Bull. Korean Math. Soc. **47** (2010), No. 2, pp. 385–399
DOI 10.4134/BKMS.2010.47.2.385

ON QUASI-RIGID IDEALS AND RINGS

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Abstract. Let σ be an endomorphism and I a σ-ideal of a ring R. Pearson and Stephenson called I a *σ-semiprime ideal* if whenever A is an ideal of R and m is an integer such that $A\sigma^t(A) \subseteq I$ for all $t \geq m$, then $A \subseteq I$, where σ is an automorphism, and Hong et al. called I a *σ-rigid ideal* if $a\sigma(a) \in I$ implies $a \in I$ for $a \in R$. Notice that R is called a σ-semiprime ring (resp., a σ-rigid ring) if the zero ideal of R is a σ-semiprime ideal (resp., a σ-rigid ideal). Every σ-rigid ideal is a σ-semiprime ideal for an automorphism σ, but the converse does not hold, in general. We, in this paper, introduce the quasi σ-rigidness of ideals and rings for an automorphism σ which is in between the σ-rigidness and the σ-semiprimeness, and study their related properties. A number of connections between the quasi σ-rigidness of a ring R and one of the Ore extension $R[x;\sigma,\delta]$ of R are also investigated. In particular, R is a (principally) quasi-Baer ring if and only if $R[x;\sigma,\delta]$ is a (principally) quasi-Baer ring, when R is a quasi σ-rigid ring.

1. Definitions

Let σ be an endomorphism of a ring R, the additive map $\delta : R \to R$ is called a *σ-derivation* if $\delta(ab) = \delta(a)b + \sigma(a)\delta(b)$ for any $a, b \in R$. For a ring R with an endomorphism σ of R and a σ-derivation δ, the *Ore extension* $R[x;\ \sigma,\ \delta]$ of R is the ring obtained by giving the polynomial ring over R with new multiplication: $xr = \sigma(r)x + \delta(r)$ for all $r \in R$. If $\delta = 0$, we write $R[x;\sigma]$ for $R[x;\sigma,0]$ and it is called the *skew polynomial ring* (or, an *Ore extension of endomorphism type*); while $R[[x;\sigma]]$ is called a *skew power series ring.*

An endomorphism σ of a ring R is called *rigid* [17] if $a\sigma(a) = 0$ implies $a = 0$ for $a \in R$. A ring R is called a *σ-rigid ring* [9] if there exists a rigid endomorphism σ of R. The Ore extension $R[x;\sigma,\delta]$ of R is reduced (i.e., it has no nonzero nilpotent elements) and σ is a monomorphism if and only if R is a σ-rigid ring if and only if $R[x;\sigma]$ is reduced by [9, Proposition 5] and [10, Proposition 3], respectively. Hence, σ-rigid rings are reduced rings, but there exists an endomorphism σ of a commutative reduced ring which is not a σ-rigid

Received November 13, 2008.
2000 *Mathematics Subject Classification.* 16N60, 16S36, 16W60.
Key words and phrases. endomorphism, rigidness, semiprimeness, Ore extension, (principally) quasi-Baer ring.

ring by [9, Example 9]. An ideal I of R is called a *σ-ideal* if $\sigma(I) \subseteq I$. In [13], a σ-ideal I of a ring R is called a *σ-rigid ideal* if $a\sigma(a) \in I$ implies $a \in I$ for $a \in R$, and the connections between σ-rigid ideals of R and the related ideals of the Ore extension $R[x;\sigma,\delta]$ of R are investigated. Obviously, R is a σ-rigid ring if and only if the zero ideal of R is a σ-rigid ideal. Following [22], for an automorphism σ of a ring R, a σ-ideal I of R is called *a σ-semiprime ideal* if whenever A is an ideal of R and m is an integer such that $A\sigma^t(A) \subseteq I$ for all $t \geq m$, then $A \subseteq I$; the ring R is called *σ-semiprime* if the zero ideal of R is a σ-semiprime ideal. Notice that R is a σ-semiprime ring if and only if the skew polynomial ring $R[x;\sigma]$ is semiprime by [22, Proposition 1.1] (also, [18, Proposition 4.6]). It is well-known that for an automorphism σ of a ring R, the ring R is σ-semiprime if and only if whenever $a \in R$ and m is an integer such that $aR\sigma^t(a) = 0$ for all $t \geq m$, then $a = 0$. It is clear that every σ-rigid ideal (ring) is a σ-semiprime ideal (ring) for an automorphism σ. Hence, for an endomorphism σ and a σ-ideal I of a ring R, we consider the following condition

$$(*) \qquad aR\sigma(a) \subseteq I \text{ implies } a \in I \text{ for } a \in R.$$

Then it can be easily checked that every σ-rigid ideal satisfies $(*)$ and every σ-ideal satisfying $(*)$ is σ-semiprime for an automorphism σ, but the converses do not hold by the next example, respectively.

Example 1.1. (1) Let $R = \mathrm{Mat}_2(\mathbb{Z}_3)$ be the 2×2 matrix ring over a field $\mathbb{Z}_3$. Then $I = \{0\}$ is a maximal (and prime) ideal of R. Let $\sigma : R \to R$ be an automorphism defined by $\sigma\left(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\right) = \left(\begin{smallmatrix} a & -b \\ -c & d \end{smallmatrix}\right)$. Suppose that $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) R\sigma\left(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\right) \subseteq I$. Since R is a prime ring, $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in I$ or $\sigma\left(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\right) \in I$. Then $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in I$, and therefore I satisfies $(*)$. However, I is not a σ-rigid ideal by [13, Example 3.3].

(2) Let $\mathbb{Z}_2$ be the ring of integers modulo 2 and $R = \mathbb{Z}_2 \oplus \mathbb{Z}_2$. Then R is a commutative reduced ring. Define $\sigma : R \to R$ by $\sigma(a,b) = (b,a)$. Then the zero ideal I of R does not satisfy $(*)$: In fact, $(1,0)R\sigma(1,0) \subseteq I$, but $(1,0) \notin I$. We now claim that $R[x;\sigma]$ is semiprime. Let $f(x)R[x;\sigma]f(x) = 0$, where $f(x) = \sum_{i=0}^{m}(a_i,b_i)x^i \in R[x;\sigma]$ with $(a_m,b_m) \neq 0$. We may assume that $a_m \neq 0$. Then $f(x)x^t f(x) = 0$ for all integer $t \geq 0$. Thus $(a_m,b_m)\sigma^{m+t}(a_m,b_m) = 0$ of all $t \geq 0$. This implies that $(a_m,b_m)\sigma^m(a_m,b_m) = 0$ and $(a_m,b_m)\sigma^{m+1}(a_m,b_m) = 0$. If m is even (or odd), then $(a_m,b_m)(a_m,b_m) = 0$ and $(a_m,b_m)(b_m,a_m) = 0$. Hence $a_m = 0 = b_m$. Thus $(a_m,b_m) = 0$; which is a contradiction. Therefore $R[x;\sigma]$ is semiprime and so R is σ-semiprime by [22, Proposition 1.1], equivalently, I is a σ-semiprime ideal.

Another generalization of σ-rigid ideals is a σ-compatible ideal. In [7], an ideal I of a ring R is called a *σ-compatible ideal* if for each $a, b \in R$, $ab \in I \Leftrightarrow a\sigma(b) \in I$. The next example shows that the class of σ-compatible ideals and the class of σ-ideals satisfying $(*)$ do not depend on each other.

Example 1.2. (1) In Example 1.1(1), the zero ideal $\{0\}$ of R satisfies $(*)$, but not a σ-compatible ideal: Indeed, $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} 2 & 0 \\ 1 & 0 \end{smallmatrix}\right) = 0$, but $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 0 \end{smallmatrix}\right)\sigma\left(\left(\begin{smallmatrix} 2 & 0 \\ 1 & 0 \end{smallmatrix}\right)\right) \neq 0$.

(2) Consider a ring $R = \left\{ \begin{pmatrix} f(x) & g(x) \\ 0 & f(x) \end{pmatrix} \mid f(x), g(x) \in F[x] \right\}$, where $F[x]$ is the polynomial ring over a field F. For a nonzero element $a \in F$, let $\sigma : R \to R$ be an endomorphism defined by $\sigma\left(\begin{pmatrix} f(x) & g(x) \\ 0 & f(x) \end{pmatrix} \right) = \begin{pmatrix} f(x) & ag(x) \\ 0 & f(x) \end{pmatrix}$. Then the ideal $I = \left\{ \begin{pmatrix} 0 & g(x) \\ 0 & 0 \end{pmatrix} \mid g(x) \in \langle p(x) \rangle \right\}$, where $\langle p(x) \rangle$ is an ideal generated by an irreducible polynomial $p(x)$ in $F[x]$, is a σ-compatible ideal of R by [7, Example 2.5]. However, I does not satisfy $(*)$: In fact, for $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \notin I$ we have $AR\sigma(A) = \{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \} \subseteq I$.

Based on these facts, we define the following.

Definition 1.3. Let σ be an automorphism of a ring R. For a σ-ideal I of R, I is called a *quasi σ-rigid ideal* (or, a *strongly σ-semiprime ideal*) of R if $aR\sigma(a) \subseteq I$ implies $a \in I$ for $a \in R$. A ring R is called a *quasi σ-rigid ring* (or, a *strongly σ-semiprime ring*) if the zero ideal of R is a quasi σ-rigid ideal.

In this paper, we study both quasi σ-rigid ideals and quasi σ-rigid rings for an automorphism σ. Several relationship between the quasi σ-rigidness of a ring R and one of the Ore extension $R[x;\sigma,\delta]$ are also investigated. In particular, we show that R is a (principally) quasi-Baer ring if and only if the Ore extension $R[x;\sigma,\delta]$ of R is (principally) quasi-Baer when R is a quasi σ-rigid ring (Theorem 4.3 and Theorem 4.4).

Throughout this paper, R denotes an associative ring with identity. We assume that every endomorphism σ of a given ring is an automorphism, unless specified otherwise.

2. Structures of quasi σ-rigid ideals

Recall that for an ideal I of a ring R, the ideal I is called *σ-invariant* if $\sigma^{-1}(I) = I$. Note that every σ-invariant ideal is a σ-ideal.

Lemma 2.1. *Every quasi σ-rigid ideal of a ring is σ-invariant and semiprime.*

Proof. Let I be a quasi σ-rigid ideal of a ring R. Let $a \in \sigma^{-1}(I)$. Then $\sigma(a) \in I$, and so $aR\sigma(a) \subseteq I$ and hence $a \in I$. Thus $\sigma^{-1}(I) \subseteq I$ and therefore I is σ-invariant. Now, assume that $aRa \subseteq I$ for $a \in R$. For any $r \in R$, $ar\sigma(a)R\sigma(ar\sigma(a)) \subseteq I$ and so $ar\sigma(a) \in I$. Thus $a \in I$ and therefore I is semiprime. □

The converse of Lemma 2.1 does not hold by following.

Example 2.2. Consider the ring $R = \mathbb{Z}_2 \oplus \mathbb{Z}_2$ and the endomorphism σ defined by $\sigma(a,b) = (b,a)$, in Example 1.1(2). Let I be the prime radical $N_*(R) = \{(0,0)\}$ of R (Note that the only proper σ-ideal is $\{(0,0)\}$). Then I is clearly a semiprime ideal and a σ-invariant ideal, but not a quasi σ-rigid ideal by Example 1.1(2).

We have the basic equivalences for quasi σ-rigid ideals as follows.

Proposition 2.3. *Let I be a σ-ideal of a ring R. The following are equivalent to a quasi σ-rigid ideal I of R :*

(1) *$\sigma(a)Ra \subseteq I$ implies $a \in I$ for any $a \in R$.*

(2) *For any ideal A of R, $A\sigma(A) \subseteq I$ implies $A \subseteq I$.*

(3) *For any $\bar{a} \in \bar{R}$, $\bar{a}\bar{R}\bar{\sigma}(\bar{a}) = \bar{0}$ implies $\bar{a} = \bar{0}$, where $\bar{R} = R/I$, $\bar{a} = a + I$ and $\bar{\sigma} : \bar{R} \to \bar{R}$ is defined by $\bar{\sigma}(a + I) = \sigma(a) + I$ for $a \in R$.*

Proof. (1) Suppose that I is a quasi σ-rigid ideal. Then I is semiprime by Lemma 2.1. If $\sigma(a)Ra \subseteq I$ for $a \in R$, then $aR\sigma(a) \subseteq I$ and thus $a \in I$. Conversely, assume that $\sigma(a)Ra \subseteq I$ implies $a \in I$ for any $a \in R$. If $bRb \subseteq I$, then $\sigma(\sigma(b)rb)R\sigma(b)rb = \sigma^2(b)\sigma(r)\sigma(bRb)rb \subseteq I$ for any $r \in R$, and so $\sigma(b)rb \in I$ and thus $b \in I$ by the assumption, entailing that I is semiprime. Hence, $aR\sigma(a) \subseteq I$ implies $\sigma(a)Ra \subseteq I$, and so $a \in I$ by the assumption, concluding that I is a quasi σ-rigid ideal. (2) Let I be a quasi σ-rigid ideal and $a \in A$. If $A\sigma(A) \subseteq I$, then $aR\sigma(a) \subseteq I$, and so $a \in I$. Consequently, $A \subseteq I$. Conversely, assume that for any ideal A of R, $A\sigma(A) \subseteq I$ implies $A \subseteq I$. If $aR\sigma(a) \subseteq I$, then $RaR\sigma(RaR) = RaR\sigma(a)R \subseteq I$ and so $RaR \subseteq I$, and hence $a \in I$, implying that I is a quasi σ-rigid ideal. (3) is obvious. □

For the remainder of this paper, let δ be a σ-derivation of a ring R. Recall that an ideal I of R is called a *δ-ideal* if $\delta(I) \subseteq I$.

Lemma 2.4. *Let I be a quasi σ-rigid ideal of a ring R.*

(1) *If $aRb \subseteq I$ for $a, b \in R$, then $aR\sigma^n(b), \sigma^n(a)Rb \subseteq I$ for every positive integer n. Conversely, if $aR\sigma^k(b)$ or $\sigma^k(a)Rb \subseteq I$ for some positive integer k, then $aRb \subseteq I$.*

(2) *If I is a δ-ideal with $aRb \subseteq I$ for $a, b \in R$, then $aR\delta^n(b), \delta^n(a)Rb \subseteq I$ for every positive integer n.*

Proof. We freely use the fact that every quasi σ-rigid ideal is σ-invariant and semiprime by Lemma 2.1. (1) Suppose that $aRb \subseteq I$ for $a, b \in R$. It is enough to show that $aR\sigma(b)$, $\sigma(a)Rb \subseteq I$. For any $r \in R$, $br\sigma(a)R\sigma(br\sigma(a)) = br\sigma(aRb)\sigma(r\sigma(a)) \subseteq I$. Since I is a quasi σ-rigid ideal, $br\sigma(a) \in I$ and so $bR\sigma(a) \subseteq I$. Hence, $\sigma(a)Rb \subseteq I$ since I is a semiprime ideal. Next, we obtain $bRa \subseteq I$ since $aRb \subseteq I$ and I is a semiprime ideal. By the same method, we get $aR\sigma(b) \subseteq I$. Conversely, suppose that $aR\sigma^k(b) \subseteq I$ for some positive integer k. Then, by the above arguments, $\sigma^k(aRb) = \sigma^k(a)R\sigma^k(b) \subseteq I$. Since I is a σ-invariant ideal, $\sigma^{k-1}(aRb) \subseteq \sigma^{-1}(I) = I$. Continuing this process, we have $aRb \subseteq I$. Similarly, $\sigma^k(a)Rb \subseteq I$ for some positive integer k implies $aRb \subseteq I$. (2) Assume that I is a δ-ideal and $aRb \subseteq I$ for $a, b \in R$. It is sufficient to show that $aR\delta(b), \delta(a)Rb \subseteq I$. Let $aRb \subseteq I$. Note that $bRa \subseteq I$ and so $bR\sigma(a) \subseteq I$ by (1). For any $r \in R$, $\delta(arb) = \sigma(ar)\delta(b) + \delta(ar)b \in I$. Thus $(\sigma(ar)\delta(b)R)^2 = \delta(arb)R\sigma(ar)\delta(b)R - \delta(ar)bR\sigma(ar)\ \delta(b)R \subseteq I$ because $\delta(arb) \in I$ and $bR\sigma(a) \subseteq I$. Since I is semiprime, we have $\sigma(a)R\delta(b) \subseteq I$ and so $aR\delta(b) \subseteq I$ by (1). Similarly, $bRa \subseteq I$ from $aRb \subseteq I$ implies $\delta(a)Rb \subseteq I$, completing proof. □

Corollary 2.5. *Let I be a σ-ideal of a ring R. The ideal I is a quasi σ-rigid ideal if and only if I is a semiprime ideal, and $aR\sigma(b) \subseteq I \Leftrightarrow aRb \subseteq I$ for $a, b \in R$.*

Proof. It follows from Lemma 2.1 and Lemma 2.4. □

Theorem 2.6. *Let I be a quasi σ-rigid ideal of a ring R.*

(1) *Let $p(x) = \sum_{i=0}^{m} a_i x^i$ and $q(x) = \sum_{j=0}^{n} b_j x^j \in R[x; \sigma, \delta]$. If I is a δ-ideal of R, then $p(x)R[x; \sigma, \delta]q(x) \subseteq I[x; \sigma, \delta]$ if and only if $a_i R b_j \subseteq I$ for all i and j.*

(2) *Let $p(x) = \sum_{i=0}^{\infty} a_i x^i$ and $q(x) = \sum_{j=0}^{\infty} b_j x^j \in R[[x; \sigma]]$. Then $p(x)R[[x; \sigma]]q(x) \subseteq I[[x; \sigma]]$ if and only if $a_i R b_j \subseteq I$ for all i and j.*

Proof. (1) Suppose that $p(x)R[x; \sigma, \delta]q(x) \subseteq I[x; \sigma, \delta]$. Then $p(x)rq(x) \in I[x; \sigma, \delta]$ for any $r \in R$, and put

$$(\sum_{i=0}^{m} a_i x^i) r (\sum_{j=0}^{n} b_j x^j) = c_{m+n} x^{m+n} + c_{m+n-1} x^{m+n-1} + \cdots + c_1 x + c_0 \in I[x; \sigma, \delta].$$

We claim that $a_i R b_j \subseteq I$ for all $0 \leq i \leq m$ and $0 \leq j \leq n$. We proceed by induction on $i + j$. When $i + j = m + n$, we have $c_{m+n} = a_m \sigma^m(r) \sigma^m(b_n) \in I$ by above, and so $a_m R b_n \subseteq I$ by Lemma 2.4. Now suppose that our claim is true for $i + j > k \geq 0$. Consider

$$c_k = \sum_{i+j=k} a_i \sigma^i(r) \sigma^i(b_j) + \sum_{i+j>k} a_i R \sigma^{i_1} \delta^{j_1} \sigma^{i_2} \delta^{j_2} \cdots \sigma^{i_l} \delta^{j_l}(b_j) \subseteq I$$

for any $r \in R$ and some nonnegative integers $i_1, \ldots, i_l, j_1, \ldots, j_l$. Since

$$\sum_{i+j>k} a_i R \sigma^{i_1} \delta^{j_1} \sigma^{i_2} \delta^{j_2} \cdots \sigma^{i_l} \delta^{j_l}(b_j) \subseteq I,$$

by induction hypothesis and Lemma 2.4, we obtain

$$c_k = \sum_{i+j=k} a_i \sigma^i(r) \sigma^i(b_j) \in I \tag{1}$$

for any $r \in R$. Multiplying Eq.(1) by Ra_k from the right hand-side, we have

$$\left(\sum_{i+j=k} a_i \sigma^i(r) \sigma^i(b_j) \right) R a_k = a_k \sigma^k(r) \sigma^k(b_0) R a_k \subseteq I$$

since $\sigma^u(b_j) R a_i \subseteq I$ for all $i+j > k$ and any nonnegative integer u, by induction hypothesis and Lemma 2.4(1). Then for any $r \in R$, $(a_k \sigma^k(r) \sigma^k(b_0) R)^2 \subseteq I$ and so $a_k \sigma^k(r) \sigma^k(b_0) \in I$ by Lemma 2.1. Hence $a_k R \sigma^k(b_0) \subseteq I$ and so $a_k R b_0 \subseteq I$ by Lemma 2.4(1). Thus Eq.(1) becomes

$$\sum_{i+j=k,\ 0 \leq i \leq k-1} a_i \sigma^i(r) \sigma^i(b_j) \in I. \tag{2}$$

Multiplying Eq.(2) by Ra_{k-1} from the right hand-side,

$$a_{k-1}\sigma^{k-1}(r)\sigma^{k-1}(b_1)Ra_{k-1} \subseteq I$$

and so $a_{k-1}Rb_1 \subseteq I$ by the same method above. Continuing in this manner, we obtain that $a_iRb_j \subseteq I$ for all i, j with $i + j = k$. Therefore $a_iRb_j \subseteq I$ for all $0 \leq i \leq m$ and $0 \leq j \leq n$. The converse follows directly from Lemma 2.4.

(2) Suppose that $p(x)R[[x;\sigma]]q(x) \subseteq I[[x;\sigma]]$. Then $p(x)rq(x) \in I[[x;\sigma]]$ for any $r \in R$. Then

$$\left(\sum_{i=0}^{\infty} a_i x^i\right) r \left(\sum_{j=0}^{\infty} b_j x^j\right) = \sum_{k=0}^{\infty}\left(\sum_{i+j=k} a_i\sigma^i(r)\sigma^i(b_j)x^{i+j}\right) \in I[[x;\sigma]]. \tag{3}$$

We claim that $a_iRb_j \subseteq I$ for all i, j. We proceed by induction on $i + j$. For $i + j = 0$, we have $a_0Rb_0 \subseteq I$ by Eq.(3). Now assume that our claim is true for $i + j \leq n - 1$. Note that $a_iR\sigma^u(b_j)$ and $\sigma^v(b_j)Ra_i \subseteq I$ for any nonnegative integers u and v, by induction hypothesis and Lemma 2.4(1), when $i+j \leq n-1$. We show that $a_iRb_j \in I$ for $i + j = n$. From Eq.(3), we have

$$\sum_{i+j=n} a_i\sigma^i(r)\sigma^i(b_j) \in I. \tag{4}$$

Multiplying Eq.(4) by Ra_0 from the right hand-side, we obtain $(a_0rb_n)Ra_0 \subseteq I$, and so $a_0rb_n \in I$ and thus $a_0Rb_n \subseteq I$. Thus Eq.(4) becomes

$$\sum_{i+j=n,\ 1\leq i\leq n} a_i\sigma^i(r)\sigma^i(b_j) \in I. \tag{5}$$

Multiplying Eq.(5) by Ra_1 from the right hand-side, we obtain $a_1Rb_{n-1} \subseteq I$ by the similar arguments as above. Continuing this process, we can prove that $a_iRb_j \subseteq I$ for all i, j with $i + j = n$. Therefore, $a_iRb_j \subseteq I$ for all i and j. The converse also follows directly from Lemma 2.4(1). □

Corollary 2.7. *Let I be a quasi σ-rigid ideal of a ring R.*

(1) *$I[x;\sigma,\delta]$ is a semiprime ideal of $R[x;\sigma,\delta]$, when I is a δ-ideal of R.*

(2) *$I[[x;\sigma]]$ is a semiprime ideal of $R[[x;\sigma]]$.*

Proof. (1) Let $p(x)R[x;\sigma,\delta]p(x) \subseteq I[x;\sigma,\delta]$, where $p(x) = \sum_{i=0}^{m} a_ix^i \in R[x;\sigma,\delta]$. Then $a_iRa_i \subseteq I$ for all $0 \leq i \leq m$ by Theorem 2.6. Since I is a semiprime ideal of R by Lemma 2.1, we have $a_i \in I$ for all $0 \leq i \leq m$, and thus $p(x) \in I[x;\sigma,\delta]$. Therefore $I[x;\sigma,\delta]$ is a semiprime ideal of $R[x;\sigma,\delta]$. (2) By the same method as (1). □

Recall from [21], a one-sided ideal I of a ring R has the *insertion of factors property* (or simply, IFP) if $ab \in I$ implies $aRb \subseteq I$ for $a, b \in R$.

Lemma 2.8. *For an ideal I of a ring R, I is a quasi σ-rigid ideal and has the IFP if and only if I is a σ-rigid ideal.*

Proof. Let I be a quasi σ-rigid ideal and have the IFP. If $a\sigma(a) \in I$ for $a \in R$, then $aR\sigma(a) \subseteq I$ by the IFP, and hence $a \in I$, concluding that I is a σ-rigid ideal. Conversely, suppose that I is a σ-rigid ideal. Then I is a completely semiprime ideal (i.e., $a^2 \in I$ implies $a \in I$ for $a \in R$) of R by [13, Proposition 2.2(1)], and so I has the IFP by [21, Lemma 3.2(a)]. □

Let $N(R)$, $N_*(R)$ and $N^*(R)$ denote the set of all nilpotent elements, the prime radical and the upper nilradical (i.e., the sum of all nil ideals) of a ring R, respectively. A ring R is called *2-primal* [3] if $N_*(R) = N(R)$, and a ring R is called *NI* [20] if $N^*(R) = N(R)$. It is well-known that a ring R is 2-primal if and only if $N_*(R)$ is a completely semiprime ideal of R, and a ring R is NI if and only if $N^*(R)$ is a completely semiprime ideal of R. Every 2-primal ring is NI, but the converse does not hold in general.

We use $R[x]$ to denote the polynomial ring with an indeterminate x over a ring R.

Theorem 2.9. (1) *R is a 2-primal ring and $N_*(R)$ is a quasi σ-rigid ideal if and only if $N_*(R)$ is a σ-rigid ideal of R. In particular, if R is a 2-primal ring, then $f(x)g(x) \in N(R)[x]$ if and only if $a_ib_j \in N(R)$ for all i and j, where $f(x) = \sum_{i=0}^{m} a_ix^i$ and $g(x) = \sum_{j=0}^{n} b_jx^j \in R[x]$.*

(2) *R is an NI ring and $N^*(R)$ is a quasi σ-rigid ideal if and only if $N^*(R)$ is a σ-rigid ideal of R.*

Proof. (1) Note that R is a 2-primal ring if and only if $N_*(R)$ has the IFP by [16, Theorem 2.1]. Hence, by Lemma 2.8, $N_*(R)$ is a quasi σ-rigid ideal if and only if $N_*(R)$ is a σ-rigid ideal, when R is a 2-primal ring. Now, let R be a 2-primal ring and $f(x) = \sum_{i=0}^{m} a_ix^i$ and $g(x) = \sum_{j=0}^{n} b_jx^j \in R[x]$. It is well-known that the polynomial ring $R[x]$ over R is 2-primal by [3, Proposition 2.6]. Assume that $f(x)g(x) \in N(R)[x]$. Then $f(x)g(x) \in N(R)[x] = N(R[x]) = N_*(R[x])$ if and only if $f(x)R[x]g(x) \subseteq N(R)[x] = N_*(R[x])$ by [3, Proposition 2.6] and [16, Theorem 2.1] if and only if $a_iRb_j \subseteq N(R)$ for all i and j by Theorem 2.6(1) if and only if $a_ib_j \in N(R)$ by [16, Theorem 2.1]. (2) It follows from the fact that $N^*(R)$ has the IFP if and only if R is an NI ring by [12, Theorem 8]. □

Let $\rho(R)$ be either $N_*(R)$ or $N^*(R)$, and put

$$\Gamma(R) = \begin{cases} \mathrm{mSpec}(R), & \text{if } \rho(R) = \mathbf{N}_*(R) \\ \mathbf{mSpec_S}(R), & \text{if } \rho(R) = \mathbf{N}^*(R), \end{cases}$$

where $\mathrm{mSpec}(R)$ and $\mathbf{mSpec_S}(R)$ denote the set of all minimal prime ideal and all minimal strongly prime ideals of R, respectively.

Corollary 2.10. *Assume that $\rho(R)$ is a completely semiprime ideal of a ring R. The following are equivalent:*

(1) *$\rho(R)$ is a quasi σ-rigid ideal.*

(2) *$\rho(R)$ is a σ-rigid ideal.*

(3) *P is σ-invariant for each $P \in \Gamma(R)$.*

(4) $\sigma^{-1}(P) \subseteq P$ *for each* $P \in \Gamma(R)$.
(5) P *is a* σ*-ideal for each* $P \in \Gamma(R)$.

Proof. It follows from Theorem 2.9 and [13, Proposition 3.4]. □

3. Extensions of quasi σ-rigid rings

Recall that a ring R is called *quasi σ-rigid* if the zero ideal of R is a quasi σ-rigid ideal. For an automorphism σ, every σ-rigid ring is a quasi σ-rigid ring and every quasi σ-rigid ring is a σ-semiprime ring, but the converses do not hold by Example 1.1, respectively. Every quasi σ-rigid ring is a semiprime ring by Lemma 2.1 (but not reduced by Example 1.1(1)). There exists a semiprime ring R with an endomorphism σ such that the skew polynomial ring $R[x;\sigma]$ is not semiprime [14, Example 4.3]. However, we have the following result by Corollary 2.7.

Corollary 3.1. *If R is a quasi σ-rigid ring, then $R[x;\sigma,\delta]$ is a semiprime ring.*

It can be easily checked that any prime ring with an automorphism σ is a quasi σ-rigid ring; while for the ring $R = \mathbb{Z}_2 \oplus \mathbb{Z}_2$ with an automorphism σ in Example 1.1(2), both R and $R[x;\sigma]$ are semiprime rings, but R is not quasi σ-rigid. Note that the minimal prime ideal $\mathbb{Z}_2 \oplus \{0\}$ of R is not a σ-ideal. However,

Proposition 3.2. *If R is a semiprime ring whose minimal prime ideals are σ-ideals, then R is a quasi σ-rigid ring.*

Proof. Suppose $aR\sigma(a) = 0$ for $a \in R$. Then $aR\sigma(a) \subseteq P$ for any minimal prime ideal P of R. So $a \in P$ or $\sigma(a) \in P$. Since P is a σ-ideal, we get $\sigma(a) \in P$ and so $\sigma(a) \in N_*(R) = 0$ because R is semiprime. Thus $a = 0$, concluding that R is a quasi σ-rigid ring. □

Recall that R is called a *σ-compatible* ring [1] (or [8]) if for each $a, b \in R$, $ab = 0 \Leftrightarrow a\sigma(b) = 0$, equivalently, the zero ideal of R is a σ-compatible ideal. Every σ-rigid ring is a σ-compatible ring, but the converse does not hold, in general. Note that the class of quasi σ-rigid rings and the class of σ-compatible rings do not depend on each other by Example 1.2(1) and the following example.

Example 3.3. We consider a ring $R = \left\{\left(\begin{smallmatrix} a & t \\ 0 & a \end{smallmatrix}\right) \mid a \in \mathbb{Z},\ t \in \mathbb{Q}\right\}$, where $\mathbb{Z}$ and $\mathbb{Q}$ are the set of all integers and all rational numbers, respectively. Let $\sigma : R \to R$ be an endomorphism defined by $\sigma\left(\left(\begin{smallmatrix} a & t \\ 0 & a \end{smallmatrix}\right)\right) = \left(\begin{smallmatrix} a & t/2 \\ 0 & a \end{smallmatrix}\right)$. Note that for $\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right) \neq 0$, we have $\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} a & t \\ 0 & a \end{smallmatrix}\right)\sigma\left(\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)\right) = 0$ for any $a \in \mathbb{Z}$ and $t \in \mathbb{Q}$. This yields that R is not a quasi σ-rigid ring. Now, we show that R is a σ-compatible ring. Suppose that $AB = 0$ for $A = \left(\begin{smallmatrix} a & t \\ 0 & a \end{smallmatrix}\right)$, $B = \left(\begin{smallmatrix} b & s \\ 0 & b \end{smallmatrix}\right) \in R$. Then $ab = 0$ and $as + tb = 0$, and so $a = 0$ or $b = 0$. If $a = 0$, then $t = 0$ and hence $A = 0$. Similarly, if $b = 0$, then $s = 0$ and so $B = 0$, entailing that $AB = 0 \Leftrightarrow A\sigma(B) = 0$. Therefore R is a σ-compatible ring.

A ring R is called *semicommutative* if $ab = 0$ implies $aRb = 0$ for $a, b \in R$, which every reduced ring is semicommutative. Notice that a ring is semicommutative if and only if the zero ideal has the IFP, and so semicommutative rings are also called IFP-rings. Recently, the concept of the semicommutativity of a ring is extended to an endomorphism of a ring. An endomorphism σ of a ring R is called *semicommutative* [2, Definition 2.1] if whenever $ab = 0$ for $a, b \in R$, $aR\sigma(b) = 0$; a ring R is called *σ-semicommutative* if there exists a semicommutative endomorphism σ of R. The semicommutativity and the σ-semicommutativity of a ring are independent each other by [2, Example 2.3 and Example 2.7]. In a semicommutative ring, the quasi σ-rigidness and the σ-rigidness of a ring coincide by Lemma 2.8. Moreover, for a σ-semicommutative ring we have the following:

Proposition 3.4. *Let R be a σ-semicommutative ring. The following are equivalent*:
(1) *R is a σ-rigid ring.*
(2) *R is a quasi σ-rigid ring.*
(3) *R is a σ-semiprime ring.*

Proof. It is enough to show that (3)⇒(1). Assume that R is a σ-semiprime ring. Let $a\sigma(a) = 0$ for $a \in R$. Then $aR\sigma^t(a) = 0$ for any positive integer t by [2, Remark 2.2]. Thus $a = 0$ and therefore R is a σ-rigid ring. □

Note that the ring R, in Example 3.3, is a σ-semicommutative ring by the same method as in [2, Example 2.5(1)]. Hence, any condition in Proposition 3.4 cannot be replaced by "R is a σ-compatible ring".

Corollary 3.5. *If R is a semiprime and semicommutative ring, then R is a reduced ring.*

For an automorphism σ of a ring R, the map $\bar{\sigma} : R[x] \to R[x]$ defined by $\bar{\sigma}(\sum_{i=0}^{m} a_i x^i) = \sum_{i=0}^{m} \sigma(a_i)x^i$ is an automorphism of the polynomial ring $R[x]$ and clearly this map extends σ. The ring of *Laurent* polynomials in x, coefficients in a ring R, consists of all formal sums $\sum_{i=k}^{n} m_i x^i$ with obvious addition and multiplication, where $m_i \in R$ and k, n are (possibly negative) integers; denote it by $R[x; x^{-1}]$. The map $\bar{\sigma} : R[x, x^{-1}] \to R[x, x^{-1}]$ defined by $\bar{\sigma}(\sum_{i=k}^{n} a_i x^i) = \sum_{i=k}^{n} \sigma(a_i)x^i$ extends σ and is also an automorphism of $R[x, x^{-1}]$.

Proposition 3.6. *For a ring R, the following are equivalent*:
(1) *R is a quasi σ-rigid ring.*
(2) *$R[x]$ is a quasi $\bar{\sigma}$-rigid ring.*
(3) *$R[x, x^{-1}]$ is a quasi $\bar{\sigma}$-rigid ring.*

Proof. (1)⇔(2) Let R be a quasi σ-rigid ring. Suppose that $R[x]$ is not a quasi $\bar{\sigma}$-rigid ring. Then there exists a nonzero polynomial $f(x) = a_0 + a_1 x + \cdots + a_n x^n$ such that $f(x)R[x]\bar{\sigma}(f(x)) = 0$. We may assume that $a_n \neq 0$. By simple computation, we have $a_nR\sigma(a_n) = 0$. Since R is a quasi σ-rigid ring, we obtain $a_n = 0$; which is a contradiction. Conversely, assume (2) and let $aR\sigma(a) = 0$ for $a \in R$. Note that $aR[x]\bar{\sigma}(a) = 0$. Since $R[x]$ is a quasi $\bar{\sigma}$-rigid ring, we have $a = 0$. Hence, R is a quasi σ-rigid ring. (1)⇔(3) can be proved by the similar arguments above. □

Note that I is a quasi σ-rigid ideal of a ring R if and only if the factor ring R/I is a quasi $\bar{\sigma}$-rigid ring by Proposition 2.3, where $\bar{\sigma} : R/I \to R/I$ is defined by $\bar{\sigma}(a+I) = \sigma(a)+I$ for $a \in R$. The following example shows that there exists a ring R with an automorphism σ such that for any nonzero proper ideal I of R, I is a σ-rigid ideal and so R/I is a quasi $\bar{\sigma}$-rigid ring, but R is not a quasi σ-rigid ring. Moreover, the next example illuminates that the subring of a quasi σ-rigid ring need not be a quasi σ-rigid ring, combining with Example 1.1(1).

Example 3.7. Let $R = \left(\begin{smallmatrix} F & F \\ 0 & F \end{smallmatrix}\right)$ where F is a field, and σ be an endomorphism of R defined by $\sigma\left(\left(\begin{smallmatrix} a & b \\ 0 & c \end{smallmatrix}\right)\right) = \left(\begin{smallmatrix} a & -b \\ 0 & c \end{smallmatrix}\right)$. For $a = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right) \in R$, $aR\sigma(a) = 0$, but $a \neq 0$, and so R is not a quasi σ-rigid ring. For the only nonzero proper ideals $I = \left(\begin{smallmatrix} F & F \\ 0 & 0 \end{smallmatrix}\right)$, $J = \left(\begin{smallmatrix} 0 & F \\ 0 & F \end{smallmatrix}\right)$ and $K = \left(\begin{smallmatrix} 0 & F \\ 0 & 0 \end{smallmatrix}\right)$ of R, it can be easily checked that I, J and K are σ-rigid ideals, thus R/I, R/J and R/K are quasi $\bar{\sigma}$-rigid rings by Proposition 2.3.

For an automorphism σ of a ring R, the map $\bar{\sigma} : \mathrm{Mat}_n(R) \to \mathrm{Mat}_n(R)$ defined by $\bar{\sigma}((a_{ij})) = (\sigma(a_{ij}))$ is an automorphism of the $n \times n$ full matrix ring $\mathrm{Mat}_n(R)$.

Theorem 3.8. *For a ring R, the following are equivalent*:

(1) *R is a quasi σ-rigid ring.*

(2) *$\mathrm{Mat}_n(R)$ is a quasi $\bar{\sigma}$-rigid ring for any $n \geq 2$.*

(3) *$\mathrm{Mat}_n(R)$ is a quasi $\bar{\sigma}$-rigid ring for some $n \geq 2$.*

Proof. (1)⇒(2) Let R be a quasi σ-rigid ring and $n \geq 2$. Suppose that $A\mathrm{Mat}_n(R)\bar{\sigma}(A) = 0$ for $A = (a_{ij}) \in \mathrm{Mat}_n(R)$. Let E_{ij} denote the matrix unit in $\mathrm{Mat}_n(R)$ with (i,j)-entry 1 and zero elsewhere. Then $A(rE_{ij})\bar{\sigma}(A) = 0$ implies $a_{ij}r\sigma(a_{ij}) = 0$ for each $i, j \in \{1, 2, \ldots, n\}$ and any $r \in R$. Hence, $a_{ij}R\sigma(a_{ij}) = 0$, entailing $a_{ij} = 0$ since R is a quasi σ-rigid ring. Therefore $A = 0$, concluding that $\mathrm{Mat}_n(R)$ is a quasi $\bar{\sigma}$-rigid ring. (2)⇒(3) is obvious. (3)⇒(1) Assume that $\mathrm{Mat}_n(R)$ is a quasi $\bar{\sigma}$-rigid ring for $n \geq 2$. Let $aR\sigma(a) = 0$ for $a \in R$ and $A = a\sum_{i=1}^{n} E_{ii}$ in $\mathrm{Mat}_n(R)$. Then $A\mathrm{Mat}_n(R)\bar{\sigma}(A) = 0$. Since $\mathrm{Mat}_n(R)$ is a quasi $\bar{\sigma}$-rigid ring, we get $A = 0$, and hence $a = 0$, proving that R is a quasi σ-rigid ring. □

From Theorem 3.8, one may conjecture that the $n \times n$ upper triangular matrix ring $U_n(R)$ over a quasi σ-rigid ring R is quasi $\bar{\sigma}$-rigid for $n \geq 2$, but the possibility is erased by the following.

Example 3.9. Let R be a ring with any endomorphism σ. Let $A = E_{1n} \in U_n(R)$, where E_{ij} is the matrix unit in $U_n(R)$. Then $AU_n(R)\bar{\sigma}(A) = 0$ (regardless of σ). Thus the $n \times n$ upper triangular matrix ring $U_n(R)$ over R is not a quasi $\bar{\sigma}$-rigid ring for any $n \geq 2$. Moreover, for a ring R and $n \geq 2$, let

$$S_n(R) = \left\{ \begin{pmatrix} a & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a & a_{23} & \cdots & a_{2n} \\ 0 & 0 & a & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a \end{pmatrix} | a, a_{ij} \in R \right\} \text{ and}$$

$$V_n(R) = \left\{ \begin{pmatrix} a_1 & a_2 & a_3 & a_4 & \cdots & a_n \\ 0 & a_1 & a_2 & a_3 & \cdots & a_{n-1} \\ 0 & 0 & a_1 & a_2 & \cdots & a_{n-2} \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & a_2 \\ 0 & 0 & 0 & 0 & \cdots & a_1 \end{pmatrix} \mid a_1, a_2, \ldots, a_n \in R \right\}.$$

By the same method, we conclude that $S_n(R)$ and $V_n(R)$ are not a quasi $\bar{\sigma}$-rigid ring for any $n \geq 2$. Since $V_n(R) \cong R[x]/\langle x^n \rangle$ by [19] where $\langle x^n \rangle$ is an ideal of $R[x]$ generated by x^n, $R[x]/\langle x^n \rangle$ is not a quasi $\bar{\sigma}$-rigid ring for $n \geq 2$ either, where $\bar{\sigma}(f(x) + \langle x^n \rangle) = \sigma(f(x)) + \langle x^n \rangle$ for $f(x) \in R[x]$.

For an automorphism σ and an idempotent e of a ring R such that $\sigma(e) = e$, the map $\bar{\sigma} : eRe \to eRe$ defined by $\bar{\sigma}(ere) = e\sigma(r)e$ is an automorphism of eRe.

Proposition 3.10. *For a ring R, assume that $\sigma(e) = e$ for $e^2 = e \in R$. If R is a quasi σ-rigid ring, then eRe is a quasi $\bar{\sigma}$-rigid ring.*

Proof. For $eae \in eRe$, suppose that $eae(eRe)\bar{\sigma}(eae) = 0$. Then

$$0 = eae(eRe)\bar{\sigma}(eae) = eae(eRe)e\sigma(a)e = (eae)R\sigma(eae).$$

Since R is a quasi σ-rigid ring, $eae = 0$ and so eRe is a quasi $\bar{\sigma}$-rigid ring. □

The condition "$\sigma(e) = e$ for $e^2 = e \in R$" in Proposition 3.10 cannot be dropped by the following example.

Example 3.11. Consider the quasi σ-rigid ring $R = \mathrm{Mat}_2(\mathbb{Z}_3)$ where σ is defined by $\sigma\left(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\right) = \left(\begin{smallmatrix} a & -b \\ -c & d \end{smallmatrix}\right)$ as in Example 1.1(1). For an idempotent $e = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 1 \end{smallmatrix}\right) \in R$, $\sigma(e) \neq e$. Let $a = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 1 \end{smallmatrix}\right) \in R$. Then $eae = \left(\begin{smallmatrix} 0 & 2 \\ 0 & 2 \end{smallmatrix}\right) \neq 0$. But for any $r = \left(\begin{smallmatrix} s & t \\ u & v \end{smallmatrix}\right) \in R$, $eae(ere)\bar{\sigma}(eae) = eaere\sigma(a)e = 0$, and so $eae(eRe)\bar{\sigma}(eae) = 0$, implying that eRe is not a quasi $\bar{\sigma}$-rigid ring.

Recall that an element u of a ring R is *right regular* if $ur = 0$ implies $r = 0$ for $r \in R$. Similarly, *left regular* elements can be defined. An element is *regular* if it is both left and right regular (and hence not a zero divisor). A ring R is called right (resp., left) *Ore* if given $a, b \in R$ with b regular there exist $a_1, b_1 \in R$ with b_1 regular such that $ab_1 = ba_1$ (resp., $b_1 a = a_1 b$). It is a well-known fact that R is a right (resp., left) Ore ring if and only if the classical right (resp., left) quotient ring of R exists.

Let σ be an automorphism of a ring R. Suppose that there exists the classical right quotient ring $Q(R)$ of R. Then for any $ab^{-1} \in Q(R)$ where $a, b \in R$ with b regular, the induced map $\bar{\sigma} : Q(R) \to Q(R)$ defined by $\bar{\sigma}(ab^{-1}) = \sigma(a)\sigma(b)^{-1}$ is also an automorphism. Note that the classical right quotient ring $Q(R)$ of a σ-rigid ring R is $\bar{\sigma}$-rigid. Similarly, we have the following.

Proposition 3.12. *Suppose that there exists the classical right quotient ring $Q(R)$ of a ring R. If R is a quasi σ-rigid ring, then $Q(R)$ is a quasi $\bar{\sigma}$-rigid ring.*

Proof. Suppose that $ab^{-1}Q(R)\bar{\sigma}(ab^{-1}) = 0$. Then $0 = ab^{-1}Q(R)\bar{\sigma}(ab^{-1}) = aQ(R)\sigma(a)\sigma(b)^{-1}$, since $b^{-1}Q(R) = Q(R)$. This implies $aQ(R)\sigma(a) = 0$, and so $aR\sigma(a) = 0$. Since R is a quasi σ-rigid ring, we get $a = 0$ and thus $Q(R)$ is a quasi $\bar{\sigma}$-rigid ring. □

4. Applications

Recall that a ring R is called *Baer* [15] if the right (left) annihilator of every nonempty subset of R is generated by an idempotent; and a ring R is called *quasi-Baer* [6] if the right annihilator of each right ideal of R is generated (as a right ideal) by an idempotent. It is well-known that the (quasi-)Baerness of a ring is left-right symmetric. A ring R is called a *right* (resp., *left*) *p.p.*-ring if the right (resp., left) annihilator of an element of R is generated by an idempotent. R is called a *p.p.*-ring if it is both a right and left p.p.-ring. From [4], a ring R is called *right* (resp., *left*) *principally quasi-Baer* (*or* simply, *right* (resp., *left*) *p.q.-Baer*) if the right (resp., left) annihilator of a principal right (resp., left) ideal of R is generated by an idempotent. R is called a *p.q.-Baer* ring if it is both right and left p.q.-Baer. The class of p.q.-Baer rings includes all biregular rings, all (quasi-)Baer rings and all abelian (i.e., its idempotents are central) p.p.-rings. The extensions of Baer, quasi-Baer, p.q.-Baer and p.p.-rings have been studied by many authors [2, 4, 6, 7, 8]. In [9], it was proved that for a σ-rigid ring R, a ring R is (quasi-)Baer if and only if $R[x; \sigma, \delta]$ is a (quasi-)Baer ring if and only if $R[[x; \sigma]]$ is a (quasi-)Baer ring; and R is a p.q.-Baer (resp., p.p.-) ring if and only if $R[x; \sigma, \delta]$ is a p.q.-Baer (resp., p.p.-) ring. Moreover, there exists a commutative von Neumann regular ring R (and so a p.q.-Baer ring and a p.p.-ring), but $R[[x; \sigma]]$ is neither a p.q.-Baer ring nor a p.p.-ring by [9, p. 225]. As parallel results to these, we have the following for a quasi σ-rigid ring.

Lemma 4.1. *Let R be a quasi σ-rigid ring.*

(1) *For any $p(x)$ and $q(x)$ in $R[x; \sigma, \delta]$ (resp., $R[[x; \sigma]]$), $p(x)R[x; \sigma, \delta]q(x) = 0$ (resp., $p(x)R[[x; \sigma]]q(x) = 0$) if and only if $aRb = 0$ for all coefficients a, b of $p(x)$ and $q(x)$, respectively.*

(2) *For $e^2 = e \in R[x; \sigma, \delta]$ (resp., $R[[x; \sigma]]$), if $eR[x; \sigma, \delta]$ (resp., $eR[[x; \sigma]]$) is an ideal of $R[x; \sigma, \delta]$ (resp., $eR[[x; \sigma]]$), then $e = e_0$ where e_0 is the constant term of e.*

Proof. (1) It follows from Theorem 2.6. (2) Now $1-e=(1-e_0)-\sum_{i=1}^{n} e_i x^i$. Since $eR[x;\sigma,\delta]$ is an ideal, we have $(1-e)R[x;\sigma,\delta]e \subseteq (1-e)eR[x;\sigma,\delta]=0$, and so $(1-e)R[x;\sigma,\delta]e=0$. By (1), $(1-e_0)Re_0=0$ and $e_iRe_i=0$ for any $1 \leq i \leq n$. Since R is semiprime by Lemma 2.1, we have $e_i=0$ for any $1 \leq i \leq n$. Therefore $e=e_0$. □

The following example shows that the condition "$eR[x;\sigma,\delta]$ is an ideal of $R[x;\sigma,\delta]$" in Lemma 4.1(2) cannot be dropped.

Example 4.2. Consider the quasi σ-rigid ring $R=\mathrm{Mat}_2(F)$ and the endomorphism σ as in Example 1.1(1). Let $e=\left(\begin{smallmatrix}0&1\\0&1\end{smallmatrix}\right)+\left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right)x \in R[x;\sigma]$. Then $e^2=e \in R[x;\sigma]$ and $e \in eR[x;\sigma]$. For $r=\left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right) \in R[x;\sigma]$, $re=\left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right)+\left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right)x \notin eR[x;\sigma]$, since the constant term of any element of $eR[x;\sigma]$ is of the form $\left(\begin{smallmatrix}a&b\\a&b\end{smallmatrix}\right)$ where $a,b \in F$. This implies that $eR[x;\sigma]$ is not an ideal. Note that $e=\left(\begin{smallmatrix}0&1\\0&1\end{smallmatrix}\right)+\left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right)x \notin R$.

For a nonempty subset S of a ring R, the right annihilator of S in R will be by $r_R(S)=\{c \in R \mid dc=0 \text{ for any } d \in S\}$.

Proposition 4.3. *Let R be a quasi σ-rigid ring. The following are equivalent*:
(1) *R is a quasi-Baer ring.*
(2) *$R[x;\sigma,\delta]$ is a quasi-Baer ring.*
(3) *$R[[x;\sigma]]$ is a quasi-Baer ring.*

Proof. Without the assumption that R is quasi σ-rigid, (1)⇒(2) and (1)⇒(3) were proved in [11, Theorem 1] and [5, Theorem 1.2] respectively. (2)⇒(1) Assume that $R[x;\sigma,\delta]$ is a quasi-Baer ring. Let J be an ideal of R. Then $r_{R[x;\sigma,\delta]}(JR[x;\sigma,\delta])=eR[x;\sigma,\delta]$ for some idempotent $e \in R$, by Lemma 4.1(2). Thus $r_R(J)=r_{R[x;\sigma,\delta]}(JR[x;\sigma,\delta]) \cap R = eR[x;\sigma,\delta] \cap R = eR$ by Lemma 4.1(1). Therefore R is a quasi-Baer ring. (3)⇒(1) is also proved by the similar arguments above. □

Observe that if R is a quasi σ-rigid ring, then R is a right p.q.-Baer ring if and only if R is a left p.q.-Baer ring since R is semiprime by [4, Corollary 1.11].

Theorem 4.4. *Let R be a quasi σ-rigid ring.*
(1) *R is a right p.q.-Baer ring if and only if $R[x;\sigma,\delta]$ is a right p.q.-Baer ring.*
(2) *If $R[[x;\sigma]]$ is a right p.q.-Baer ring, then R is a right p.q.-Baer ring.*

Proof. (1) Assume that R is a right p.q.-Baer ring. For any principal right ideal $I=p(x)R[x;\sigma,\delta]$ of $R[x;\sigma,\delta]$ where $p(x)=a_0+a_1x+\cdots+a_mx^m$, we take $I^*=a_0R+\cdots+a_mR$ as the finitely generated right ideal generated by $a_0,\ldots,a_m$. Since R is right p.q.-Baer, $r_R(I^*)=eR$ for some $e^2=e \in R$. Note that e is central since R is semiprime. Then $I^*Re=0$, and so $p(x)R[x;\sigma,\delta]e=0$ by Lemma 4.1(1). Hence, $Ie=0$ and so $e \in r_{R[x;\sigma,\delta]}(I)$. Thus $eR[x;\sigma,\delta] \subseteq r_{R[x;\sigma,\delta]}(I)$. Now we let $q(x)=b_0+b_1x+\cdots+b_nx^n \in r_{R[x;\sigma,\delta]}(I)$. Then $p(x)R[x;\sigma,\delta]q(x)=0$ and thus $b_0,b_1,\ldots,b_n \in r_R(I^*)=eR$ by Lemma 4.1(1).

Hence there exist $c_0, c_1, \ldots, c_n$ such that $q(x) = ec_0 + ec_1x + \cdots + ec_nx^n = e(c_0 + c_1x + \cdots + c_nx^n) \in eR[x;\sigma,\delta]$. Consequently, $r_{R[x;\sigma,\delta]}(I) = eR[x;\sigma,\delta]$. Therefore $R[x;\sigma,\delta]$ is right p.q.-Baer. The proofs of the converses of both (1) and (2) follow the proof (2)⇒(1) of Proposition 4.3. □

Remark 4.5. (1) The condition "quasi-Baer rings" in Proposition 4.3 can neither be replaced by "Baer rings" nor "right p.p.-rings": For example, let $R = \mathrm{Mat}_2(\mathbb{Z})$. Then R is a Baer ring, but $R[x]$ is not right p.p. by [9, Example 10(2)]. Also R is a quasi σ-rigid ring, but $R[x;\sigma]$ is neither Baer nor right p.p., in case σ is the identity endomorphism of R.

(2) There exists a quasi σ-rigid and p.q.-Baer ring which is not quasi-Baer, letting σ be the identity endomorphism of R by [4, Lemma 1.4 and Example 1.5(i)].

(3) The converse of Theorem 4.4(2) does not hold by [9, p. 225].

(4) The condition "R is a quasi σ-rigid ring" in Proposition 4.3 and Theorem 4.4 is not superfluous by [9, Example 9].

From [9, Example 9], we see that there exists a semiprime ring R with $\sigma(e) = e$ for any central idempotent $e \in R$ such that $R[x;\sigma,\delta]$ is p.q.-Baer, but R is not quasi σ-rigid. However, we have the following which is compared with Proposition 3.2.

Proposition 4.6. *Let R be a semiprime ring with $\sigma(e) = e$ for any central idempotent $e \in R$. If R is a right p.q.-Baer ring, then R is quasi σ-rigid.*

Proof. Suppose that R is right p.q.-Baer and $aR\sigma(a) = 0$ for $a \in R$. Then $\sigma(a) \in r_R(aR) = eR = \sigma(eR)$ where $e = e^2 \in R$ is central since R is semiprime. It follows that $a \in eR$, entailing $aRa = 0$ and hence $a = 0$ since R is semiprime. Therefore R is quasi σ-rigid. □

The condition "$\sigma(e) = e$ for any central idempotent $e \in R$" in Proposition 4.6 cannot be dropped. For the ring $R = \mathbb{Z}_2 \oplus \mathbb{Z}_2$ with an automorphism σ in Example 1.1(2), R is semiprime and right p.q.-Baer but not quasi σ-rigid and $\sigma(1,0) \neq (1,0)$.

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