and only if M has the positive $\mathcal{A}$-capacity, i.e., there exists a compact subset K of M such that

$$\mathrm{Cap}_{\mathcal{A}}(K, \infty, M) = \inf_{u} \mathbf{J}(u, M) > 0,$$

where the infimum is taken over all functions u in $C_0^\infty(M)$ with $u = 1$ on K.

We now introduce additional conditions on $\mathbf{F}$ as follows:

$(A7)$ $\mathcal{A}_x(\lambda\xi) = \lambda|\lambda|^{p-2}\mathcal{A}_x(\xi)$ whenever $\lambda \in \mathbb{R} \setminus \{0\}$;
for any ξ, ξ' in T_xM,

$(A8)$ in case $2 \le p < \infty$,

$$\mathbf{F}_x\Big(\frac{\xi+\xi'}{2}\Big) + \mathbf{F}_x\Big(\frac{\xi-\xi'}{2}\Big) \le \frac{1}{2}(\mathbf{F}_x(\xi) + \mathbf{F}_x(\xi')),$$

in case $1 < p \le 2$,

$$\mathbf{F}_x\Big(\frac{\xi+\xi'}{2}\Big)^{\tilde{p}} + \mathbf{F}_x\Big(\frac{\xi-\xi'}{2}\Big)^{\tilde{p}} \le \Big(\frac{1}{2}(\mathbf{F}_x(\xi) + \mathbf{F}_x(\xi'))\Big)^{\tilde{p}},$$

where $\tilde{p} = 1/(p-1)$.

For $\mathbf{F}(\xi) = \frac{1}{p}|\xi|^p$, the condition $(A8)$ is called the Clarkson inequality (See [3]).

Let $\mathcal{BD}_p(M)$ be the set of all bounded continuous functions u on a complete Riemannian manifold M whose distributional gradient ∇u belongs to $L^p(M)$. Then, by using the Minkowski inequality, it is easy to see that $\mathcal{BD}_p(M)$ forms an algebra over the real numbers with the usual addition and multiplication of functions and scalar multiplication defined pointwise. The function space $\mathcal{BD}_p(M)$ is called the Royden p-algebra of M (See [9]). We say that a sequence $\{f_n\}$ of functions in $\mathcal{BD}_p(M)$ converges to a function $f \in \mathcal{BD}_p(M)$ in the $\mathcal{BD}_p$-topology if

(i) $\{f_n\}$ is uniformly bounded;
(ii) f_n converges uniformly to f on each compact subset of M;
(iii) $\lim_{n\to\infty} \int_M |\nabla(f_n - f)|^p = 0$.

It is well known that $\mathcal{BD}_p(M)$ is complete in the $\mathcal{BD}_p$-topology. Let $\mathcal{BD}_{p,0}(M)$ be the closure of the set of all compactly supported smooth functions in $\mathcal{BD}_p(M)$. It is easy to see that $\mathcal{BD}_{p,0}(M)$ is not only a subalgebra but also an ideal of $\mathcal{BD}_p(M)$. We denote by $\mathcal{HBD}_{\mathcal{A}}(M)$ the subset of all bounded energy finite $\mathcal{A}$-harmonic functions in $\mathcal{BD}_p(M)$, where $\mathcal{A}$ is an elliptic operator on M satisfying conditions $(A1), (A2), (A7)$ and $(A8)$. Adopting the arguments in [6], one can prove the following $\mathcal{A}$-harmonic version of the Royden decomposition theorem:

Proposition 1.1. *Let $\mathcal{A}$ be an elliptic operator on a p-nonparabolic complete Riemannian manifold M satisfying conditions* $(A1), (A2), (A7)$ *and* $(A8)$. *Then for each $f \in \mathcal{BD}_p(M)$, there exist unique $u \in \mathcal{HBD}_{\mathcal{A}}(M)$ and $g \in \mathcal{BD}_{p,0}(M)$ such that $f = u + g$.*

For a complete Riemannian manifold M, there exists a locally compact Hausdorff space $\hat{M}$, called the Royden p-compactification of M, which contains

M as an open dense subset. In particular, every function $f \in \mathcal{BD}_p(M)$ can be extended to a continuous function, denoted again by f, on $\hat{M}$ and the class of such extended functions separates points in $\hat{M}$. The Royden p-boundary of $\hat{M}$ is the set $\hat{M} \setminus M$ and will be denoted by $\partial\hat{M}$. Throughout the paper, for a subset Ω of M, we denote by $\overline{\Omega}$ the closure of Ω in M and $\hat{\Omega}$ the closure of Ω in $\hat{M}$. An important part of the Royden p-boundary $\partial\hat{M}$ is the p-harmonic boundary Δ_M defined by

$$\Delta_M = \{\mathbf{x} \in \partial\hat{M} : f(\mathbf{x}) = 0 \text{ for all } f \in \mathcal{BD}_{p,0}(M)\}.$$

Let $\mathcal{F}$ be a family of locally rectifiable curves in a complete Riemannian manifold M. Let us fix a real number p such that $1 < p < \infty$. A nonnegative Borel measurable function $\rho : M \to \mathbb{R}$ is called admissible with respect to $\mathcal{F}$ if $\int_\gamma \rho \geq 1$ for all curves γ in $\mathcal{F}$. The p-extremal length of $\mathcal{F}$, denoted by $\lambda_p(\mathcal{F})$, is defined as

$$\lambda_p(\mathcal{F}) = \Big(\inf_\rho \int_M \rho^p\Big)^{-1},$$

where the infimum is taken over the set of all admissible functions ρ with respect to $\mathcal{F}$. A property is said to hold for p-almost every curve in $\mathcal{F}$ if it holds for all curves in $\mathcal{F} \setminus \mathcal{F}_0$, where $\mathcal{F}_0$ is a subfamily of $\mathcal{F}$ with p-extremal length ∞.

Under the above setting, the value at the p-harmonic boundary of each function of the Royden p-algebra is completely determined by its asymptotic behavior along p-almost every curve as follows:

Theorem 1.2. *Let us denote $\mathcal{G}$ to be the family of all locally rectifiable curves in a complete Riemannian manifold M joining $\overline{B}_R(o)$ to infinity of M, where $B_R(o)$ denotes a geodesic ball of radius $R > 0$ centered at o. Then for any function f in $\mathcal{BD}_p(M)$, the following conditions are equivalent*:

(i) $f|_{\Delta_M} = 0$.

(ii) *f converges to a constant 0 for p-almost every curve in $\mathcal{G}$, that is,*

$$\lim_{t\to\infty} f(\gamma(t)) = 0$$

along each curve γ in $\mathcal{G}$ except a subfamily of $\mathcal{G}$ whose p-extremal length is ∞.

Applying the comparison principle in Lemma 2.3 together with Theorem 1.2, one can prove that each bounded $\mathcal{A}$-harmonic function with finite energy is uniquely determined by the behavior of the function along p-almost every curve as follows:

Corollary 1.3. *Let $\mathcal{G}$ be given as in Theorem 1.2. Suppose that $f, h \in \mathcal{HBD}_\mathcal{A}(M)$ and*

$$\lim_{t\to\infty} f(\gamma(t)) = \lim_{t\to\infty} h(\gamma(t))$$

for p-almost every curve in $\mathcal{G}$. Then $f \equiv h$ on M.

Corollary 1.4. *Let $\mathcal{G}$ be given as in Theorem 1.2. Suppose that $h \in \mathcal{HBD}_{\mathcal{A}}(M)$ and $c \in \mathbb{R}$ with*

$$\lim_{t\to\infty} h(\gamma(t)) = c$$

for p-almost every curve in $\mathcal{G}$. Then $h \equiv c$ on M.

2. The maximum principle and the p-extremal length

We now study the relation between a sort of asymptotic behavior of functions in the Royden p-algebra $\mathcal{BD}_p(M)$ near infinity of M and the values of the functions at the p-harmonic boundary Δ_M of M. We first give a characterization of the p-parabolicity in terms of the p-harmonic boundary as follows (See [6]):

Lemma 2.1. *A complete Riemannian manifold M is p-parabolic if and only if the p-harmonic boundary Δ_M of M is empty.*

Furthermore, there is a useful duality relation between $\mathcal{BD}_{p,0}(M)$ and Δ_M (See [6]):

Lemma 2.2. *For any complete Riemannian manifold M,*

$$\mathcal{BD}_{p,0}(M) = \{f \in \mathcal{BD}_p(M) : f = 0 \ \text{ on } \ \Delta_M\}.$$

We now give the comparison principle and the maximum principle for $\mathcal{A}$-harmonic functions as follows:

Lemma 2.3. *Let $\mathcal{A}$ be an elliptic operator on a p-nonparabolic complete Riemannian manifold M satisfying conditions $(A1), (A2), (A7)$ and $(A8)$.*

(C) *Suppose that there exist $\mathcal{A}$-harmonic functions $u, v \in \mathcal{HBD}_{\mathcal{A}}(M)$ such that*

$$v \leq u \ \text{ on } \ \Delta_M.$$

Then we have $v \leq u$ on M.

(M) *Suppose that there exists an $\mathcal{A}$-harmonic function $u \in \mathcal{HBD}_{\mathcal{A}}(M)$ such that*

$$a \leq u \leq b \ \text{ on } \ \Delta_M$$

for some constants a and b with $a \leq b$. Then we have $a \leq u \leq b$ on M.

Proof. Let us consider a function $\phi = \min\{u - v, 0\}$. Since $v \leq u$ on Δ_M, we conclude that $\phi = 0$ on Δ_M. Thus by Lemma 2.2, we conclude that ϕ belongs to $\mathcal{BD}_{p,0}(M)$. Since u and v are $\mathcal{A}$-harmonic on M and there is a sequence of compactly supported smooth functions converging to ϕ in $\mathcal{BD}_p(M)$, we have

$$\int_M \langle \mathcal{A}_x(\nabla u), \nabla \phi\rangle = 0$$

and

$$\int_M \langle \mathcal{A}_x(\nabla v), \nabla \phi\rangle = 0.$$

Let 1_Ω be the characteristic function of the set $\Omega = \{x \in M : u(x) \le v(x)\}$. Since $\nabla\phi = 1_\Omega \nabla(u - v)$ almost everywhere in M, we conclude that

$$\int_\Omega \langle \mathcal{A}_x(\nabla u) - \mathcal{A}_x(\nabla v), \nabla(u - v)\rangle = 0.$$

By the condition $(A6)$, $u-v$ is almost everywhere constant in Ω. Since u and v are continuous, $u - v$ is constant in Ω. Hence we have (C) from the continuity of u and v.

On the other hand, since every constant function is also $\mathcal{A}$-harmonic, we have (M) from (C). □

We now introduce the notion of the p-capacity of a condenser: Let $\Omega \subset M$ be a nonempty open set and let E_1 and E_2 be mutually disjoint closed subsets of $\overline{\Omega}$. The p-capacity for a triple (E_1, E_2, Ω) is defined by

$$\mathrm{Cap}_p(E_1, E_2, \Omega) = \inf_v \int_\Omega |\nabla v|^p,$$

where the infimum is taken over all smooth functions v on $\Omega \cup E_1 \cup E_2$ such that $0 \le v \le 1$ on Ω, $v = 0$ on E_1 and $v = 1$ on E_2. Such a triple (E_1, E_2, Ω) is called a condenser. For an unbounded open set $\Omega \subset M$ and a nonempty compact set $E \subset \overline{\Omega}$, we define

$$\mathrm{Cap}_p(E, \infty, \Omega) = \lim_{r\to\infty} \mathrm{Cap}_p(E, \overline{\Omega} \setminus B_r(o), \Omega),$$

where $B_r(o)$ denotes the geodesic ball of radius $r > 0$ centered at a fixed point o in M. It is needed to note that $\mathrm{Cap}_p(E, \overline{\Omega} \setminus B_r(o), \Omega)$ is monotone decreasing in $r > 0$. On the other hand, an unbounded open set $\Omega \subset M$ is called p-hyperbolic if there exists a nonempty compact set $E \subset \overline{\Omega}$ such that

$$\mathrm{Cap}_p(E, \infty, \Omega) > 0.$$

From the properties of the p-capacity, it is easy to see that any open set Ω is p-hyperbolic if there exists a p-hyperbolic subset Ω' of Ω. An unbounded open proper set $\Omega \subset M$ is called $\mathcal{A}$-massive if there exists a function u in $\mathcal{BD}_p(M)$ such that

$$\begin{cases} \mathcal{A}u = 0 & \text{in } \Omega; \\ u = 0 & \text{on } M \setminus \Omega; \\ \sup_\Omega u = 1. \end{cases}$$

Such a function u is called an inner potential of Ω. In fact, for each nonconstant function u in $\mathcal{HBD}_\mathcal{A}(M)$, the set $\{x \in M : u(x) > c\}$ is $\mathcal{A}$-massive, where $\inf_M u < c < \sup_M u$. There is a useful property of $\mathcal{A}$-massive sets (See [4], [5] and [6]):

Lemma 2.4. *Let $\mathcal{A}$ be an elliptic operator on a complete Riemannian manifold M satisfying conditions $(A1)$, $(A2)$, $(A7)$ and $(A8)$. If Ω is $\mathcal{A}$-massive, then there exists a proper p-hyperbolic subset Ω_0 of Ω such that $\overline{\Omega}_0 \subset \Omega$ and a continuous function v on $\overline{\Omega}$ such that $\mathcal{A}v = 0$ in $\Omega \setminus \overline{\Omega}_0$, $v = 0$ on $\partial\Omega$ and v has finite energy, that is, $\mathbf{J}(v, M) < \infty$.*

On the other hand, the p-capacity of a condenser is closely related to the p-extremal length of a family of curves as follows: Let Ω be an unbounded open subset of M and E be a compact set in Ω. Let $\mathcal{F}_{\Omega,E}$ be the family of all locally rectifiable curves in Ω joining E to infinity of Ω. This means that γ is a curve in $\mathcal{F}_{\Omega,E}$ if $\gamma : [\alpha, \beta) \to \Omega$ (β may be ∞) is locally rectifiable, $\gamma(\alpha)$ belongs to E, and for any compact set K of M, there exists $t_K \in [\alpha, \beta)$ such that $\gamma(t)$ does not belong to K for all $t > t_K$. Then, by results in [2], we have

$$\Big(\lambda_p(\mathcal{F}_{\Omega,E})\Big)^{-1} = \mathrm{Cap}_p(E, \infty, \Omega) \tag{2.1}$$

(See also [12] or [8]). In particular, if Ω is $\mathcal{A}$-massive, then by Lemma 2.4, there exists a proper p-hyperbolic subset Ω_0 of Ω such that $\overline{\Omega}_0 \subset \Omega$. Since Ω_0 is p-hyperbolic, there exists a nonempty compact subset $E \subset \overline{\Omega}_0$ such that $\mathrm{Cap}_p(E, \infty, \Omega_0) > 0$. Therefore, combining (2.1) and the monotone property of the p-capacity, we conclude that

$$\Big(\lambda_p(\mathcal{F}_{\Omega,E})\Big)^{-1} = \mathrm{Cap}_p(E, \infty, \Omega) > 0. \tag{2.2}$$

Let us denote $\mathcal{G}$ to be the family of all locally rectifiable curves in M joining $\overline{B}_R(o)$ to infinity of M. For an unbounded set Ω of M, $\mathcal{G}_\Omega$ denotes the subfamily of $\mathcal{G}$ which consists of all locally rectifiable curves in Ω joining $\overline{B}_R(o) \cap \Omega$ to infinity of Ω, where $R > 0$ is sufficiently large such that $\overline{B}_R(o) \cap \Omega \neq \emptyset$. From now on, $\mathcal{G}$ and $\mathcal{G}_\Omega$ mean those appear in the above setting unless otherwise specified. In particular, if Ω is a $\mathcal{A}$-massive set of M, we have $\lambda_p(\mathcal{G}_\Omega) < \infty$.

In fact, the p-extremal length of a family of curves in an unbounded set is closely related to the p-harmonic boundary as follows: Let us denote $e(\gamma)$ to be the end part of a curve $\gamma \in \mathcal{G}$ in $\partial\hat{M}$, this means that $e(\gamma) = \hat{\gamma} \cap \partial\hat{M}$, where $\hat{\gamma}$ denotes the closure in $\hat{M}$ of the image set under γ. The following lemmas give a tractable criterion for a family of curves in an unbounded set to have infinite p-extremal length:

Lemma 2.5. *Let Ω be an unbounded open subset of M such that $\overline{B}_R(o) \cap \Omega \neq \emptyset$ for sufficiently large $R > 0$. Let $\mathcal{G}_0$ be a subfamily of $\mathcal{G}_\Omega$ and K be the closure of the set $\bigcup_{\gamma \in \mathcal{G}_0} e(\gamma)$ in $\partial\hat{M}$. Suppose that K is disjoint from $\hat{\Omega} \cap \Delta_M$. Then $\lambda_p(\mathcal{G}_0) = \infty$.*

Proof. If M is p-parabolic, then $\Delta_M = \emptyset$ and $\mathrm{Cap}_p(\overline{B}_R(o), \infty, M) = 0$. Thus by (2.1), we have

$$\Big(\lambda_p(\mathcal{G})\Big)^{-1} = \mathrm{Cap}_p(\overline{B}_R(o), \infty, M) = 0.$$

So we may assume that M is p-nonparabolic. By the result in [11], it suffices to show that there exists a function ρ in $L^p(M)$ such that

$$\int_\gamma \rho = \infty \quad \text{for each curve} \ \ \gamma \in \mathcal{G}_0.$$

Since K is a nonempty compact subset in $\partial\hat{M}\setminus\Delta_M$, there exists a continuous function f such that $f|_K=\infty$ and $\int_M|\nabla f|^p<\infty$ (See [10], [1], and [13]). Hence the lemma follows. To be precise, from the definition of K, we have $e(\gamma)\in K$ for any curve $\gamma\in\mathcal{G}_0$. Thus we conclude that $f(\gamma)=\infty$ for any curve $\gamma\in\mathcal{G}_0$, where $f(\gamma)=\lim_{t\to\infty}f(\gamma(t))$. Then for any $\varepsilon>0$, the function $\varepsilon|\nabla f|$ is admissible with respect to $\mathcal{G}_0$. Consequently,

$$\lambda_p(\mathcal{G}_0)\geq\left(\varepsilon^p\int_M|\nabla f|^p\right)^{-1}.$$

This completes the proof. □

Lemma 2.6. *Let $\mathcal{G}_0$ be a subfamily of $\mathcal{G}$ such that $\lambda_p(\mathcal{G}_0)=\infty$ and K be the closure of the set $\bigcup_{\gamma\in\mathcal{G}\setminus\mathcal{G}_0}e(\gamma)$ in $\partial\hat{M}$. Then K contains Δ_M.*

Proof. If $\Delta_M=\emptyset$, then nothing to prove. So we may assume that $\Delta_M\neq\emptyset$. That is, M is p-nonparabolic. Then by (2.1), we have

$$\Big(\lambda_p(\mathcal{G})\Big)^{-1}=\mathrm{Cap}_p(\overline{B}_R(o),\infty,M)>0.$$

Since $\lambda_p(\mathcal{G}\setminus\mathcal{G}_0)<\infty$, by Lemma 2.5, we have $K\cap\Delta_M\neq\emptyset$. Suppose that the lemma is not true. We may assume that $\mathbf{x}\in\Delta_M\setminus K$. Let us choose a function $f\in\mathcal{BD}_p(M)$ such that $0<f<1$ on M and

$$\begin{cases} f(\mathbf{x})=1;\\ f|_{K\cap\Delta_M}=0.\end{cases}$$

By Proposition 1.1, there exist unique $h\in\mathcal{HBD}_{\mathcal{A}}(M)$ and $g\in\mathcal{BD}_{p,0}(M)$ such that $f=h+g$, where $\mathcal{A}$ is an elliptic operator on M satisfying conditions $(A1),(A2),(A7)$ and $(A8)$. Since g belongs to $\mathcal{BD}_{p,0}(M)$, by Lemma 2.2, $g=0$ on Δ_M. From this fact together with the maximum principle, one can conclude that $0<h<1$ on M and

$$\begin{cases} h(\mathbf{x})=1;\\ h|_{K\cap\Delta_M}=0.\end{cases}$$

Let us consider the set

$$\Omega=\{x\in M:h(x)>1-\varepsilon\},$$

where ε is a positive constant so small that $1-\varepsilon>0$. Clearly, Ω is an $\mathcal{A}$-massive subset of M. Similarly arguing as (2.2), we have $\lambda_p(\mathcal{G}_\Omega)<\infty$. Let us denote K_1 to be the closure of the set $\bigcup_{\gamma\in\mathcal{G}_\Omega\setminus\mathcal{G}_0}e(\gamma)$ in $\partial\hat{M}$. Since $\lambda_p(\mathcal{G}_\Omega\setminus\mathcal{G}_0)<\infty$, by Lemma 2.5 again, we have $K_1\cap\Delta_M\neq\emptyset$. Since $K_1\cap\Delta_M$ is a subset of $K\cap\Delta_M$, we conclude that

$$h|_{K_1\cap\Delta_M}=0.$$

On the other hand, from the definition of Ω, we see that $h(\gamma)\geq 1-\varepsilon$ for all curves $\gamma\in\mathcal{G}_\Omega$, where $h(\gamma)=\lim_{t\to\infty}h(\gamma(t))$. Hence we have

$$h|_{K_1}\geq 1-\varepsilon$$

which is a contradiction. This completes the proof. □

3. The proof of Theorem 1.2

We are ready to prove the main theorem which gives a connection between a sort of asymptotic behavior of functions in $\mathcal{BD}_p(M)$ near infinity of M and the values of the functions at Δ_M:

Proof of Theorem 1.2. Suppose that $f|_{\Delta_M} = 0$. By considering the positive part and the negative part of f separately, we may assume that $f \geq 0$. For each positive integer n, let us consider the family of curves

$$\mathcal{G}_n = \left\{ \gamma \in \mathcal{G} : f(\gamma) \geq \frac{1}{n} \right\},$$

where $f(\gamma) = \lim_{t\to\infty} f(\gamma(t))$. Since $f|_{\Delta_M} = 0$, we conclude that $K_n \cap \Delta_M = \emptyset$ for each positive integer n, where K_n is the closure of the set $\bigcup_{\gamma\in\mathcal{G}_n} e(\gamma)$ in $\partial\hat{M}$. Hence, by Lemma 2.5, one can conclude that $\lambda_p(\mathcal{G}_n) = \infty$ for each positive integer n. Then $\lim_{t\to\infty} f(\gamma(t)) = 0$ for all curves $\gamma \in \mathcal{G} \setminus \mathcal{G}_\infty$, where $\mathcal{G}_\infty = \bigcup_{n=1}^{\infty} \mathcal{G}_n$. Since $\lambda_p(\mathcal{G}_\infty) = \infty$, we have $\lim_{t\to\infty} f(\gamma(t)) = 0$ for p-almost every curve $\gamma \in \mathcal{G}$.

On the other hand, the converse follows immediately from Lemma 2.6. This completes the proof. □

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Seok Woo Kim
Department of Mathematics Education
Konkuk University
Seoul 143-701, Korea
E-mail address: swkim@konkuk.ac.kr

Sang Moon Lee
Department of Mathematics
Konkuk University
Seoul 143-701, Korea
E-mail address: upsm99@konkuk.ac.kr

Yong Hah Lee
Department of Mathematics Education
Ewha Womans University
Seoul 120-750, Korea
E-mail address: yonghah@ewha.ac.kr