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COMPLETE CONVERGENCE FOR WEIGHTED SUMS OF RANDOM ELEMENTS

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Abstract. We obtain a result on complete convergence of weighted sums for arrays of rowwise independent Banach space valued random elements. No assumptions are given on the geometry of the underlying Banach space. The result generalizes the main results of Ahmed et al. [1], Chen et al. [2], and Volodin et al. [14].

1. Introduction

The concept of complete convergence of a sequence of random variables was introduced by Hsu and Robbins [4] as follows. A sequence $\{U_n, n \geq 1\}$ of random variables converges completely to the constant θ if $\sum_{n=1}^{\infty} P(|U_n - \theta| > \epsilon) < \infty$ for all $\epsilon > 0$. By the Borel-Cantelli lemma, this implies that $U_n \to \theta$ almost surely (a.s.). The converse is true if $\{U_n, n \geq 1\}$ are independent random variables. Hsu and Robbins [4] proved that the sequence of arithmetic means of independent and identically distributed (i.i.d.) random variables converges completely to the expected value if the variance of the summands is finite.

This result has been generalized and extended in several directions. Some of these generalizations are in a Banach space setting, for example, see Ahmed et al. [1], Hu et al. [5, 6], Kuczmaszewska and Szynal [7], Sung [10], Volodin et al. [14], and Wang et al. [15]. A sequence of Banach space valued random elements is said to converge completely to the 0 element of the Banach space if the corresponding sequence of norms converges completely to 0.

Hu et al. [6] presented a general result establishing complete convergence for the row sums of an array of rowwise independent but not necessarily identically distributed Banach space valued random elements. Using this, Hu et al. [5] obtained the following complete convergence result. Theorem 1.1 generalizes

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results of Hsu and Robbins [4], Kuczmaszewska and Szynal [7], Sung [10], and Wang et al. [15].

Rowwise independence means that the random elements within each row are independent but that no independence is assumed between rows.

In the following, we assume that $\{X_{ni}, i \geq 1, n \geq 1\}$ is an array of rowwise independent random elements in a real separable Banach space and $\{a_{ni}, i \geq 1, n \geq 1\}$ is an array of real numbers.

Theorem 1.1 (Hu et al. [5]). *Suppose that the array $\{X_{ni}, i \geq 1, n \geq 1\}$ is stochastically dominated by a random variable X. That is,*

$$P(||X_{ni}|| > x) \leq DP(|X| > x) \quad \text{for all } x > 0 \text{ and for all } i \geq 1 \text{ and } n \geq 1,$$

where D is a positive constant. Assume that

$$\sup_{i\geq 1} |a_{ni}| = O(n^{-\gamma}) \quad \text{for some } \gamma > 0 \tag{1.1}$$

and

$$\sum_{i=1}^{\infty} |a_{ni}| = O(n^{\mu}) \quad \text{for some } \mu \in [0, \gamma).$$

If $E|X|^{1+(1+\mu+t)/\gamma} < \infty$ for some $t \in (-1, \gamma - \mu - 1]$ and $\sum_{i=1}^{\infty} a_{ni}X_{ni} \to 0$ in probability, then

$$\sum_{n=1}^{\infty} n^t P\left(||\sum_{i=1}^{\infty} a_{ni}X_{ni}|| > \epsilon\right) < \infty \quad \text{for all } \epsilon > 0. \tag{1.2}$$

It is assumed in Theorem 1.1 that $\sum_{i=1}^{\infty} a_{ni}X_{ni}$ is finite a.s., since the a.s. convergence is not automatic from the hypotheses. Ahmed et al. [1] established the following more general result than Theorem 1.1.

Theorem 1.2 (Ahmed et al. [1]). *Suppose that the array $\{X_{ni}, i \geq 1, n \geq 1\}$ is stochastically dominated by a random variable X. Assume that* (1.1) *holds and*

$$\sum_{i=1}^{\infty} |a_{ni}| = O(n^{\mu}) \quad \text{for some } \mu < \gamma.$$

Let t be such that $t + \mu \neq -1$ and fix $\delta > 1$ such that $1 + \mu/\gamma < \delta \leq 2$. If $E|X|^{\nu} < \infty$, where $\nu = \max\{1 + (1 + \mu + t)/\gamma, \delta\}$, and $\sum_{i=1}^{\infty} a_{ni}X_{ni} \to 0$ in probability, then (1.2) *holds.*

Theorem 1.2 was slightly generalized by Volodin et al. [14] as follows. Theorem 1.2 corresponds to the case $\theta = 1$ in Theorem 1.3.

Theorem 1.3 (Volodin et al. [14]). *Suppose that the array $\{X_{ni}, i \geq 1, n \geq 1\}$ is stochastically dominated by a random variable X. Assume that* (1.1) *holds and*

$$\sum_{i=1}^{\infty} |a_{ni}|^{\theta} = O(n^{\mu}) \quad \text{for some } 0 < \theta \leq 2 \text{ and } \mu \text{ such that } \theta + \mu/\gamma < 2.$$

Let t be such that $t+\mu \neq -1$ and fix $\delta > \theta$ such that $\theta + \mu/\gamma < \delta \leq 2$. If $E|X|^{\nu} < \infty$, where $\nu = \max\{\theta + (1+\mu+t)/\gamma, \delta\}$, and $\sum_{i=1}^{\infty} a_{ni}X_{ni} \to 0$ in probability, then (1.2) *holds.*

Note that (1.2) holds clearly if $t < -1$. Hence it is of interest only for $t \geq -1$. In particular, the case $t = -1$ is of interest. Ahmed et al. [1] conjectured that when $t = -1$, the assumption $E|X|^{\nu} < \infty$ can be replaced by $E|X|^{1+\mu/\gamma}\log^{\rho}(|X|) < \infty$ $(\rho > 0)$ in Theorem 1.2. Sung and Volodin [13] gave a positive answer as follows:

Theorem 1.4 (Sung and Volodin [13])**.** *Suppose that the array $\{X_{ni}, i \geq 1, n \geq 1\}$ is stochastically dominated by a random variable X. Assume that* (1.1) *holds and*

$$\sum_{i=1}^{\infty} |a_{ni}|^{\theta} = O(n^{\mu}) \quad \text{for some } \mu > 0 \text{ and } \theta > 0 \text{ such that } \theta + \mu/\gamma < 2. \tag{1.3}$$

If $E|X|^{\theta+\mu/\gamma}\log^{\rho}(|X|) < \infty$ for some $\rho > 0$ and $\sum_{i=1}^{\infty} a_{ni}X_{ni} \to 0$ in probability, then

$$\sum_{n=1}^{\infty} \frac{1}{n} P\left(||\sum_{i=1}^{\infty} a_{ni}X_{ni}|| > \epsilon\right) < \infty \quad \text{for all } \epsilon > 0. \tag{1.4}$$

Chen et al. [2] improved Theorem 1.4 by proving that the condition

$$E|X|^{\theta+\mu/\gamma}\log^{\rho}(|X|) < \infty$$

can be replaced by the weaker condition $E|X|^{\theta+\mu/\gamma} < \infty$.

Theorem 1.5 (Chen et al. [2])**.** *Suppose that the array $\{X_{ni}, i \geq 1, n \geq 1\}$ is stochastically dominated by a random variable X. Assume that* (1.1) *and* (1.3) *hold. If $E|X|^{\theta+\mu/\gamma} < \infty$ and $\sum_{i=1}^{\infty} a_{ni}X_{ni} \to 0$ in probability, then* (1.4) *holds.*

In this paper, we obtain a new complete convergence result which generalizes the above all results. No assumptions are made concerning the geometry of the underlying Banach space.

The plan of the paper is as follows. In Section 2, we recall well known inequalities and give some elementary results pertaining to the current work. The main result is given in Section 3.

The symbol C denotes a positive constant which is not necessarily the same one in each appearance.

2. Preliminaries

In this section, we present some inequalities and elementary results which will be useful in the proof of our main result.

Let B be a real separable Banach space with norm $||\cdot||$. Let $(\Omega, \mathcal{F}, P)$ be a probability space. A random element (or B-valued random element) is defined to be a $\mathcal{F}$-measurable mapping from Ω to B equipped with the Borel σ-algebra (the σ-algebra generated by the open sets determined by $||\cdot||$).

The first two lemmas are well known and their proofs are standard.

Lemma 2.1. *Let* $\{X_n, n \geq 1\}$ *be a sequence of random variables which are stochastically dominated by a random variable* X. *Then for any* $r > 0$ *and* $b > 0$, *the following statements hold*:

(i) $E|X_n|^r I(|X_n| \leq b) \leq D\{E|X|^r I(|X| \leq b) + b^r P(|X| > b)\}$.

(ii) $E|X_n|^r I(|X_n| > b) \leq DE|X|^r I(|X| > b)$.

Lemma 2.2. *Let* X *be a random variable with* $E|X|^r < \infty$ *for some* $r > 0$. *Then for any* $p > 0$, *the following statements hold*:

(i) $\sum_{n=1}^{\infty}(1/n^{1+\delta/p})E|X|^{r+\delta}I(|X| \leq n^{1/p}) \leq CE|X|^r$ *for any* $\delta > 0$.

(ii) $\sum_{n=1}^{\infty}(1/n^{1-\delta/p})E|X|^{r-\delta}I(|X| > n^{1/p}) \leq CE|X|^r$ *for any* $\delta > 0$ *such that* $r - \delta > 0$.

(iii) $\sum_{n=1}^{\infty}(1/n^{1-r/p})P(|X| > n^{1/p}) \leq CE|X|^r$.

Lemma 2.3. *Let* X *and* Y *be random variables with the same distribution. Then for any* $r > 0$ *and* $b > 0$, *the following statements hold*:

(i) $E|X - Y|^r I(|X - Y| \leq b) \leq \max\{2, 2^{2r-1}\}\big(E|X|^r I(|X| \leq b) + b^r P(|X| > b)\big)$.

(ii) $E|X - Y|^r I(|X - Y| > b) \leq 2^{r+1}E|X|^r I(|X| > b/2)$.

Proof. (i) Observe that

$$|X-Y|I(|X-Y| \leq b) \leq |X|I(|X| \leq b)+bI(|X| > b)+|Y|I(|Y| \leq b)+bI(|Y| > b).$$

If $0 < r \leq 1$, then we have by the c_r-inequality that

$$\begin{aligned} &E|X - Y|^r I(|X - Y| \leq b)\\ &\leq E(|X|^r I(|X| \leq b) + b^r I(|X| > b) + |Y|^r I(|Y| \leq b) + b^r I(|Y| > b))\\ &= 2\big(E|X|^r I(|X| \leq b) + b^r P(|X| > b)\big). \end{aligned}$$

If $r > 1$, then we have by the Hölder inequality that

$$\begin{aligned} &E|X - Y|^r I(|X - Y| \leq b)\\ &\leq 4^{r-1}E\big(|X|^r I(|X| \leq b) + b^r I(|X| > b) + |Y|^r I(|Y| \leq b) + b^r I(|Y| > b)\big)\\ &= 2^{2r-1}\big(E|X|^r I(|X| \leq b) + b^r P(|X| > b)\big). \end{aligned}$$

(ii) Since $E|X| = \int_0^\infty P(|X| > x)\, dx$, it follows that

$$\begin{aligned} &E|X - Y|^r I(|X - Y| > b)\\ &= b^r P(|X - Y| > b) + \int_{b^r}^{\infty} P(|X - Y|^r > x)\, dx\\ &\leq b^r\big(P(|X| > b/2) + P(|Y| > b/2)\big)\\ &\quad + \int_{b^r}^{\infty} P(|X| > x^{1/r}/2) + P(|Y| > x^{1/r}/2)\, dx\\ &= 2^{r+1}E|X|^r I(|X| > b/2). \end{aligned}$$

□

Combining Lemma 2.1 and Lemma 2.3 gives the following:

Lemma 2.4. *Let $\{X_n, n \geq 1\}$ be a sequence of random variables which are stochastically dominated by a random variable X. Assume that X_n and Y_n have the same distribution. Then for any $r > 0$ and $b > 0$, the followings hold*:

(i) $E|X_n - Y_n|^r I(|X_n - Y_n| \leq b) \leq D \max\{2, 2^{2r-1}\}\big(E|X|^r I(|X| \leq b) + 2b^r P(|X| > b)\big)$.

(ii) $E|X_n - Y_n|^r I(|X_n - Y_n| > b) \leq D2^{r+1} E|X|^r I(|X| > b/2)$.

Note that Lemmas 2.1-2.4 are still valid for random elements.

The following lemma gives us a useful contraction principle and can be found in Lemma 6.5 of Ledoux and Talagrand [9].

Lemma 2.5. *Let $\{X_i, i \geq 1\}$ be a sequence of symmetric random elements. Let further $\{\xi_i, i \geq 1\}$ and $\{\zeta_i, i \geq 1\}$ be real random variables such that $\xi_i = \phi_i(X_i)$, where $\phi_i : B \to R$ is symmetric (even), and similarly for ζ_i. Then, if $|\xi_i| \leq |\zeta_i|$ almost surely for every i, for every $t > 0$*

$$P\left(||\sum_i \xi_i X_i|| > t\right) \leq 2P\left(||\sum_i \zeta_i X_i|| > t\right).$$

In particular, this inequality applies when $\xi_i = I_{\{X_i \in A_i\}} \leq 1 \equiv \zeta_i$, where the sets A_i are symmetric in B (in particular $A_i = \{||x|| \leq a_i\}$).

The next lemma is a modification of a result of Kuelbs and Zinn [8] concerning the relationship between convergence in probability and mean convergence for sums of independent bounded random variables. We refer to Lemma 2.1 of Hu et al. [6] for the proof.

Lemma 2.6. *Let $\{X_{ni}, i \geq 1, n \geq 1\}$ be an array of rowwise independent symmetric random elements. Suppose there exists $\delta > 0$ such that $||X_{ni}|| \leq \delta$ almost surely for all $i \geq 1$ and $n \geq 1$. Put $S_n = \sum_{i=1}^{\infty} X_{ni}$. If $S_n \to 0$ in probability, then $E||S_n|| \to 0$ as $n \to \infty$.*

The following inequalities are Banach space analogues of the classical Marcinkiewicz-Zygmund and Rosenthal inequalities and are due to de Acosta [3].

Lemma 2.7. *Let $\{X_i, 1 \leq i \leq n\}$ be a sequence of independent random elements. Then there exists a positive constant C_p depending only on p such that*

(i) *for* $1 \leq p \leq 2$,

$$E\big|||\sum_{i=1}^n X_i|| - E||\sum_{i=1}^n X_i||\big|^p \leq C_p \sum_{i=1}^n E||X_i||^p,$$

(ii) *for* $p > 2$,

$$E\big|||\sum_{i=1}^n X_i|| - E||\sum_{i=1}^n X_i||\big|^p \leq C_p \Big\{\Big(\sum_{i=1}^n E||X_i||^2\Big)^{p/2} + \sum_{i=1}^n E||X_i||^p\Big\}.$$

3. Main results

Throughout this section, let $\{X_{ni}, i \geq 1, n \geq 1\}$ be an array of rowwise independent random elements which are stochastically dominated by a random X satisfying $E|X|^{p(t+\beta+1)} < \infty$ for some $p > 0$, $t \geq -1$, $\beta \in R$ such that $p(t+\beta+1) \geq 1$.

Let $\{a_{ni}, i \geq 1, n \geq 1\}$ be a bounded array of real numbers such that

$$\sum_{i=1}^{\infty} |a_{ni}|^q = O(n^\beta) \quad \text{for some } q < p(t+\beta+1). \tag{3.1}$$

Note that (3.1) implies

$$\sum_{i=1}^{\infty} |a_{ni}|^{q+\gamma} = O(n^\beta) \quad \text{for any } \gamma > 0,$$

since $|a_{ni}| = O(1)$.

We now state our main results. For the case $p(t+\beta+1) > 1$, it is assumed that the series $\sum_{i=1}^{\infty} a_{ni}X_{ni}$ converges a.s..

Theorem 3.1. *Suppose that* (3.1) *holds and*

$$\sum_{i=1}^{\infty} \frac{a_{ni}X_{ni}}{n^{1/p}} \to 0 \quad \textit{in probability.} \tag{3.2}$$

Furthermore, assume that

$$\sum_{i=1}^{\infty} a_{ni}^2 = O(n^\alpha) \quad \textit{for some } \alpha < \frac{2}{p} \tag{3.3}$$

if $p(t+\beta+1) \geq 2$. *Then*

$$\sum_{n=1}^{\infty} n^t P\left(||\sum_{i=1}^{\infty} a_{ni}X_{ni}|| > n^{1/p}\epsilon\right) < \infty \quad \textit{for all } \epsilon > 0. \tag{3.4}$$

Remark 3.1. (i) If $t < -1$, then (3.4) is immediate.

(ii) When $p(t+\beta+1) = 1$, the series $\sum_{i=1}^{\infty} a_{ni}X_{ni}$ converges a.s., since

$$\sum_{i=1}^{\infty} E||a_{ni}X_{ni}|| \leq DE|X| \max_{i,n} |a_{ni}|^{1-q} \sum_{i=1}^{\infty} |a_{ni}|^q \leq Cn^\beta < \infty.$$

(iii) Sung [11] proved Theorem 3.1 for the random variable case. When $0 < p(t+\beta+1) < 1$, Theorem 3.1 is still valid without the independence condition and the weak law of large numbers condition (see Theorem 2(i) of Sung [11]).

To prove our main results, we need the following lemmas.

Lemma 3.1. *Let* $\{X^s_{ni}, i \geq 1, n \geq 1\}$ *be an array of the symmetrized version of* $\{X_{ni}\}$, *i.e.,* $X^s_{ni} = X_{ni} - X^*_{ni}$, *where* X_{ni} *and* X^*_{ni} *are independent and have the same distribution. If the bounded array* $\{a_{ni}\}$ *satisfies* (3.1), *then the following statements hold*:

(i)

$$\sum_{n=1}^{\infty} n^t \sum_{i=1}^{\infty} E||n^{-1/p} a_{ni} X^s_{ni} I(||X^s_{ni}|| \leq n^{1/p})||^{p(t+\beta+1)+\delta} \leq CE|X|^{p(t+\beta+1)}$$

for any $\delta > 0$.

(ii)

$$\sum_{n=1}^{\infty} n^t \sum_{i=1}^{\infty} E||n^{-1/p} a_{ni} X^s_{ni} I(||X^s_{ni}|| > n^{1/p})||^{p(t+\beta+1)-\delta} \leq CE|X|^{p(t+\beta+1)}$$

for any $\delta > 0$ *such that* $p(t+\beta+1)-\delta \geq q$ *and* $p(t+\beta+1)-\delta > 0$.

Proof. By Lemmas 2.1-2.3, we get that

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^t \sum_{i=1}^{\infty} E||n^{-1/p} a_{ni} X^s_{ni} I(||X^s_{ni}|| \leq n^{1/p})||^{p(t+\beta+1)+\delta} \\
&\leq C \sum_{n=1}^{\infty} \frac{1}{n^{\beta+1+\delta/p}} \sum_{i=1}^{\infty} |a_{ni}|^{p(t+\beta+1)+\delta} \big\{E||X_{ni} I(||X_{ni}|| \leq n^{1/p})||^{p(t+\beta+1)+\delta} \\
&\qquad\qquad + n^{t+\beta+1+\delta/p} P(||X_{ni}|| > n^{1/p})\big\} \\
&\leq C \sum_{n=1}^{\infty} \frac{1}{n^{\beta+1+\delta/p}} \sum_{i=1}^{\infty} |a_{ni}|^{p(t+\beta+1)+\delta} \big\{E|XI(|X| \leq n^{1/p})|^{p(t+\beta+1)+\delta} \\
&\qquad\qquad + n^{t+\beta+1+\delta/p} P(|X| > n^{1/p})\big\} \\
&\leq C \sum_{n=1}^{\infty} \frac{1}{n^{1+\delta/p}} E|XI(|X| \leq n^{1/p})|^{p(t+\beta+1)+\delta} + C \sum_{n=1}^{\infty} n^{t+\beta} P(|X| > n^{1/p}) \\
&\leq CE|X|^{p(t+\beta+1)} < \infty.
\end{aligned}$$

Thus (i) is proved. The proof of (ii) is similar to that of (i) and is omitted. □

Lemma 3.2. *Let* $\{X^s_{ni} = X_{ni} - X^*_{ni}, i \geq 1, n \geq 1\}$ *be an array of the symmetrized version of* $\{X_{ni}\}$. *If* $\sum_{i=1}^{\infty} a_{ni} X_{ni}/n^{1/p} \to 0$ *in probability, then the following statements hold*:

(i) $\sum_{i=1}^{\infty} a_{ni} X^s_{ni} I(||X^s_{ni}|| \leq n^{1/p})/n^{1/p} \to 0$ *in probability.*
(ii) $\sum_{i=1}^{\infty} a_{ni} X^s_{ni} I(||X^s_{ni}|| > n^{1/p})/n^{1/p} \to 0$ *in probability.*

Proof. Applying Lemma 2.5, we have that

$$
\begin{aligned}
& P\left(||\sum_{i=1}^{\infty} a_{ni} X_{ni}^s I(||X_{ni}^s|| \le n^{1/p})|| > n^{1/p}\epsilon\right) \\
& \le 2P\left(||\sum_{i=1}^{\infty} a_{ni} X_{ni}^s|| > n^{1/p}\epsilon\right) \\
& \le 2\left\{P\left(||\sum_{i=1}^{\infty} a_{ni} X_{ni}|| > n^{1/p}\epsilon/2\right) + P\left(||\sum_{i=1}^{\infty} a_{ni} X_{ni}^*|| > n^{1/p}\epsilon/2\right)\right\} \\
& = 4P\left(||\sum_{i=1}^{\infty} a_{ni} X_{ni}|| > n^{1/p}\epsilon/2\right) \to 0
\end{aligned}
$$

as $n \to \infty$. Thus (i) is proved. The proof of (ii) is similar to that of (i) and is omitted. □

Lemma 3.2 can be strengthen by imposing some additional conditions.

Lemma 3.3. *Let $\{X_{ni}^s = X_{ni} - X_{ni}^*, i \ge 1, n \ge 1\}$ be an array of the symmetrized version of $\{X_{ni}\}$. Let $\{a_{ni}, i \ge 1, n \ge 1\}$ be a bounded array of real numbers satisfying* (3.1). *Assume that $\sum_{i=1}^{\infty} a_{ni} X_{ni}/n^{1/p} \to 0$ in probability. Furthermore, suppose that* (3.3) *holds if $p(t+\beta+1) \ge 2$. Then the following statements hold*:

(i) $E||\sum_{i=1}^{\infty} a_{ni} X_{ni}^s I(||X_{ni}^s|| \le n^{1/p})||/n^{1/p} \to 0$ *as* $n \to \infty$.
(ii) $E||\sum_{i=1}^{\infty} a_{ni} X_{ni}^s I(||X_{ni}^s|| > n^{1/p})||/n^{1/p} \to 0$ *as* $n \to \infty$.

Proof. (i) By Lemma 3.2, $\sum_{i=1}^{\infty} a_{ni} X_{ni}^s I(||X_{ni}^s|| \le n^{1/p})/n^{1/p} \to 0$ in probability. Since $|a_{ni}| = O(1)$, we have that $||a_{ni} X_{ni}^s I(||X_{ni}^s|| \le n^{1/p})||/n^{1/p} = O(1)$. Thus (i) follows by Lemma 2.6.

(ii) We proceed with three cases. First we consider the case of $q < 1$. We get by Lemma 2.4 that

$$
\begin{aligned}
& E||\sum_{i=1}^{\infty} a_{ni} X_{ni}^s I(||X_{ni}^s|| > n^{1/p})||/n^{1/p} \\
& \le C\sum_{i=1}^{\infty} |a_{ni}| E|X| I(|X| > n^{1/p}/2)/n^{1/p} \\
& \le Cn^{\beta-1/p} E|X| I(|X| > n^{1/p}/2) \\
& \le Cn^{-t-1} E|X|^{p(t+\beta+1)} I(|X| > n^{1/p}/2) \to 0
\end{aligned}
$$

as $n \to \infty$, since $t+1 \ge 0$ and $E|X|^{p(t+\beta+1)} I(|X| > n^{1/p}/2) \to 0$ as $n \to \infty$.

Next we consider the case of $1 \le q \le 2$. Using Lemma 2.4 and Lemma 2.7, we get that

$$E\Big| \|\sum_{i=1}^{\infty} a_{ni} X_{ni}^s I(\|X_{ni}^s\| > n^{1/p})\|/n^{1/p} - E\|\sum_{i=1}^{\infty} a_{ni} X_{ni}^s I(\|X_{ni}^s\| > n^{1/p})\|/n^{1/p}\Big|^q$$

$$\le C_q n^{-q/p} \sum_{i=1}^{\infty} |a_{ni}|^q E\|X_{ni}^s\|^q I(\|X_{ni}^s\| > n^{1/p})$$

$$\le C n^{\beta - q/p} E|X|^q I(|X| > n^{1/p}/2)$$

$$\le C n^{-t-1} E|X|^{p(t+\beta+1)} I(|X| > n^{1/p}/2) \to 0$$

as $n \to \infty$. It follows that

$$\|\sum_{i=1}^{\infty} a_{ni} X_{ni}^s I(\|X_{ni}^s\| > n^{1/p})\|/n^{1/p} - E\|\sum_{i=1}^{\infty} a_{ni} X_{ni}^s I(\|X_{ni}^s\| > n^{1/p})\|/n^{1/p} \to 0$$

in probability. On the other hand, $\|\sum_{i=1}^{\infty} a_{ni} X_{ni}^s I(\|X_{ni}^s\| > n^{1/p})\|/n^{1/p} \to 0$ in probability by Lemma 3.2. So $E\|\sum_{i=1}^{\infty} a_{ni} X_{ni}^s I(\|X_{ni}^s\| > n^{1/p})\|/n^{1/p} \to 0$ as $n \to \infty$.

Finally we consider the case of $q > 2$. Using Lemma 2.4, Lemma 2.7, and (3.3), we get that

$$E\Big| \|\sum_{i=1}^{\infty} a_{ni} X_{ni}^s I(\|X_{ni}^s\| > n^{1/p})\|/n^{1/p} - E\|\sum_{i=1}^{\infty} a_{ni} X_{ni}^s I(\|X_{ni}^s\| > n^{1/p})\|/n^{1/p}\Big|^q$$

$$\le C_q n^{-q/p} \Big\{ \Big(\sum_{i=1}^{\infty} a_{ni}^2 E\|X_{ni}^s\|^2 I(\|X_{ni}^s\| > n^{1/p}) \Big)^{q/2} + \sum_{i=1}^{\infty} |a_{ni}|^q E\|X_{ni}^s\|^q I(\|X_{ni}^s\| > n^{1/p}) \Big\}$$

$$\le C n^{-q/p} \Big\{ \Big(\sum_{i=1}^{\infty} a_{ni}^2 E|X|^2 I(|X| > n^{1/p}/2) \Big)^{q/2} + \sum_{i=1}^{\infty} |a_{ni}|^q E|X|^q I(|X| > n^{1/p}/2) \Big\}$$

$$\le C n^{-q/p} \Big\{ \Big(n^{\alpha} E|X|^2 I(|X| > n^{1/p}/2) \Big)^{q/2} + n^{\beta} E|X|^q I(|X| > n^{1/p}/2) \Big\}$$

$$\le C \Big\{ n^{q(\alpha-(t+\beta+1))/2} \Big(E|X|^{p(t+\beta+1)} I(|X| > n^{1/p}/2) \Big)^{q/2} + n^{-t-1} E|X|^{p(t+\beta+1)} I(|X| > n^{1/p}/2) \Big\}$$

$$=: A_n + B_n.$$

Since $2 < q < p(t+\beta+1)$ and $\alpha < 2/p$, $q(\alpha-(t+\beta+1))/2 < 0$ and so $A_n \to 0$ as $n \to \infty$. Since $t+1 \ge 0$ and $E|X|^{p(t+\beta+1)} I(|X| > n^{1/p}/2) \to 0$, $B_n \to 0$ as $n \to \infty$. The rest of the proof is same as that of the case $1 \le q \le 2$ and is omitted. □

Finally, we need the following lemma which is due to Sung et al. [12].

Lemma 3.4 (Sung et al. [12]). *Let $\{X_{ni}, i \ge 1, n \ge 1\}$ be an array of rowwise independent random elements. Let $\{c_n, n \ge 1\}$ be a sequence of positive numbers. Suppose that for every $\epsilon > 0$ and some $\delta > 0$,*

(i) $\sum_{n=1}^{\infty} c_n \sum_{i=1}^{\infty} P(\|X_{ni}\| > \epsilon) < \infty$,

(ii) *there exists* $J \geq 2$ *such that*

$$\sum_{n=1}^{\infty} c_n \left(\sum_{i=1}^{\infty} E||X_{ni}||^2 I(||X_{ni}|| \leq \delta) \right)^J < \infty,$$

(iii) $\sum_{i=1}^{\infty} X_{ni} \to 0$ *in probability.*

Then $\sum_{n=1}^{\infty} c_n P(||\sum_{i=1}^{\infty} X_{ni}|| > \epsilon) < \infty$ *for all* $\epsilon > 0$.

With the preliminary lemmas, we prove our main result.

Proof of Theorem 3.1. Let $\{X_{ni}^s = X_{ni} - X_{ni}^*, i \geq 1, n \geq 1\}$ be an array of the symmetrized version of $\{X_{ni}\}$. Put μ_n be a median of $||\sum_{i=1}^{\infty} a_{ni} X_{ni}||/n^{1/p}$. Since $||\sum_{i=1}^{\infty} a_{ni} X_{ni}||/n^{1/p} \to 0$ in probability, $\mu_n \to 0$ as $n \to \infty$. Then we have by the weak symmetrization inequality that for all large n

$$\begin{aligned} P\left(\frac{||\sum_{i=1}^{\infty} a_{ni} X_{ni}||}{n^{1/p}} > \epsilon \right) &\leq P\left(\frac{||\sum_{i=1}^{\infty} a_{ni} X_{ni}||}{n^{1/p}} - \mu_n > \frac{\epsilon}{2} \right) \\ &\leq 2P\left(\frac{||\sum_{i=1}^{\infty} a_{ni} X_{ni}^s||}{n^{1/p}} > \frac{\epsilon}{2} \right) \\ &\leq 2P\left(\frac{||\sum_{i=1}^{\infty} a_{ni} X_{ni}^s I(||X_{ni}^s|| \leq n^{1/p})||}{n^{1/p}} > \frac{\epsilon}{4} \right) \\ &\quad + 2P\left(\frac{||\sum_{i=1}^{\infty} a_{ni} X_{ni}^s I(||X_{ni}^s|| > n^{1/p})||}{n^{1/p}} > \frac{\epsilon}{4} \right). \end{aligned}$$

Hence it is enough to show that

$$\sum_{n=1}^{\infty} n^t P\left(||\sum_{i=1}^{\infty} a_{ni} X_{ni}^s I(||X_{ni}^s|| \leq n^{1/p})|| > n^{1/p}\epsilon/4 \right) < \infty \tag{3.5}$$

and

$$\sum_{n=1}^{\infty} n^t P\left(||\sum_{i=1}^{\infty} a_{ni} X_{ni}^s I(||X_{ni}^s|| > n^{1/p})|| > n^{1/p}\epsilon/4 \right) < \infty. \tag{3.6}$$

We proceed with three cases.

Case 1: $1 < p(t + \beta + 1) < 2$.
By Lemma 3.3, there exists a positive integer N such that

$$E||\sum_{i=1}^{\infty} a_{ni} X_{ni}^s I(||X_{ni}^s|| \leq n^{1/p})|| < n^{1/p}\epsilon/8 \quad \text{if } n \geq N.$$

Taking $\delta > 0$ such that $p(t+\beta+1)+\delta \le 2$, we get by Markov's inequality, Lemma 2.7, and Lemma 3.1 that

(3.7)

$$\begin{aligned}
&\sum_{n=N}^{\infty} n^t P\left(\frac{||\sum_{i=1}^{\infty} a_{ni}X_{ni}^s I(||X_{ni}^s|| \le n^{1/p})||}{n^{1/p}} > \frac{\epsilon}{4}\right) \\
&\le \sum_{n=N}^{\infty} n^t P\left(\left|\frac{||\sum_{i=1}^{\infty} a_{ni}X_{ni}^s I(||X_{ni}^s|| \le n^{1/p})||}{n^{1/p}} - \frac{E||\sum_{i=1}^{\infty} a_{ni}X_{ni}^s I(||X_{ni}^s|| \le n^{1/p})||}{n^{1/p}}\right| > \frac{\epsilon}{8}\right) \\
&\le C\sum_{n=N}^{\infty} n^t E\left|||\sum_{i=1}^{\infty} n^{-1/p}a_{ni}X_{ni}^s I(||X_{ni}^s|| \le n^{1/p})|| - E||\sum_{i=1}^{\infty} n^{-1/p}a_{ni}X_{ni}^s I(||X_{ni}^s|| \le n^{1/p})||\right|^{p(t+\beta+1)+\delta} \\
&\le C\sum_{n=N}^{\infty} n^t \sum_{i=1}^{\infty} E||n^{-1/p}a_{ni}X_{ni}^s I(||X_{ni}^s|| \le n^{1/p})||^{p(t+\beta+1)+\delta} \\
&\le CE|X|^{p(t+\beta+1)} < \infty.
\end{aligned}$$

Hence (3.5) holds when $1 < p(t+\beta+1) < 2$.

By Lemma 3.3, there exists a positive integer M such that

$$E||\sum_{i=1}^{\infty} a_{ni}X_{ni}^s I(||X_{ni}^s|| > n^{1/p})|| < n^{1/p}\epsilon/8 \quad \text{if } n \ge M.$$

Taking $\delta > 0$ such that $p(t+\beta+1)-\delta \ge \max\{1,q\}$, we get by Markov's inequality, Lemma 2.7, and Lemma 3.1 that

(3.8)

$$\begin{aligned}
&\sum_{n=M}^{\infty} n^t P\left(\frac{||\sum_{i=1}^{\infty} a_{ni}X_{ni}^s I(||X_{ni}^s|| > n^{1/p})||}{n^{1/p}} > \frac{\epsilon}{4}\right) \\
&\le \sum_{n=M}^{\infty} n^t P\left(\left|\frac{||\sum_{i=1}^{\infty} a_{ni}X_{ni}^s I(||X_{ni}^s|| > n^{1/p})||}{n^{1/p}} - \frac{E||\sum_{i=1}^{\infty} a_{ni}X_{ni}^s I(||X_{ni}^s|| > n^{1/p})||}{n^{1/p}}\right| > \frac{\epsilon}{8}\right) \\
&\le C\sum_{n=M}^{\infty} n^t E\left|||\sum_{i=1}^{\infty} n^{-1/p}a_{ni}X_{ni}^s I(||X_{ni}^s|| > n^{1/p})|| - E||\sum_{i=1}^{\infty} n^{-1/p}a_{ni}X_{ni}^s I(||X_{ni}^s|| > n^{1/p})||\right|^{p(t+\beta+1)-\delta} \\
&\le C\sum_{n=M}^{\infty} n^t \sum_{i=1}^{\infty} E||n^{-1/p}a_{ni}X_{ni}^s I(||X_{ni}^s|| > n^{1/p})||^{p(t+\beta+1)-\delta} \\
&\le CE|X|^{p(t+\beta+1)} < \infty.
\end{aligned}$$

Hence (3.6) holds when $1 < p(t+\beta+1) < 2$.

Case 2: $p(t+\beta+1) = 1$.

By the proof of (3.7), (3.5) holds when $p(t+\beta+1) = 1$. To prove (3.6), take $\delta > 0$ such that $p(t+\beta+1)-\delta > \max\{0,q\}$. Then we have by Markov's inequality and Lemma 3.1 that

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^t P\left(||\sum_{i=1}^{\infty} a_{ni}X_{ni}^s I(||X_{ni}^s|| > n^{1/p})|| > n^{1/p}\epsilon/4\right)\\
&\le C\sum_{n=1}^{\infty} n^t E||\sum_{i=1}^{\infty} n^{-1/p}a_{ni}X_{ni}^s I(||X_{ni}^s|| > n^{1/p})||^{p(t+\beta+1)-\delta}\\
&\le C\sum_{n=1}^{\infty} n^t \sum_{i=1}^{\infty} E||n^{-1/p}a_{ni}X_{ni}^s I(||X_{ni}^s|| > n^{1/p})||^{p(t+\beta+1)-\delta}\\
&\le CE|X|^{p(t+\beta+1)} < \infty.
\end{aligned}$$

Hence (3.6) holds when $p(t+\beta+1) = 1$.

Case 3: $p(t+\beta+1) \ge 2$.

We apply Lemma 3.4 to the random element $n^{-1/p}a_{ni}X_{ni}^s I(||X_{ni}^s|| \le n^{1/p})$. Taking $\delta > 0$, we have by Lemma 3.1 that

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^t \sum_{i=1}^{\infty} P\left(||n^{-1/p}a_{ni}X_{ni}^s I(||X_{ni}^s|| \le n^{1/p})|| > \epsilon\right)\\
&\le C\sum_{n=1}^{\infty} n^t \sum_{i=1}^{\infty} E||n^{-1/p}a_{ni}X_{ni}^s I(||X_{ni}^s|| \le n^{1/p})||^{p(t+\beta+1)+\delta}\\
&\le CE|X|^{p(t+\beta+1)} < \infty.
\end{aligned}$$

Since $2/p > \alpha$, we can take $J \ge 2$ such that $J(2/p-\alpha) - t > 1$. It follows that

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^t \left(\sum_{i=1}^{\infty} E||n^{-1/p}a_{ni}X_{ni}^s I(||X_{ni}^s|| \le n^{1/p})||^2 I(||n^{-1/p}a_{ni}X_{ni}^s I(||X_{ni}^s|| \le n^{1/p})|| \le 1)\right)^J\\
&\le \sum_{n=1}^{\infty} n^t \left(n^{-2/p}\sum_{i=1}^{\infty} a_{ni}^2 E||X_{ni}^s||^2\right)^J\\
&\le C\sum_{n=1}^{\infty} n^t \left(n^{\alpha-2/p}E|X|^2\right)^J < \infty.
\end{aligned}$$

By Lemma 3.2, $\sum_{i=1}^{\infty} a_{ni}X_{ni}^s I(||X_{ni}^s|| \le n^{1/p})/n^{1/p} \to 0$ in probability. Thus all conditions of Lemma 3.4 are satisfied. Hence (3.5) holds when $p(t+\beta+1) \ge 2$. Similarly, (3.6) holds when $p(t+\beta+1) \ge 2$. □

Now we prove Theorems 1.1-1.5 by using our main result (Theorem 3.1). Since Theorem 1.1 and Theorem 1.2 follow by Theorem 1.3, and Theorem 1.4 follows by Theorem 1.5, we will prove only Theorem 1.3 and Theorem 1.5.

Proof of Theorem 1.3. We will apply Theorem 3.1 with $p = 1/\gamma$ and a_{ni} replaced by $n^\gamma a_{ni}$. Then we have that

$$|n^\gamma a_{ni}| = O(1) \quad \text{and} \quad \sum_{i=1}^{\infty} |n^\gamma a_{ni}|^\theta = O(n^{\theta\gamma+\mu}).$$

When $1 + \mu + t > 0$, take $q = \theta$ and $\beta = \theta\gamma + \mu$. Then $p(t + \beta + 1) = \frac{1}{\gamma}(t + (\theta\gamma + \mu) + 1) = \theta + (1 + \mu + t)/\gamma$ and so $q < p(t+\beta+1) \le \nu$. Hence (3.1) with a_{ni} replaced by $n^\gamma a_{ni}$ holds.
When $1 + \mu + t < 0$, take $q = \theta$ and $\beta = \theta\gamma + \mu + \gamma(\delta - (\theta + (1 + \mu + t)/\gamma))$. Since $\delta > \theta$, $\beta > \theta\gamma + \mu$ and $p(t + \beta + 1) = \delta$, and so $q < p(t + \beta + 1) \le \nu$. Hence (3.1) with a_{ni} replaced by $n^\gamma a_{ni}$ holds.
Furthermore,

$$\sum_{i=1}^{\infty} |n^\gamma a_{ni}|^2 = n^{2\gamma} \sum_{i=1}^{\infty} a_{ni}^2 \le n^{2\gamma} \sup_{i,n} |a_{ni}|^{2-\theta} \sum_{i=1}^{\infty} |a_{ni}|^\theta \le Cn^{\theta\gamma+\mu}.$$

Since $\theta + \mu/\gamma < 2$, $\theta\gamma + \mu < 2/p$ and so (3.3) with a_{ni} replaced by $n^\gamma a_{ni}$ holds for $\alpha = \theta\gamma + \mu$. Thus the result follows by Theorem 3.1. □

Proof of Theorem 1.5. We will apply Theorem 3.1 with $t = -1, p = 1/\gamma, q = \theta, \alpha = \theta\gamma + \mu, \beta = \theta\gamma + \mu$, and a_{ni} replaced by $n^\gamma a_{ni}$. Since $\theta + \mu/\gamma < 2$ and $\mu > 0$, $p(t+\beta+1) = \theta + \mu/\gamma > q$ and $\alpha < 2/p$. From the proof of Theorem 1.3, we have that

$$|n^\gamma a_{ni}| = O(1), \quad \sum_{i=1}^{\infty} |n^\gamma a_{ni}|^\theta = O(n^{\theta\gamma+\mu}), \text{ and } \sum_{i=1}^{\infty} |n^\gamma a_{ni}|^2 = O(n^{\theta\gamma+\mu}).$$

Thus the result follows by Theorem 3.1. □

Finally, we give an example showing that our result is more general than Theorems 1.1-1.5.

Example 3.1. Let $\{X_{ni}, i \ge 1, n \ge 1\}$ be an array of rowwise independent random elements which are stochastically dominated by a random variable X satisfying $E|X|^3 < \infty$. If $\sum_{i=1}^n X_{ni}/(n\sqrt{i}) \to 0$ in probability, then

$$\sum_{n=1}^{\infty} n^2 P\left(||\sum_{i=1}^{n} X_{ni}/\sqrt{i}|| > n\epsilon\right) < \infty \quad \text{for all } \epsilon > 0. \tag{3.9}$$

To prove (3.9), let $t = 2, p = 1$, and

$$a_{ni} = \begin{cases} 1/\sqrt{i}, & 1 \le i \le n \text{ and } n \ge 1, \\ 0, & \text{otherwise.} \end{cases}$$

Observe that

$$\sum_{i=1}^{\infty} |a_{ni}|^q = \sum_{i=1}^{n} (1/\sqrt{i})^q = O(1) \text{ if } q > 2$$

and

$$\sum_{i=1}^{\infty} a_{ni}^2 = \sum_{i=1}^{n} \frac{1}{i} = O(\log n).$$

If we take $\beta = 0, 2 < q < 3$, and $0 < \alpha < 2$, then $q < p(t + \beta + 1) = 3$ and $\alpha < 2/p$, and so (3.1) and (3.3) hold. Thus (3.9) holds by Theorem 3.1.

We next show that (3.9) can not be proved by using Theorem 1.3 or Theorem 1.5 (Note that Theorem 1.3 is stronger than Theorems 1.1 and 1.2, and Theorem 1.5 is stronger than Theorem 1.4). Since $t = 2$, we can not apply Theorem 1.5.

We now apply Theorem 1.3 to prove (3.9). Since $|\frac{1}{n\sqrt{i}}| = O(n^{-1})$, $\gamma \leq 1$. Noting that

$$\sum_{i=1}^{n} \frac{1}{(n\sqrt{i})^{\theta}} = \begin{cases} n^{1-3\theta/2} & \text{if } 0 < \theta < 2, \\ n^{-2} \log n & \text{if } \theta = 2, \end{cases}$$

we have that $\mu \geq 1 - 3\theta/2$ if $0 < \theta < 2$ and $\mu > -2$ if $\theta = 2$. If $0 < \theta < 2$, then $\theta+(1+\mu+t)/\gamma \geq \theta+\mu+3 \geq 4-\theta/2 > 3$. If $\theta = 2$, then $\theta+(1+\mu+t)/\gamma \geq 5+\mu > 3$. In both cases, $\theta + (1 + \mu + t)/\gamma > 3$. Thus we can not apply Theorem 1.3.

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