COMPARISON THEOREMS ON THE OSCILLATION OF A CLASS OF NEUTRAL DIFFERENCE EQUATIONS WITH CONTINUOUS VARIABLES

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Abstract. In this paper, we introduce an iterative method to study oscillatory properties of delay difference equations of the following form

$$\nabla_\alpha \left[x(t) - r(t) x(t - \kappa) \right] + p(t) x(t - \tau) - q(t) x(t - \sigma) = 0, \quad t \geq t_0,$$

where $t_0 \in \mathbb{R}$, t varies in the real interval $[t_0, \infty)$, $\alpha > 0$, $\kappa, \tau, \sigma \geq 0$, $r \in C\left([t_0 - \alpha, \infty), \mathbb{R}^+\right)$, $p, q \in C\left([t_0, \infty), \mathbb{R}^+\right)$ and $\nabla_\alpha x(t) = x(t) - x(t - \alpha)$ for $t \geq t_0$.

1. Introduction

There are so many studies developed on the oscillation of differential equations and difference equations in the past two decades. However, there are very few studies investigating the oscillatory behavior of difference equations with continuous variables. The readers are referred to [1] for the fundamentals of the oscillation theory of differential equations and [2]-[9] for fundamental results on difference equations with continuous arguments. In this paper, we study oscillatory behavior of a neutral difference equation of the form

$$(1) \qquad \nabla_\alpha \left[x(t) - r(t) x(t - \kappa) \right] + p(t) x(t - \tau) - q(t) x(t - \sigma) = 0,$$

where $t \geq t_0$ and t travels through reals, $r \in C\left([t_0 - \alpha, \infty), \mathbb{R}^+\right)$, $p, q \in C\left([t_0, \infty), \mathbb{R}^+\right)$, $\kappa > \alpha > 0$, $\tau > \sigma > \alpha$. Here, ∇_α denotes the backward difference operator with the step α, that is, $\nabla_\alpha x(t) = x(t) - x(t - \alpha)$ for $t \geq t_0$. For convenience in the paper, we set

$$\beta := \begin{cases} \tau, & r \equiv 0 \\ \max\{\kappa + \alpha, \tau\}, & r \not\equiv 0 \end{cases} \quad \text{and} \quad \gamma := \begin{cases} \sigma, & r \equiv 0 \\ \min\{\kappa, \sigma\}, & r \not\equiv 0. \end{cases}$$

A function $x \in C\left([t_0 - \beta, \infty), \mathbb{R}\right)$ is called a *solution* of (1) if x satisfies (1) on $[t_0, \infty)$. As is customary, a solution of (1) is called *oscillatory* if it

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has arbitrary large zeros; otherwise, such a solution is called *nonoscillatory*. Throughout the work, eventually trivial solutions of (1) are out of our interest.

2. Main results

For an arbitrary continuous function f denote the n-th minimized function by

$$f^{(n)}(t) := \begin{cases} f(t), & n = 0 \\ \min\left\{f^{(n-1)}(\eta) : t - \alpha \le \eta \le t\right\}, & n \in \mathbb{N}, \end{cases}$$

and let $h(t) := p(t) - q(t - \tau + \sigma)$.

Before stating our main results, we need to give some lemmas as follows. We start with the following lemma which is an extension of [5, Lemma 1].

Lemma 2.1. *Assume that*

$$(2) \qquad h(t) \ge 0 \ (\not\equiv 0) \quad and \quad \int_{t-\alpha}^{t} r(\eta) \, d\eta + \int_{t-\tau+\sigma}^{t} q(\eta) \, d\eta \le \alpha$$

hold for all sufficiently large t. Let x be an eventually positive solution of (1). Then the companion function of x given by
(3)

$$z_x(t) := \int_{t-\alpha}^{t} x(\eta) \, d\eta - \int_{t-\alpha}^{t} r(\eta) x(\eta - \kappa) \, d\eta - \int_{t-\tau+\sigma}^{t} q(\eta) x(\eta - \sigma) \, d\eta, \ t \ge t_0 + \beta$$

satisfies

$$(4) \qquad\qquad\qquad z_x' \le 0 \ (\not\equiv 0), \quad z_x > 0$$

on a subhalfline of $[t_0, \infty)$. Moreover,

$$(5) \qquad\qquad\qquad \nabla_\alpha z_x(t) + h^{(1)}(t) z_x(t - \tau) \le 0$$

holds on a subhalfline of $[t_0, \infty)$.

Proof. Let $t_1 \ge t_0$ satisfy (2) for all $t \ge t_1$. Using the fact that x is a solution of (1), we have

$$(6) \quad z_x'(t) = \nabla_\alpha \left[x(t) - r(t) x(t - \kappa)\right] - q(t) x(t - \sigma) + q(t - \tau + \sigma) x(t - \tau)$$
$$= -h(t) x(t - \tau) \le 0 \ (\not\equiv 0)$$

for all $t \ge t_2$, where $t_2 \ge t_1 + \tau$. Hence, there exists $t_3 \ge t_2$ such that z_x is of constant sign on $[t_3, \infty)$. We claim that z_x is of positive sign. Assume the contrary that $z_x(t) \le 0$ holds for all $t \ge t_3$. Then, there exist $t_4 \ge t_3$ and a constant $\mu > 0$ satisfying $z_x(t) < -\mu$ for all $t \ge t_4$. Let $\phi : [t_4, \infty) \to \mathbb{R}$ be the function satisfying the mean value

$$\alpha x(\phi(t)) = \int_{t-\alpha}^{t} x(\eta) \, d\eta$$

for all $t \geq t_4$. Note that $\gamma \geq \alpha$ and $t - \alpha \leq \phi(t) \leq t$ for all $t \geq t_4$. In view of the definition of z_x in (3), we obtain

$$(7) \qquad \alpha x(\phi(t)) < -\mu + \left(\int_{t-\alpha}^{t} r(\eta) \, d\eta + \int_{t-\tau+\sigma}^{t} q(\eta) \, d\eta \right) \max_{t-\beta \leq \eta \leq t-\gamma} x(\eta)$$

$$\leq -\mu + \alpha \max_{t-\beta-\alpha \leq \eta \leq t} x(\phi(\eta))$$

for all $t \geq t_4$. By the well-known result [1, Lemma 1.5.4], (7) yields a contradiction that $x \circ \phi$ can not be a positive function. This proves $z_x > 0$ on $[t_3, \infty)$, that is, (4) holds. Integrating (6) over $[t - \alpha, t]$, we get

$$\nabla_\alpha z_x(t) + \int_{t-\alpha}^{t} h(\eta) x(\eta - \tau) \, d\eta = 0,$$

or

$$(8) \qquad \nabla_\alpha z_x(t) + h^{(1)}(t) \int_{t-\alpha}^{t} x(\eta - \tau) \, d\eta \leq 0$$

for all $t \geq t_3$. Taking (3) into account, we see that

$$(9) \qquad \int_{t-\alpha}^{t} x(\eta) \, d\eta \geq z_x(t)$$

holds for all $t \geq t_4$, where $t_4 \geq t_3 + \tau$. Substituting (9) into (8), we see that (5) holds on $[t_4, \infty)$. Hence, the proof is completed. $\qquad\square$

Now, we state the following result extracted from [3]:

Lemma 2.2. *Consider the following delay differential equation*

$$(10) \qquad x'(t) + A(t) x(t - \rho) = 0,$$

where $\rho > 0$ and $A \in C([t_0, \infty), \mathbb{R}^+)$, and the corresponding delay differential inequality

$$(11) \qquad x'(t) + A(t) x(t - \rho) \leq 0.$$

(10) possesses eventually positive solutions if and only if so does (11).

We give the following comparison theorem:

Theorem 2.1. *Assume that (2) holds for all sufficiently large t. If every solution of*

$$(12) \qquad y'(t) + \frac{1}{\alpha} h^{(1)}(t) y(t - \tau + \alpha) = 0$$

is oscillatory, then every solution of (1) is oscillatory.

Proof. Assume the contrary that (1) has nonoscillatory solution x. Since (1) is linear, there is no loss in assuming x as an eventually positive solution. Then,

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we have by Lemma 2.1 that z_x introduced in (3) is nonincreasing. Let $t_1 \geq t_0$ satisfy (2), $x(t) > 0$, $z'_x(t) \leq 0$ ($\not\equiv 0$) and $z_x(t) > 0$ for all $t \geq t_1$. Now, we set

$$(13) \qquad y(t) := \int_{t-\alpha}^{t} z_x(\eta)\,\mathrm{d}\eta$$

for $t \geq t_2$, where $t_2 \geq t_1 + \alpha$, then we have

$$(14) \qquad y'(t) = \nabla_\alpha z_x(t) \leq 0$$

for all $t \geq t_2$. Since z_x is nonincreasing and positive on $[t_2, \infty)$, we deduce from (13) that

$$(15) \qquad 0 < \frac{1}{\alpha} y(t) \leq z_x(t - \alpha)$$

holds for all $t \geq t_2$. Considering (5), (14) and (15), we get that

$$y'(t) + \frac{1}{\alpha} h^{(1)} y(t - \tau + \alpha) \leq 0,$$

which indicates that the corresponding differential equation (12) has an eventually positive solution by Lemma 2.2. This contradiction completes the proof. □

The following corollary allows us to test oscillatory behavior of all solutions of (1). For convenience, we introduce

$$\overline{h} := \limsup_{t \to \infty} \frac{1}{\alpha} \int_{t-\tau+\alpha}^{t} h^{(2)}(\eta)\,\mathrm{d}\eta \quad \text{and} \quad \underline{h} := \liminf_{t \to \infty} \frac{1}{\alpha} \int_{t-\tau+\alpha}^{t} h^{(2)}(\eta)\,\mathrm{d}\eta.$$

Corollary 2.1. *Assume that* (2) *holds for all sufficiently large t. If*

$$(16) \qquad \overline{h} > 1 \quad or \quad \underline{h} > \frac{1}{e}$$

or

$$(17) \qquad \underline{h} \leq \frac{1}{e} \quad and \quad \overline{h} > 1 - \frac{1 - \underline{h} - \sqrt{1 - 2\underline{h} - \underline{h}^2}}{2}$$

holds, then every solution of (1) *is oscillatory.*

Proof. (12) can not have eventually positive solutions under either one of the conditions (16) or (17). □

Now, we introduce

$$\Lambda(t) := \{\lambda > 0 : 1 - \lambda h(s) > 0 \text{ for all } s \geq t\}$$

and

$$\tau_1 := \left\lfloor \frac{\tau}{\alpha} \right\rfloor,$$

where $\lfloor \cdot \rfloor$ denotes the lowest integer function.

Corollary 2.2. *Assume that* (2) *holds for all sufficiently large* t, *and*

$$\limsup_{t \to \infty} h(t) > 0$$

holds. Furthermore, assume that $\tau_1 \in \mathbb{N}$ *and*

$$\limsup_{t \to \infty} \sup_{\lambda \in \Lambda(t)} \left(\lambda \prod_{i=1}^{\tau_1 - 1} \left[1 - \lambda h^{(2)}(t - i\alpha) \right] \right) < 1.$$

Then every solution of (1) *is oscillatory.*

Proof. The proof follows from the papers [2, 4, 7]. $\qquad\square$

3. Iterative results

In this section, we improve the results of the previous section by iteration.

We first improve Lemma 2.1 by introducing $\tau_2 := \lfloor (\tau - \sigma)/\alpha \rfloor$ and the recursion

(18)
$$h_n(t) := \begin{cases} 1, & n = 0 \\ r^{(1)}(t) h_{n-1}(t - \kappa) + \displaystyle\sum_{j=0}^{\tau_2 - 1} q^{(1)}(t - j\alpha) h_{n-1}(t - j\alpha - \sigma), & n \in \mathbb{N}. \end{cases}$$

Lemma 3.1. *Assume that* $\tau_2 \in \mathbb{N}$ *and all conditions of Lemma 2.1 are held. Then, for each* $n \in \mathbb{N}$, *there exists a halfline* $J_n \subset [t_0, \infty)$ *such that* z_x *introduced in* (3) *satisfies*

(19)
$$\nabla_\alpha z_x(t) + h^{(1)}(t) \sum_{i=0}^{n} h_i(t - \tau) z_x(t - \tau) \leq 0$$

for all $t \in J_n$.

Proof. Assume that conclusions of Lemma 2.1 hold for all $t \geq t_1$, where $t_1 \geq t_0$. Let h_n function in (18) be defined for $t \geq t_1 + n\beta$, where $n \in \mathbb{N}$. We see from (3) for $t \geq t_1$ that

(20)
$$\int_{t-\alpha}^{t} x(\eta)\,d\eta = z_x(t) + \int_{t-\alpha}^{t} r(\eta) x(\eta - \kappa)\,d\eta + \int_{t-\tau+\sigma}^{t} q(\eta) x(\eta - \sigma)\,d\eta$$

$$\geq z_x(t) + r^{(1)}(t) \int_{t-\alpha}^{t} x(\eta - \kappa)\,d\eta$$

$$+ \sum_{j=0}^{\tau_2 - 1} q^{(1)}(t - j\alpha) \int_{t-(j+1)\alpha}^{t-j\alpha} x(\eta - \sigma)\,d\eta$$

holds. Now, we prove by induction that

(21)
$$\int_{t-\alpha}^{t} x(\eta)\,d\eta \geq \sum_{i=0}^{m} h_i(t) z_x(t)$$

holds for all $t \geq t_1 + m\beta$ and $m = 0, 1, \ldots, n$. Clearly, the above inequality holds for $m = 0$ trivially from (9). Suppose that (21) holds for $m = n - 1$ and all $t \geq t_1 + (n-1)\beta$. Now, we show that (21) also holds for $m = n$ and all $t \geq t_1 + n\beta$. Consequently, from (20), (21) and the nonincreasing nature of z_x, we have

(22)

$$
\int_{t-\alpha}^{t} x(\eta)\,\mathrm{d}\eta
$$

$$
\geq z_x(t) + r^{(1)}(t) \sum_{i=0}^{n-1} h_i(t-\kappa) z_x(t-\kappa)
$$

$$
+ \sum_{i=0}^{n-1} \sum_{j=0}^{\tau_2-1} q^{(1)}(t-j\alpha) h_i(t-j\alpha-\sigma) z_x(t-j\alpha-\sigma)
$$

$$
\geq \left(1 + \sum_{i=0}^{n-1} \left[r^{(1)}(t) h_i(t-\kappa) + \sum_{j=0}^{\tau_2-1} q^{(1)}(t-j\alpha) h_i(t-j\alpha-\sigma) \right]\right) z_x(t)
$$

$$
= \left(1 + \sum_{i=0}^{n-1} h_{i+1}(t)\right) z_x(t) = \sum_{i=0}^{n} h_i(t) z_x(t)
$$

for all $t \geq t_1 + n\beta$. Substituting (22) into (8), we obtain that (19) holds on $J_n := [t_1 + n\beta, \infty)$. This completes the proof of the lemma.   □

Remark 3.1. Note that Lemma 2.1 is a particular case of Lemma 3.1 with $n = 0$.

Now, we give the following result which is an extension of Theorem 2.1:

Theorem 3.1. *Assume that assumptions of Lemma 3.1 are held. If there exists $n_0 \in \mathbb{N}$ such that every solution of*

(23)
$$
y'(t) + \frac{1}{\alpha} h^{(1)}(t) \sum_{i=0}^{n_0} h_i(t-\tau) y(t-\tau+\alpha) = 0
$$

is oscillatory, then every solution of (1) *is also oscillatory.*

Proof. Since all assumptions of Lemma 2.1, Theorem 2.1 and Lemma 3.1 are held. Considering (14), (15), and substituting (13) into (19), we see that the corresponding equation (23) has an eventually positive solution by Lemma 2.2, and this contradiction completes the proof.   □

For convenience, we need to introduce the followings:

$$
\overline{h}(n) := \limsup_{t \to \infty} \frac{1}{\alpha} \int_{t-\tau+\alpha}^{t} h^{(2)}(\eta) \sum_{i=0}^{n} h_i(\eta-\tau)\,\mathrm{d}\eta,
$$

$$\underline{h}(n) := \liminf_{t \to \infty} \frac{1}{\alpha} \int_{t-\tau+\alpha}^{t} h^{(2)}(\eta) \sum_{i=0}^{n} h_i(\eta - \tau)\,\mathrm{d}\eta.$$

Corollary 3.1. *Assume that assumptions of Lemma 3.1 are held. If there exists $n_0 \in \mathbb{N}$ satisfying*

$$\overline{h}(n_0) > 1 \quad or \quad \underline{h}(n_0) > \frac{1}{\mathrm{e}}$$

or

$$\underline{h}(n_0) \leq \frac{1}{\mathrm{e}} \quad and \quad \overline{h}(n_0) > 1 - \frac{1 - \underline{h}(n_0) - \sqrt{1 - 2\underline{h}(n_0) - \underline{h}(n_0)^2}}{2},$$

then every solution of (1) *is oscillatory.*

Corollary 3.2. *Assume that assumptions of Lemma 3.1 are held. If $\underline{h}(\infty) > 1/\mathrm{e}$ or $\overline{h}(\infty) > 1$ holds, then every solution of* (1) *is oscillatory.*

Proof. The claim follows by Corollary 3.1, since $\underline{h}(n)$ is nondecreasing and $\underline{h}(\infty) > 1/\mathrm{e}$ or $\overline{h}(\infty) > 1$ holds. $\qquad\square$

The following theorem is useful for testing oscillatory behavior of (1) when r and q are nonincreasing functions.

Theorem 3.2. *Assume that assumptions of Lemma 3.1 are held. Moreover, r and q are nonincreasing. If there exists $n_0 \in \mathbb{N}$ such that*

$$\liminf_{t \to \infty} \frac{1}{\alpha} \int_{t-\tau+\alpha}^{t} h^{(2)}(\eta) \sum_{i=0}^{n_0} [r(\eta) + \tau_2 q(\eta)]^i \,\mathrm{d}\eta > \frac{1}{\mathrm{e}}$$

or

$$\limsup_{t \to \infty} \frac{1}{\alpha} \int_{t-\tau+\alpha}^{t} h^{(2)}(\eta) \sum_{i=0}^{n_0} [r(\eta) + \tau_2 q(\eta)]^i \,\mathrm{d}\eta > 1$$

holds, then every solution of (1) *is oscillatory.*

Proof. Since, r and q are nonincreasing, we eventually have

$$h_0(t) = 1,$$

$$h_1(t) = r^{(1)}(t) + \sum_{j=0}^{\tau_2 - 1} q^{(1)}(t - j\alpha) \geq r(t) + \tau_2 q(t),$$

and in general, we can see that

$$h_n(t) \geq [r(t) + \tau_2 q(t)]^n, \quad n \in \mathbb{N}$$

holds. Applying Corollary 3.1, we see that every solution of (1) is oscillatory. $\qquad\square$

Now, we consider the following scalar equation

(24) $$\nabla_\alpha [x(t) - rx(t - \kappa)] + px(t - \tau) - qx(t - \sigma) = 0.$$

Theorem 3.3. *Assume that*

$$\tau_2 = \left\lfloor \frac{\tau - \sigma}{\alpha} \right\rfloor \in \mathbb{N},$$

$$r \geq 0, \quad p > q \geq 0,$$

$$0 \leq \alpha r + q\left(\tau - \sigma\right) \leq \alpha$$

hold. If

(25) $$\frac{\left(\tau - \alpha\right)\left(p - q\right)}{\alpha\left(1 - \left(r + \tau_2 q\right)\right)} > \frac{1}{e}$$

holds, then every solution of (24) *is oscillatory.*

Proof. As is estimated in Theorem 3.2, we have $h_n\left(t\right) = \left[r + \tau_2 q\right]^n$ for $n \in \mathbb{N}$. Now, we consider the following two possible cases:

Case 1. $r + q\tau_2 \geq 1$. In this case, we see that

$$\underline{h}\left(n\right) = \frac{\left(\tau - \alpha\right)\left(p - q\right)}{\alpha} \sum_{i=0}^{n} \left[r + \tau_2 q\right]^i \geq \frac{\left(\tau - \alpha\right)\left(p - q\right)}{\alpha} \sum_{i=0}^{n} 1$$

$$= \frac{\left(\tau - \alpha\right)\left(p - q\right)}{\alpha}\left(n + 1\right)$$

which implies $\underline{h}\left(\infty\right) = \infty$, therefore Corollary 3.2 is applicable.

Case 2. $0 \leq r + q\tau_2 < 1$. In this case, we see that

$$\underline{h}\left(n\right) = \frac{\left(\tau - \alpha\right)\left(p - q\right)}{\alpha} \sum_{i=0}^{n} \left[r + \tau_2 q\right]^i$$

holds. Therefore, we obtain

$$\underline{h}\left(\infty\right) = \frac{\left(\tau - \alpha\right)\left(p - q\right)}{\alpha\left(1 - \left(r + \tau_2 q\right)\right)} > \frac{1}{e}.$$

We complete the proof by considering (25) and applying Corollary 3.2. $\square$

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