

POSITIVE SOLUTIONS OF BOUNDARY VALUE PROBLEMS FOR p -LAPLACIAN WITH SIGN-CHANGING NONLINEAR TERMS

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ABSTRACT. By using the fixed point index theory, we investigate the existence of at least two positive solutions for p -Laplace equation with sign-changing nonlinear terms $(\varphi_p(u'))' + a(t)f(t, u(t), u'(t)) = 0$, subject to some boundary conditions. As an application, we also give an example to illustrate our results.

1. Introduction

The existence of positive solutions on boundary value problems for ordinary differential equations has been extensively studied by many authors, one may see [2, 3, 4, 5, 6], [8, 9, 11, 12, 14] and references therein. So far as we know, most results have been obtained mainly by using the monotone iterative and upper-lower solutions technique [2, 11, 14], the fixed point index theory [8], the nonlinear alternative of Leray-Schauder [5], and some fixed point theorems [3, 4, 6, 9], and some new existence principles [12], and so on. In order to apply the concavity of solutions in the proofs, almost all existing works were done under the assumption that the nonlinear term is nonnegative. In this paper, we eliminate the nonnegative condition imposed on the nonlinearity f , which is a crucial condition in the proof of these literatures. Clearly, comparing to the equations with nonnegative conditions imposed on f , the equations with sign-changing nonlinearities that we shall consider, to a certain extent, reflect even more exactly the physical reality. Very recently, when $p \neq 2$, in [10], Lü et al. studied the existence of positive solution for singular p -Laplace equation with sign changing nonlinearities

$$-(\varphi_p(u'))' = q(t)f(u) + r(t)g(u), \quad 0 < t < 1$$

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subject to the boundary condition

$$u(0) = u(1) = 0.$$

The main tool is the upper and lower solutions method. When $p = 2$, in [5] Li and Sun considered the existence of nontrivial solution for three-point boundary value problem

$$\begin{aligned} u'' + f(t, u) &= 0, \quad 0 < t < 1 \\ au(0) - bu'(0) &= 0, \quad u(1) - \alpha u(\eta) = 0, \end{aligned}$$

where $\eta \in (0, 1)$, $\alpha, a, b \in \mathbb{R}$, $f \in C([0, 1] \times \mathbb{R}, \mathbb{R})$, $\mathbb{R} = (-\infty, +\infty)$. In [7], Li investigated the existence of positive solution for second-order boundary value problem

$$\begin{aligned} u'' + f(t, u) &= 0, \quad 0 < t < 1 \\ u(0) &= u(1) = 0, \end{aligned}$$

where $f \in C([0, 1] \times \mathbb{R}^+, \mathbb{R})$, the main tool is the fixed point index theorem.

Motivated by [10, 5, 7], in this paper, we consider the existence of two positive solutions for p -Laplace equation with sign-changing nonlinear terms

$$(1.1) \quad (\varphi_p(u'))' + a(t)f(t, u(t), u'(t)) = 0, \quad 0 < t < 1$$

subject to one of the following boundary conditions

$$(1.2) \quad u(0) - B_1(u'(\eta)) = 0, \quad u'(1) = 0$$

or

$$(1.3) \quad u'(0) = 0, \quad u(1) + B_2(u'(\eta)) = 0,$$

where $\varphi_p(x) = |x|^{p-2}x$, $p > 1$, $0 < \eta < 1$, φ_p is an odd, increasing homeomorphism on \mathbb{R} . $\varphi_q(x) = |x|^{q-2}x$ is the inverse function to φ_p , $\frac{1}{p} + \frac{1}{q} = 1$, and the following conditions are satisfied throughout this paper.

(H₁) $f \in C([0, 1] \times [0, \infty) \times \mathbb{R}, \mathbb{R})$;

(H₂) $a(t) \in C([0, 1], [0, \infty))$, $f(t, 0, \cdot) \geq 0$ and $a(t)f(t, \cdot, \cdot)$ is not identical zero on any compact subinterval of $(0, 1)$. Furthermore, $a(t)$ satisfies

$$0 < \int_{\eta}^{1-\eta} a(t) dt < +\infty, \quad \eta \in (0, 1/2).$$

(H₃) B_1, B_2 are both increasing continuous odd functions defined on \mathbb{R} , and there exist nonnegative numbers l, L such that

$$lx \leq B_i(x) \leq Lx, \quad x \in \mathbb{R}^+, \quad i = 0, 1$$

holds.

Equations of the above form occur in the study of radial solutions for the n -dimensional p -Laplace equations, non-Newtonian fluid mechanics and the turbulent flow of gas in porous medium [1] and so on. But to our knowledge, when the nonlinear term f is involved in first-order derivative explicitly, in

particular, involved in changing sign nonlinear terms, few results have been seen in literature for Eq.(1.1) subject to some boundary conditions.

The purpose of this paper is to establish some simple criteria for the existence of at least two positive solutions for Eq.(1.1), (1.2) and (1.1), (1.3). The key tool is the fixed point index theory. Moreover, our works essentially improve and generalize the results in the present literatures.

2. Preliminaries and lemmas

In order to prove our main results, we provide some definitions and lemmas as follows.

Definition 2.1. Let $(E, \|\cdot\|)$ be a real Banach space. A nonempty, closed, convex set $P \subset E$ is called a cone if it satisfies the following two conditions:

- (i) $u \in P, \lambda \geq 0$, implies $\lambda u \in P$;
- (ii) $u \in P, -u \in P$, implies $u = 0$.

Every cone $P \subset E$ induces an ordering in E given by $u \leq v$ if and only if $v - u \in P$.

Definition 2.2. The map α is said to be a nonnegative continuous concave functional on cone P of a real Banach space E if

$$\alpha : P \rightarrow [0, \infty)$$

is continuous and

$$\alpha(\tau u + (1 - \tau)v) \geq \tau\alpha(u) + (1 - \tau)\alpha(v)$$

for all $u, v \in P$ and $\tau \in [0, 1]$. Similarly, we say the map β is a nonnegative continuous convex functional on a cone P of a real Banach space E if

$$\beta : P \rightarrow [0, \infty)$$

is continuous and

$$\beta(\tau u + (1 - \tau)v) \leq \tau\beta(u) + (1 - \tau)\beta(v)$$

for all $u, v \in P$ and $\tau \in [0, 1]$.

Let the Banach space $E = C[0, 1]$ be endowed with the maximum norm $\|u\| = \max\{|u(t)|, t \in [0, 1]\}$. Let $P = \{u \in E : u(t) \geq 0, t \in [0, 1]\}$, $P' = \{u \in E : u(t) \text{ is a nonnegative continuous concave function on } [0, 1]\}$, it is easy to see that P, P' are cones in E . Let $P_r = \{u \in P : \|u\| < r\}$. Then $\partial P_r = \{u \in P : \|u\| = r\}$, $\overline{P}_r = \{u \in P : \|u\| \leq r\}$.

For notational convenience, we write

$$u^+(t) = \max_{0 \leq t \leq 1} \{u(t), 0\}, \quad \forall u(t) \in E = C[0, 1].$$

Thus, we have:

Lemma 2.1. *Assume that $A : E \rightarrow E$ is a completely continuous operator. Defined operator $A^+ : A(E) \rightarrow P$ given by $(A^+u)(t) = u^+(t)$ for $u(t) \in A(E), t \in [0, 1]$. Then $A^+ \circ A : E \rightarrow P$ is also a completely continuous operator.*

Proof. By the completely continuity of A , we can know that there are $u_i \in E, i = 1, 2, \dots, m$, for any bounded set $S \subset E$ and any $\varepsilon > 0$, such that

$$A(S) \subset \bigcup_{i=1}^m B(u_i, \varepsilon),$$

where $B(u_i, \varepsilon) = \{u \in E : \|u - u_i\| < \varepsilon\}$. Therefore, for any $\bar{u} \in (A^+ \circ A)(S)$, there is $u \in A(S)$ such that $\bar{u} = \max_{0 \leq t \leq 1} \{u(t), 0\} = u^+(t)$. Thus, there exists $u_i, i = 1, 2, \dots, m$, such that $\max_{0 \leq t \leq 1} |u(t) - u_i(t)| < \varepsilon$. And we have

$$\max_{0 \leq t \leq 1} |\bar{u}(t) - \bar{u}_i(t)| \leq \max_{0 \leq t \leq 1} |u(t) - u_i(t)| < \varepsilon,$$

namely, $\bar{u} \in B(\bar{u}_i, \varepsilon)$. Hence, $(A^+ \circ A)(S)$ is relatively compact.

On the other hand, for arbitrary $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\|Au - Av\| < \varepsilon, \quad \text{when } \|u - v\| < \delta, \quad \forall u, v \in E,$$

hence,

$$\begin{aligned} \|(A^+ \circ A)(u) - (A^+ \circ A)(v)\| &= \max_{0 \leq t \leq 1} |(\overline{Au})(t) - (\overline{Av})(t)| \\ &\leq \max_{0 \leq t \leq 1} |(Au)(t) - (Av)(t)| \\ &= \|Au - Av\| < \varepsilon. \end{aligned}$$

Therefore, $A^+ \circ A$ is continuous on E , that is, $A^+ \circ A$ is completely continuous. \square

Lemma 2.2 ([8, Lemma 2.2]). *Let $u \in P', \omega \in (0, \frac{1}{2})$. Then*

$$u(t) \geq \omega \|u\|, t \in [\omega, 1 - \omega].$$

Lemma 2.3 ([13, Theorem 2.1]). *Let E be a Banach space, P be a cone in E . Assume that $T : P_r \rightarrow P$ is completely continuous, and such that $Tx \neq x$ for $x \in \partial P_r$.*

- (i) *If $\|x\| \leq \|Tx\|$ for $x \in \partial P_r$, then $i(T, P_r, P) = 0$;*
- (ii) *If $\|x\| \geq \|Tx\|$ for $x \in \partial P_r$, then $i(T, P_r, P) = 1$.*

3. Existence of two positive solutions of Problem(1.1)-(1.2)

In order to state and prove the our main result, we need the following lemma and operators.

Lemma 3.1. *Let $x(t) \in C^1[0, 1], x(t) \geq 0$. Then p -Laplace boundary value problem*

$$(3.1) \quad (\varphi_p(u'(t)))' + f(t, x(t), x'(t)) = 0, \quad 0 < t < 1,$$

$$(3.2) \quad u(0) - B_1(u'(\eta)) = 0, \quad u'(1) = 0,$$

has a unique solution

$$(3.3) \quad u(t) = B_1 \left(\varphi_q \left(\int_{\eta}^1 f(r, x(r), x'(r)) \, dr \right) \right) + \int_0^t \varphi_q \left(\int_s^1 f(r, x(r), x'(r)) \, dr \right) \, ds.$$

Proof. First, we prove that the existence of solution is satisfied. Integrating (3.1) from t to 1, and by (3.2), we have

$$\varphi_p(u'(t)) = \int_t^1 f(r, x(r), x'(r)) \, dr,$$

i.e.,

$$(3.4) \quad u'(t) = \varphi_q \left(\int_t^1 f(r, x(r), x'(r)) \, dr \right).$$

Integrating (3.4) from 0 to t , we get

$$u(t) - u(0) = \int_0^t \varphi_q \left(\int_s^1 f(r, x(r), x'(r)) \, dr \right) \, ds.$$

By (3.4), we have

$$u'(\eta) = \varphi_q \left(\int_{\eta}^1 f(r, x(r), x'(r)) \, dr \right).$$

Hence, by (3.2), we get

$$u(0) = B_1(u'(\eta)) = B_1 \left(\varphi_q \left(\int_{\eta}^1 f(r, x(r), x'(r)) \, dr \right) \right).$$

Therefore,

$$u(t) = B_1 \left(\varphi_q \left(\int_{\eta}^1 f(r, x(r), x'(r)) \, dr \right) \right) + \int_0^t \varphi_q \left(\int_s^1 f(r, x(r), x'(r)) \, dr \right) \, ds.$$

Next, we claim that the uniqueness of solution is also held. Let (3.1)-(3.2) have another solution v . Then

$$(3.5) \quad (\varphi_p(v'(t)))' + f(t, x(t), x'(t)) = 0, \quad 0 < t < 1,$$

$$(3.6) \quad v(0) - B_1(v'(\eta)) = 0, \quad v'(1) = 0.$$

Similarly, we can obtain

$$v(t) = B_1 \left(\varphi_q \left(\int_{\eta}^1 f(r, x(r), x'(r)) \, dr \right) \right) + \int_0^t \varphi_q \left(\int_s^1 f(r, x(r), x'(r)) \, dr \right) \, ds.$$

Thus, $u(t) = v(t)$ for $t \in [0, 1]$.

The proof of Lemma 3.1 is complete. □

Now, we define operators A, B, F as follows:

$A : P \rightarrow E$ given by

$$(Au)(t) = B_1 \left(\varphi_q \left(\int_{\eta}^1 a(r) f(r, u(r), u'(r)) \, dr \right) \right) \\ + \int_0^t \varphi_q \left(\int_s^1 a(r) f(r, u(r), u'(r)) \, dr \right) \, ds, \quad t \in [0, 1].$$

Thanks to Lemma 3.1 and the condition (H_2) , each fixed point of A in P is a positive solution of $BVP(1.1)$, (1.2).

$B : P \rightarrow P$ given by

$$(Bu)(t) = \left[B_1 \left(\varphi_q \left(\int_{\eta}^1 a(r) f(r, u(r), u'(r)) \, dr \right) \right) \right. \\ \left. + \int_0^t \varphi_q \left(\int_s^1 a(r) f(r, u(r), u'(r)) \, dr \right) \, ds \right]^+, \quad t \in [0, 1].$$

Obviously, $B = A^+ \circ A$.

$F : P' \rightarrow P$ given by

$$(Fu)(t) = B_1 \left(\varphi_q \left(\int_{\eta}^1 a(r) f^+(r, u(r), u'(r)) \, dr \right) \right) \\ + \int_0^t \varphi_q \left(\int_s^1 a(r) f^+(r, u(r), u'(r)) \, dr \right) \, ds, \quad t \in [0, 1].$$

It is easy to prove that $F(P') \subset P'$, and $\|Fu\| = (Fu)(1)$.

For notational convenience, we introduce the following constants:

$$M = L\varphi_q \left(\int_{\eta}^1 a(r) \, dr \right) + \int_0^1 \varphi_q \left(\int_s^1 a(r) \, dr \right) \, ds,$$

$$N = l\varphi_q \left(\int_{\eta}^{1-\eta} a(r) \, dr \right) + \int_{\eta}^{1-\eta} \varphi_q \left(\int_s^{1-\eta} a(r) \, dr \right) \, ds.$$

The first main result of this paper is as follows:

Theorem 3.1. *Assume that $(H_1), (H_2), (H_3)$ hold, and suppose that there exist positive constants a, b, d such that $0 < \frac{1}{\eta}d < a < b$. Also assume that f satisfies the following conditions:*

- (H_4) $f(t, u, u') \geq 0$ for $(t, u, u') \in [\eta, 1 - \eta] \times [d, b] \times \mathbb{R}$;
- (H_5) $f(t, u, u') < \varphi_p(a/M)$ for $(t, u, u') \in [0, 1] \times [0, a] \times \mathbb{R}$;
- (H_6) $f(t, u, u') > \varphi_p(b/N)$ for $(t, u, u') \in [\eta, 1 - \eta] \times [\eta b, b] \times \mathbb{R}$.

Then $BVP(1.1)$, (1.2) has at least two positive solutions u_1 and u_2 such that

$$0 < \|u_1\| < a < \|u_2\| < b.$$

Proof. Firstly, in view of the definitions of the operators A, F and the continuity of f , applying the Lebesgue's dominated convergence theorem and the Arzela-Ascoli theorem, it is easy to prove that $A : P \rightarrow E$ and $F : P' \rightarrow P'$ are completely continuous. Thus, it follows from Lemma 2.1 that $A^+ \circ A : P \rightarrow P$ is also completely continuous operator, i.e., $B : P \rightarrow P$ is a completely continuous operator.

Secondly, we show that B has a fixed point $u_1 \in P$ with $0 < \|u_1\| < a$. We choose $u \in \partial P_a$, then $\|u\| = a$ and $0 \leq u(t) \leq a$ for $t \in [0, 1]$. By (H_3) and (H_5) , we get

$$\begin{aligned} \|(Bu)\| &= \max_{0 \leq t \leq 1} \left[B_1 \left(\varphi_q \left(\int_{\eta}^1 a(r)f(r, u(r), u'(r)) \, dr \right) \right) \right. \\ &\quad \left. + \int_0^t \varphi_q \left(\int_s^1 a(r)f(r, u(r), u'(r)) \, dr \right) \, ds \right]^+ \\ &\leq \max_{0 \leq t \leq 1} \left(L\varphi_q \left(\int_{\eta}^1 a(r)f(r, u(r), u'(r)) \, dr \right) \right. \\ &\quad \left. + \int_0^t \varphi_q \left(\int_s^1 a(r)f(r, u(r), u'(r)) \, dr \right) \, ds \right) \\ &< \frac{a}{M} \left(L\varphi_q \left(\int_{\eta}^1 a(r) \, dr \right) + \int_0^1 \varphi_q \left(\int_s^1 a(r) \, dr \right) \, ds \right) \\ &= a = \|u\|. \end{aligned}$$

It follows from Lemma 2.3 that

$$i(B, P_a, P) = 1.$$

Consequently, B has a fixed point u_1 in P_a such that $0 < \|u_1\| < a$.

We show that u_1 is also a fixed point of A in P_a below.

Suppose that u_1 is not a fixed point of A in P_a . Then there exists $t_0 \in [0, 1]$ such that

$$(Au_1)(t_0) \neq u_1(t_0) = (Bu_1)(t_0) = \max\{(Au_1)(t_0), 0\}.$$

Therefore, $(Au_1)(t_0) < 0 = u_1(t_0)$. In view of the continuity of A , we know that there exists the neighborhood of t_0 denoted by $N(t_0, \delta) \subset [0, 1]$, such that

$$(Au_1)(t) < 0 = u_1(t) = (Bu_1)(t), \forall t \in N(t_0, \delta).$$

It follows from the definition of B that

$$\begin{aligned} (Bu_1)(t) &= \left[B_1 \left(\varphi_q \left(\int_{\eta}^1 a(r)f(r, u_1(r), u_1'(r)) \, dr \right) \right) \right. \\ &\quad \left. + \int_0^t \varphi_q \left(\int_s^1 a(r)f(r, u_1(r), u_1'(r)) \, dr \right) \, ds \right]^+ \\ &= u_1(t) = 0, \quad t \in N(t_0, \delta). \end{aligned}$$

This contradicts the condition (H_2) . Hence, u_1 is a fixed point of A in P_a .

Finally, we show the existence of another fixed point of A such that $a < \|u_2\| < b$. For this, let $u \in \partial P'_a$, similar to the proof above, it follows from condition (H_3) and (H_5) that

$$\begin{aligned} \|Fu\| &= B_1 \left(\varphi_q \left(\int_{\eta}^1 a(r) f^+(r, u(r), u'(r)) \, dr \right) \right) \\ &\quad + \int_0^1 \varphi_q \left(\int_s^1 a(r) f^+(r, u(r), u'(r)) \, dr \right) \, ds \\ &\leq L\varphi_q \left(\int_{\eta}^1 a(r) f^+(r, u(r), u'(r)) \, dr \right) \\ &\quad + \int_0^1 \varphi_q \left(\int_s^1 a(r) f^+(r, u(r), u'(r)) \, dr \right) \, ds \\ &< \frac{a}{M} \left(L\varphi_q \left(\int_{\eta}^1 a(r) \, dr \right) + \int_0^1 \varphi_q \left(\int_s^1 a(r) \, dr \right) \, ds \right) \\ &= a = \|u\|. \end{aligned}$$

In addition, let $u \in \partial P'_b$, then $\|u\| = b$. Thanks to Lemma 2.2, there exists $\eta > 0$ such that $\eta b \leq u(t) \leq b$ for $t \in [\eta, 1 - \eta]$. It follows from condition (H_3) and (H_6) that

$$\begin{aligned} \|Fu\| &= B_1 \left(\varphi_q \left(\int_{\eta}^1 a(r) f^+(r, u(r), u'(r)) \, dr \right) \right) \\ &\quad + \int_0^1 \varphi_q \left(\int_s^1 a(r) f^+(r, u(r), u'(r)) \, dr \right) \, ds \\ &\geq l\varphi_q \left(\int_{\eta}^1 a(r) f^+(r, u(r), u'(r)) \, dr \right) \\ &\quad + \int_0^1 \varphi_q \left(\int_s^1 a(r) f^+(r, u(r), u'(r)) \, dr \right) \, ds \\ &> \frac{b}{N} \left(l\varphi_q \left(\int_{\eta}^{1-\eta} a(r) \, dr \right) + \int_{\eta}^{1-\eta} \varphi_q \left(\int_s^{1-\eta} a(r) \, dr \right) \, ds \right) \\ &= b = \|u\|. \end{aligned}$$

By Lemma 2.3, we have

$$i(F, P'_a, P') = 1, \quad i(F, P'_b, P') = 0.$$

Thus $i(F, P'_b \setminus \overline{P'_a}, P') = -1$, that is, F has a fixed point u_2 in $P'_b \setminus \overline{P'_a}$, and $a < \|u_2\| < b$.

In the following we prove that u_2 is also a fixed point of A in $P'_b \setminus \overline{P'_a}$. In fact, for $u_2 \in (P'_b \setminus \overline{P'_a}) \cap \{u : Fu = u\}$, we have $u_2(1) = \|u_2\| > a$. Thus it follows from Lemma 2.2 that

$$\min_{\eta \leq t \leq 1-\eta} u_2(t) \geq \eta u_2(1) > \eta a > d.$$

Therefore, we have $d \leq u_2(t) \leq b$ for $t \in [\eta, 1 - \eta]$, from condition (H_4) , we get $f^+(t, u_2(t)) = f(t, u_2(t))$. It implies $Au_2 = Fu_2 = u_2$. Consequently, u_2 is a fixed point of A in cone P' . Namely, u_2 is another positive solution $BVP(1.1)$, (1.2).

The proof of Theorem 3.1 is complete. □

4. Existence of two positive solutions of Problem(1.1)-(1.3)

In this section, we give another main result of this paper and a lemma that is used in the proof of main result.

Lemma 4.1. *Let $x(t) \in C^1([0, 1], x(t) \geq 0$. Then p -Laplacian boundary value problem*

$$(4.1) \quad (\varphi_p(u'(t)))' + f(t, x(t), x'(t)) = 0, \quad 0 < t < 1,$$

$$(4.2) \quad u'(0) = 0, \quad u(1) + B_2(u'(\eta)) = 0,$$

has a unique solution

$$(4.3) \quad u(t) = B_2 \left(\varphi_q \left(\int_0^\eta f(r, x(r), x'(r)) \, dr \right) \right) + \int_t^1 \varphi_q \left(\int_0^s f(r, x(r), x'(r)) \, dr \right) \, ds.$$

Proof. The proof of Lemma 4.1 is similar to that of Lemma 3.1, we omit it here. □

Now, we define operators $\widehat{A}, \widehat{B}, \widehat{F}$ as follows:

$\widehat{A} : P \rightarrow E$ given by

$$(\widehat{A}u)(t) = B_2 \left(\varphi_q \left(\int_0^\eta a(r)f(r, u(r), u'(r)) \, dr \right) \right) + \int_t^1 \varphi_q \left(\int_0^s a(r)f(r, u(r), u'(r)) \, dr \right) \, ds, \quad t \in [0, 1].$$

According to Lemma 4.1 and the condition (H_2) , each fixed point of \widehat{A} in P is a positive solution of $BVP(1.1)$, (1.3).

$\widehat{B} : P \rightarrow P$ given by

$$(\widehat{B}u)(t) = \left[B_2 \left(\varphi_q \left(\int_0^\eta a(r)f(r, u(r), u'(r)) \, dr \right) \right) + \int_t^1 \varphi_q \left(\int_0^s a(r)f(r, u(r), u'(r)) \, dr \right) \, ds \right]^+, \quad t \in [0, 1].$$

Obviously, $\widehat{B} = \widehat{A}^+ \circ \widehat{A}$.

$\widehat{F} : P' \rightarrow P$ given by

$$(\widehat{F}u)(t) = B_2 \left(\varphi_q \left(\int_0^\eta a(r)f^+(r, u(r), u'(r)) \, dr \right) \right)$$

$$+ \int_t^1 \varphi_q \left(\int_0^s a(r) f^+(r, u(r), u'(r)) \, dr \right) \, ds, \quad t \in [0, 1].$$

It is easy to see that $\widehat{F}(P') \subset P'$, and $\|\widehat{F}u\| = (\widehat{F}u)(0)$.

For notational convenience, we introduce the following constants:

$$\begin{aligned} \widehat{M} &= L\varphi_q \left(\int_0^\eta a(r) \, dr \right) + \int_0^1 \varphi_q \left(\int_0^s a(r) \, dr \right) \, ds, \\ \widehat{N} &= l\varphi_q \left(\int_\eta^{1-\eta} a(r) \, dr \right) + \int_\eta^{1-\eta} \varphi_q \left(\int_\eta^s a(r) \, dr \right) \, ds. \end{aligned}$$

Theorem 4.1. *Assume that $(H_1), (H_2), (H_3)$ hold, and suppose that there exist positive constants a, b, d such that $0 < \frac{1}{\eta}d < a < b$. Also assume that f satisfies the following conditions:*

- (H_7) $f(t, u, u') \geq 0$ for $(t, u, u') \in [\eta, 1 - \eta] \times [d, b] \times \mathbb{R}$;
- (H_8) $f(t, u, u') < \varphi_p(a/\widehat{M})$ for $(t, u, u') \in [0, 1] \times [0, a] \times \mathbb{R}$;
- (H_9) $f(t, u, u') > \varphi_p(b/\widehat{N})$ for $(t, u, u') \in [\eta, 1 - \eta] \times [\eta b, b] \times \mathbb{R}$.

Then BVP(1.1), (1.3) has at least two positive solutions u_1 and u_2 such that

$$0 < \|u_1\| < a < \|u_2\| < b.$$

Proof. The proof of Theorem 4.1 is similar to that of Theorem 3.1, we omit it here. □

5. Application

In order to illustrate our results, we present an example as follows.

Example. Consider the boundary value problem involving p -Laplacian with sign-changing nonlinear terms

$$(5.1) \quad \begin{cases} (\varphi_p(u'(t)))' + a(t)f(t, u(t), u'(t)) = 0, & 0 < t < 1, \\ u(0) - B_1(u'(\eta)) = 0, & u'(1) = 0, \end{cases}$$

where

$$p = \frac{3}{2}, \quad \eta = \frac{1}{3}, \quad a(t) = 2t, \quad l = 0, \quad L = 1, \quad B_1(x) = \arctan x,$$

and

$$f(t, u, u') = \begin{cases} \frac{1}{4}t^2 + \frac{1}{10}(u - 1)^2 + \frac{\sin u'}{100}, & (t, u, u') \in [0, 1] \times [0, 6] \times \mathbb{R}, \\ \frac{1}{4}t^2 - \frac{3}{2}(u - 16)^2 + \frac{305}{2} + \frac{\sin u'}{100}, & (t, u, u') \in [0, 1] \times [6, +\infty) \times \mathbb{R}. \end{cases}$$

Then (5.1) has at least two positive solutions.

Proof. In this example, it follows from a direct calculation that

$$M = \varphi_q \left(\int_{\frac{1}{3}}^s a(r) \, dr \right) + \int_0^1 \varphi_q \left(\int_s^1 a(r) \, dr \right) \, ds = \frac{536}{405},$$

$$N = \int_{\frac{1}{3}}^{\frac{2}{3}} \varphi_q \left(\int_s^{\frac{2}{3}} a(r) dr \right) ds = \frac{53}{3645},$$

Clearly, conditions $(H_1), (H_2), (H_3)$ are satisfied.

Let $a = 4, b = 24, d = 1$, thus, we have

$$\begin{aligned} f(t, u, u') &\geq 0 \quad \text{for } (t, u, u') \in \left[\frac{1}{3}, \frac{2}{3}\right] \times [1, 24] \times \mathbb{R}, \\ f(t, u, u') &\leq \max f(t, u, u') = \frac{1}{4} + \frac{9}{10} + \frac{1}{100} < \varphi_p \left(\frac{a}{M} \right) \\ &= \varphi_p \left(\frac{405 \times 4}{536} \right) \quad \text{for } (t, u, u') \in [0, 1] \times [0, 4] \times \mathbb{R}, \end{aligned}$$

and

$$\begin{aligned} f(t, u, u') &\geq \min f(t, u, u') = \frac{1}{36} + \frac{113}{2} \\ &> \varphi_p \left(\frac{b}{N} \right) = \varphi_p \left(\frac{3645}{53} \times 24 \right) \quad \text{for } (t, u, u') \in \left[\frac{1}{3}, \frac{2}{3}\right] \times [8, 24] \times \mathbb{R}. \end{aligned}$$

Consequently, conditions $(H_4), (H_5), (H_6)$ of Theorem 3.1 are satisfied. Then by Theorem 3.1, the boundary value problem involving p -Laplacian with sign-changing nonlinear terms (5.1) has two positive solutions, and such that

$$0 < \|u_1\| < a < \|u_2\| < b. \quad \square$$

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References

- [1] R. P. Agarwal, D. O'Regan, and J. Y. Wong, *Positive Solutions of Differential, Difference and Integral Equations*, Kluwer Academic Publishers, Dordrecht, 1999.
- [2] X. Du and Z. Zhao, *On existence theorems of positive solutions to nonlinear singular differential equations*, Appl. Math. Comput. **190** (2007), no. 1, 542–552.
- [3] X. He, *Double positive solutions of a three-point boundary value problem for the one-dimensional p -Laplacian*, Appl. Math. Lett. **17** (2004), no. 8, 867–873.
- [4] J. Li and J. Shen, *Existence of three positive solutions for boundary value problems with p -Laplacian*, J. Math. Anal. Appl. **311** (2005), no. 2, 457–465.
- [5] S. Li and Y. Sun, *Nontrivial solution of a nonlinear second-order three-point boundary value problem*, Appl. Math. J. Chinese Univ. Ser. B **22** (2007), no. 1, 37–47.
- [6] X. Li, *Multiple positive solutions for some four-point boundary value problems with p -Laplacian*, Appl. Math. Comput. **202** (2008), no. 1, 413–426.
- [7] Y. Li, *Positive solutions of second-order boundary value problems with sign-changing nonlinear terms*, J. Math. Anal. Appl. **282** (2003), no. 1, 232–240.
- [8] B. Liu, *Positive solutions of three-point boundary value problems for the one-dimensional p -Laplacian with infinitely many singularities*, Appl. Math. Lett. **17** (2004), no. 6, 655–661.
- [9] Y. Liu and W. Ge, *Multiple positive solutions to a three-point boundary value problem with p -Laplacian*, J. Math. Anal. Appl. **277** (2003), no. 1, 293–302.
- [10] H. Lü, D. O'Regan, and R. P. Agarwal, *Positive solutions for singular p -Laplacian equations with sign changing nonlinearities using inequality theory*, Appl. Math. Comput. **165** (2005), no. 3, 587–597.

- [11] D. Ma and W. Ge, *Existence and iteration of positive pseudo-symmetric solutions for a three-point second-order p -Laplacian BVP*, Appl. Math. Lett. **20** (2007), no. 12, 1244–1249.
- [12] D. Ma, J. Han, and X. Chen, *Positive solution of three-point boundary value problem for the one-dimensional p -Laplacian with singularities*, J. Math. Anal. Appl. **324** (2006), no. 1, 118–133.
- [13] H. Su, Z. Wei, and B. Wang, *The existence of positive solutions for a nonlinear four-point singular boundary value problem with a p -Laplacian operator*, Nonlinear Anal. **66** (2007), no. 10, 2204–2217.
- [14] B. Sun and W. Ge, *Successive iteration and positive pseudo-symmetric solutions for a three-point second-order p -Laplacian boundary value problems*, Appl. Math. Comput. **188** (2007), no. 2, 1772–1779.

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