PERIODIC SOLUTIONS FOR A KIND OF p-LAPLACIAN HAMILTONIAN SYSTEMS

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ABSTRACT. In this paper, the existence of periodic solutions is obtained for a kind of p-Laplacian systems by the minimax methods in critical point theory. Moreover, the existence of infinite periodic solutions is also obtained.

1. Introduction

This paper is concerned with the existence of periodic solutions for the following system

$$(1.1) \qquad \begin{cases} \dfrac{d}{dt}(\phi_p(\dot{u}(t))) = \nabla F(t, u(t)), \text{ a.e. } t \in [0, T], \\ u(0) - u(T) = \dot{u}(0) - \dot{u}(T) = 0, \end{cases}$$

where $p > 1$, $\phi_p(u) = |u|^{p-2}u = \left(\sqrt{\sum_{i=1}^{N} u_i^2}\right)^{p-2} \cdot (u_1, u_2, \ldots, u_N)$, $T > 0$ and $F : [0, T] \times \mathbb{R}^N \to \mathbb{R}$ satisfies the following assumption:

(A) $F(t, u)$ is measurable in t for each $u \in \mathbb{R}^N$ and continuously differentiable in u for a.e. $t \in [0, T]$, and there exist $a \in C(\mathbb{R}^+, \mathbb{R}^+)$, $b \in L^1(0, T; \mathbb{R}^+)$ such that

$$|F(t, u)| \leq a(|u|)b(t), \ |\nabla F(t, u)| \leq a(|u|)b(t)$$

for all $u \in \mathbb{R}^N$ and a.e. $t \in [0, T]$.

When $p = 2$, problem (1.1) becomes the second order Hamiltonian systems

$$(1.2) \qquad \begin{cases} \ddot{u}(t) = \nabla F(t, u(t)), \text{ a.e. } t \in [0, T], \\ u(0) - u(T) = \dot{u}(0) - \dot{u}(T) = 0. \end{cases}$$

Many existence results for (1.2) are obtained by using variational methods, such as [2, 4, 5, 6, 7] and the references therein. Tang [4] gets some existence

results of (1.2) when $F(t, u) = F_1(t, u) + F_2(t, u)$ and $\int_0^T F(t, u)dt \to +\infty$ as $|u| \to \infty$, $F(t, \cdot)$ is subadditive for a.e. $t \in [0, T]$ and there exists $g \in L^1[0, T]$ such that $|\nabla F_2(t, u)| \leq g(t)$ for all $u \in \mathbb{R}^N$ and a.e. $t \in [0, T]$. Moreover, by the least action principle, Wu and Tang [6] obtain some existence results of (1.2) while $F_1(t, \cdot)$ is (λ, μ)-subconvex and $\nabla F_2(t, u) \leq f(t)|u|^\alpha + g(t)$ for some $0 \leq \alpha < 1$, $f, g \in L^1(0, T; \mathbb{R}^+)$.

However, while $F(t, u) \to -\infty$ as $|u| \to +\infty$ uniformly for a.e. $t \in [0, T]$, we do not know whether the question is a positive answer. Moreover, for the general case $p > 1$, some papers calling it vector p-Laplacian, there are not so many results. Differential equations with p-Laplacian have many important applications such as non-Newtonian fluid theory and the turbulent flow of the gas in porous medium. Motivated by the work mentioned above, in this paper, we consider the existence of periodic solutions for problem (1.1).

2. Preliminary

The Sobolev space $W_T^{1,p}$ is defined by

$$(2.1) \qquad W_T^{1,p} = \{u : [0, T] \to \mathbb{R}^N \mid u \text{ is absolutely continuous}, u(0) = u(T),$$
$$\text{and } \dot{u} \in L^p(0, T; \mathbb{R}^N)\}$$

and is endowed with the norm

$$(2.2) \qquad \|u\| = \left(\int_0^T |u(t)|^p dt + \int_0^T |\dot{u}(t)|^p dt \right)^{\frac{1}{p}}.$$

It follows from [2] that $W_T^{1,p}$ is a reflexive Banach space. Moreover, $W_T^{1,p}$ is a closed subspace of $W_T^{1,p}$ under (2.2). From [1, Theorem 1.21] and [1, Theorem 3.5], we obtain $W_T^{1,p}$ is uniformly convex.

Define a functional φ on $W_T^{1,p}$ by

$$(2.3) \qquad \varphi(u) = \frac{1}{p} \int_0^T |\dot{u}(t)|^p dt + \int_0^T F(t, u(t))dt, \ u \in W_T^{1,p}.$$

It follows from assumption (A) that the functional φ is continuously differentiable on $W_T^{1,p}$. Moreover,

$$(2.4) \qquad \langle \varphi'(u), v \rangle = \int_0^T [(|\dot{u}(t)|^{p-2}\dot{u}(t), \dot{v}(t)) + (\nabla F(t, u(t)), v(t))]dt$$

for all $u, v \in W_T^{1,p}$. It is well known that the solutions of problem (1.1) correspond to the critical points of φ.

The following lemmas are basic in our paper.

Lemma 2.1 ([2]). *Set* $||u||_\infty = \max_{t \in [0,T]} |u(t)|$, $||u||_{L^p} = (\int_0^T |u(t)|^p dt)^{\frac{1}{p}}$. *If* $u \in W_T^{1,p}$ *and* $\int_0^T u(t)dt = 0$, *then*

$$(2.5) \qquad ||u||_\infty \le T^{\frac{1}{q}} ||\dot{u}||_{L^p}, \quad \left(\frac{1}{p} + \frac{1}{q} = 1\right).$$

Moreover, we have

$$(2.6) \qquad ||u||_{L^p} \le T ||\dot{u}||_{L^p}.$$

Lemma 2.2 ([5]). *Suppose* $F(t,u)$ *satisfies the assumption* (A) *and* E *is a measurable subset of* $[0,T]$. *Assume*

$$F(t,u) \to -\infty, \quad |u| \to \infty$$

for a.e. $t \in E$. *Then, for every* $\delta > 0$, *there exists a subset* E_δ *of* E *with* $\mathrm{meas}(E \setminus E_\delta) < \delta$ *such that*

$$F(t,u) \to -\infty, \quad |u| \to \infty$$

uniformly for all $t \in E_\delta$.

Theorem 2.1 ([2, The Saddle Point Theorem]). *Let* X *be a Banach space and let* $\varphi \in C^1(X,R)$. *Assume that* X *splits into a direct sum of closed subspaces* $X = X^- \oplus X^+$ *with*

$$\dim X^- < \infty$$

and

$$\sup_{S_R^-} \varphi < \inf_{X^+} \varphi,$$

where $S_R^- = \{u \in X^- : |u| = R\}$. *Let*

$$B_R^- = \{u \in X^- : |u| \le R\},$$

$$M = \{g \in C(B_R^-, X) : g(s) = s \text{ if } s \in S_R^-\}$$

and

$$c = \inf_{g \in M} \max_{s \in B_R^-} \varphi(g(s)).$$

Then if φ *satisfies the* $(PS)_c$-*condition,* c *is a critical value of* φ.

Theorem 2.2 ([3]). *Assume* E *is a real Banach space,* $I \in C^1(E,R)$ *is an even functional satisfying* PS-*condition and* $I(0) = 0$. *If* $E = V \oplus X$, $\dim V < \infty$ *and* I *satisfies*

(A_1) *there exist constants* ρ, $\alpha > 0$ *such that* $I|_{\partial B_\rho \cap X} \ge \alpha$;
(A_2) *for* $\forall V_1 \subset E$ *with* $\dim V_1 < \infty$, *the set* $\{x \in V_1 : I(x) \ge 0\}$ *is bounded,*

then I *has an unbounded critical value sequence.*

3. Main results

Suppose $F(t, u) = G(u) + H(t, u)$ and
(B_1) there exist $f, g \in L^1(0, T; R^+)$ such that

$$|\nabla H(t, u)| \leq f(t)|u|^\alpha + g(t)$$

holds for all $u \in \mathbb{R}^N$ and a.e. $t \in [0, T]$. Here α satisfies

$$0 \leq \alpha < 1, \text{ if } p \geq q \quad \text{and} \quad 0 \leq \alpha < \frac{p}{q}, \text{ if } p < q;$$

(B_2) there exists a constant $r < \frac{1 - T^{\frac{p}{q}}}{T^p}$ such that

$$(\nabla G(u) - \nabla G(v), u - v) \geq -r|u - v|^p, \ u, v \in \mathbb{R}^N;$$

(B_3) there exists a subset E of $[0, T]$ with $\text{meas}(E) > 0$ such that

$$|u|^{-\max\{p,q\}\alpha} F(t, u) \to -\infty \text{ as } |u| \to \infty$$

for a.e. $t \in E$;
(B_4) there exist $c(t) \in L^1(0, T; \mathbb{R}^+)$, $d(t) \in L^1(0, T)$, $\gamma \geq 0$ such that

$$F(t, u) \leq c(t)|u|^{-\gamma} + d(t)$$

for $u \in \mathbb{R}^N$, $u \neq \theta$, and a.e. $t \in [0, T]$.

In the following, for $u \in W_T^{1,p}$, let $\bar{u} = \frac{1}{T} \int_0^T u(t)dt$ and $\tilde{u} = u - \bar{u}$.

Lemma 3.1. *Suppose $F(t, u)$ satisfy (A) and (B_1)-(B_4). Then φ satisfies the PS-condition, that is, $(u_n)_{n \in \mathbb{N}}$ has a convergent subsequence whenever it satisfies $\varphi'(u_n) \to 0$ as $n \to \infty$ and $\varphi(u_n)$ is bounded.*

Proof. By (2.6), we have

$$(3.1) \qquad \left(\int_0^T |\dot{u}(t)|^p dt \right)^{\frac{1}{p}} = \left(\int_0^T |\dot{\tilde{u}}(t)|^p dt \right)^{\frac{1}{p}} \leq ||\tilde{u}||$$

$$= \left(\int_0^T |\tilde{u}(t)|^p dt + \int_0^T |\dot{\tilde{u}}(t)|^p dt \right)^{\frac{1}{p}}$$

$$\leq ||\tilde{u}||_{L^p} + ||\dot{\tilde{u}}||_{L^p} \leq (T + 1)||\dot{\tilde{u}}||_{L^p}.$$

By (B_2) and (2.6), we obtain

$$\int_0^T (\nabla G(u(t)), \tilde{u}(t))dt = \int_0^T (\nabla G(u(t)) - \nabla G(\bar{u}(t)), \tilde{u}(t))dt$$

$$= \int_0^T (\nabla G(u(t)) - \nabla G(\bar{u}(t)), u(t) - \bar{u}(t))dt$$

$$\geq -r \int_0^T |u - \bar{u}|^p dt$$

$$= -r||\tilde{u}||_{L^p}^p \geq -rT^p||\dot{u}||_{L^p}^p, \ \forall u \in W_T^{1,p}.$$

Suppose $(u_n)_{n\in\mathbb{N}}$ is a sequence satisfying $\varphi'(u_n) \to 0$ as $n \to \infty$ and $\varphi(u_n)$ is bounded. In order to prove

$$(3.2) \qquad |u_n(t)| \geq C_0|\bar{u}_n| - C_0', \ \forall t \in [0, T]$$

holds for some positive constants C_0, C_0', there are two cases.

Case 1. $p \geq q$.

From (B_1) and (2.5),

(3.3)

$$\left| \int_0^T (\nabla H(t, u(t)), \tilde{u}(t))dt \right|$$

$$\leq \int_0^T f(t)|u(t)|^\alpha|\tilde{u}(t)|dt + \int_0^T g(t)|\tilde{u}(t)|dt$$

$$\leq (|\bar{u}|^\alpha + ||\tilde{u}(t)||_\infty^\alpha)||\tilde{u}(t)||_\infty \int_0^T f(t)dt + ||\tilde{u}(t)||_\infty \int_0^T g(t)dt$$

$$= |\bar{u}|^\alpha||\tilde{u}(t)||_\infty \int_0^T f(t)dt + ||\tilde{u}(t)||_\infty^{\alpha+1} \int_0^T f(t)dt + ||\tilde{u}(t)||_\infty \int_0^T g(t)dt$$

$$\leq ||\tilde{u}(t)||_\infty^q + |\bar{u}|^{p\alpha}\left(\int_0^T f(t)dt\right)^p + ||\tilde{u}(t)||_\infty^{\alpha+1} \int_0^T f(t)dt + ||\tilde{u}(t)||_\infty \int_0^T g(t)dt$$

$$\leq T||\dot{u}||_{L^p}^q + C_1|\bar{u}|^{p\alpha} + C_2||\dot{u}||_{L^p}^{\alpha+1} + C_3||\dot{u}||_{L^p}$$

holds for all $u \in W_T^{1,p}$ and some positive constants C_1, C_2 and C_3. Hence, for large n, one has

(3.4)

$$||\tilde{u}_n|| \geq |\langle\varphi'(u_n), \tilde{u}_n\rangle| = \left|\int_0^T |\dot{u}_n(t)|^p dt + \int_0^T (\nabla F(t, u_n(t)), \tilde{u}_n(t))dt\right|$$

$$= \left|\int_0^T |\dot{u}_n(t)|^p dt + \int_0^T (\nabla G(u_n(t)), \tilde{u}_n(t))dt + \int_0^T (\nabla H(t, u_n(t)), \tilde{u}_n(t))dt\right|$$

$$\geq ||\dot{u}_n||_{L^p}^p - rT^p||\dot{u}_n||_{L^p}^p - T||\dot{u}_n||_{L^p}^q - C_1|\bar{u}_n|^{p\alpha} - C_2||\dot{u}_n||_{L^p}^{\alpha+1} - C_3||\dot{u}_n||_{L^p}.$$

Then, by (3.1) and (3.4)

$$C_1|\bar{u}_n|^{p\alpha} \geq (1 - rT^p)||\dot{u}_n||_{L^p}^p - T||\dot{u}_n||_{L^p}^q - C_2||\dot{u}_n||_{L^p}^{\alpha+1} - (1 + T + C_3)||\dot{u}_n||_{L^p}.$$

Obviously, $p \geq 2$ and $\alpha < 1$. Moreover, from (B_2) one has $1 - rT^p > 0$ for $p > q$, and $1 - rT^p - T > 0$ for $p = q$. Hence,

$$(3.5) \qquad C_4|\bar{u}_n|^\alpha \geq ||\dot{u}_n||_{L^p} - C_5$$

holds for some constants C_4, C_5 and large n. This implies that

$$(3.6) \qquad ||\tilde{u}_n||_\infty \leq C_6(|\bar{u}_n|^\alpha + 1)$$

for some constants $C_6 \neq 1$ and large n. Then one has

$$
\begin{aligned}
|u_n(t)| \geq |\bar{u}_n| - |\tilde{u}_n(t)| &\geq |\bar{u}_n| - ||\tilde{u}_n(t)||_\infty \\
&\geq |\bar{u}_n| - C_6(|\bar{u}_n|^\alpha + 1) \\
&= |\bar{u}_n|[1 - C_6(|\bar{u}_n|^{\alpha-1} + |\bar{u}_n|^{-1})]
\end{aligned}
$$
(3.7)

for large n and $t \in [0, T]$. If $(|\bar{u}_n|)_{n \in \mathbb{N}}$ is bounded, we can easily obtain (3.2). If $(|\bar{u}_n|)_{n \in \mathbb{N}}$ is unbounded, without loss of generality, we may suppose $|\bar{u}_n| \to \infty$ as $n \to \infty$, then, for sufficiently small $\varepsilon > 0$ and large n, $|\bar{u}_n|^{\alpha-1} < \varepsilon$, $|\bar{u}_n|^{-1} < \varepsilon$. Hence, (3.2) holds.

Case 2. $p < q$.

Similarly, we have

$$
\left| \int_0^T (\nabla H(t, u(t)), \tilde{u}(t)) dt \right|
$$

$$
\leq \int_0^T f(t)|u(t)|^\alpha |\tilde{u}(t)| dt + \int_0^T g(t)|\tilde{u}(t)| dt
$$

$$
\leq ||\tilde{u}(t)||_\infty^p + |\bar{u}|^{q\alpha} \left(\int_0^T f(t) dt \right)^q + ||\tilde{u}(t)||_\infty^{\alpha+1} \int_0^T f(t) dt + ||\tilde{u}(t)||_\infty \int_0^T g(t) dt
$$

$$
\leq T^{\frac{p}{q}} ||\dot{u}||_{L^p}^p + C_1' |\bar{u}|^{q\alpha} + C_2 ||\dot{u}||_{L^p}^{\alpha+1} + C_3 ||\dot{u}||_{L^p}
$$

holds for all $u \in W_T^{1,p}$ and some positive constant C_1'. Hence, for large n, one has

$$
\begin{aligned}
||\tilde{u}_n|| &\geq |\langle \varphi'(u_n), \tilde{u}_n \rangle| \\
&\geq (1 - rT^p - T^{\frac{p}{q}})||\dot{u}_n||_{L^p}^p - C_1'|\bar{u}_n|^{q\alpha} - C_2||\dot{u}_n||_{L^p}^{\alpha+1} - C_3||\dot{u}_n||_{L^p}.
\end{aligned}
$$

Then,

$$
\begin{aligned}
C_1'|\bar{u}_n|^{q\alpha} &\geq (1 - rT^p - T^{\frac{p}{q}})||\dot{u}_n||_{L^p}^p - C_2||\dot{u}_n||_{L^p}^{\alpha+1} - (1 + T + C_3)||\dot{u}_n||_{L^p} \\
&= ||\dot{u}_n||_{L^p}^p[(1 - rT^p - T^{\frac{p}{q}}) - C_2||\dot{u}_n||_{L^p}^{\alpha+1-p} - (1 + T + C_3)||\dot{u}_n||_{L^p}^{1-p}].
\end{aligned}
$$

From (B_1), (B_2), one has $\alpha + 1 < \frac{p}{q} + 1 = p$ and $1 - rT^p - T^{\frac{p}{q}} > 0$. Hence, with the similar discussion above,

$$
C_4'|\bar{u}_n|^{q\alpha} \geq ||\dot{u}_n||_{L^p}^p - C_5'
$$
(3.8)

holds for some constants C_4', C_5' and large n, that is,

$$
||\tilde{u}_n||_\infty^{\frac{p}{q}} \leq C_6'(|\bar{u}_n|^\alpha + 1)
$$
(3.9)

for some constants $C_6' \neq 1$ and large n. Then,

(3.10) $|u_n(t)|^{\frac{p}{q}} \geq |\bar{u}_n|^{\frac{p}{q}} - ||\tilde{u}_n(t)||_\infty^{\frac{p}{q}} \geq |\bar{u}_n|^{\frac{p}{q}} - C_6'(|\bar{u}_n|^\alpha + 1)$

for large n and $t \in [0, T]$.

Then from (3.7) and (3.10), whenever $p \geq q$ or $p < q$, (3.2) holds for some constants C_0, C_0' and large n. If $(|\bar{u}_n|)_{n \in \mathbb{N}}$ is unbounded, we may assume that, going to a subsequence if necessary,

$$(3.11) \qquad |\bar{u}_n| \to \infty \quad \text{as } n \to \infty.$$

Set $\delta = \frac{m(E)}{2}$. It follows from Lemma 2.2 and (B_3) that there exists a subset E_δ of E with $\text{meas}(E \setminus E_\delta) < \delta$ such that

$$|u|^{-\max\{p,q\}\alpha} F(t, u) \to -\infty \quad \text{as } |u| \to \infty$$

uniformly for all $t \in E_\delta$, which implies that

$$m(E_\delta) = m(E) - m(E \setminus E_\delta) > \delta > 0.$$

Moreover, for every $N > 0$, there exists $M \geq 1$ such that

$$|u|^{-\max\{p,q\}\alpha} F(t, u) \leq -N$$

for all $|u| \geq M$ and $t \in E_\delta$. By (3.2) and (3.11), one has

$$|u_n(t)| \geq M$$

for large n and every $t \in [0, T]$. Then

$$
\begin{aligned}
\varphi(u_n) &= \frac{1}{p} \int_0^T |\dot{u}_n(t)|^p dt + \int_0^T F(t, u_n(t)) dt \\
&= \frac{1}{p} \|\dot{u}_n(t)\|_{L^p}^p + \int_{E_\delta} F(t, u_n(t)) dt + \int_{[0,T] \setminus E_\delta} F(t, u_n(t)) dt \\
&\leq \frac{1}{p} \|\dot{u}_n(t)\|_{L^p}^p - N \int_{E_\delta} |u_n(t)|^{\max\{p,q\}\alpha} dt + \int_{[0,T] \setminus E_\delta} c(t) |u_n|^{-\gamma} dt \\
&\quad + \int_{[0,T] \setminus E_\delta} d(t) dt \\
&\leq \frac{1}{p} \|\dot{u}_n(t)\|_{L^p}^p - 2^{-\max\{p,q\}\alpha} N \delta C_0^{\max\{p,q\}\alpha} |\bar{u}_n|^{\max\{p,q\}\alpha} \\
&\quad + N \delta C_0'^{\max\{p,q\}\alpha} + M^{-\gamma} \int_{[0,T] \setminus E_\delta} c(t) dt + \int_{[0,T] \setminus E_\delta} d(t) dt
\end{aligned}
$$

$$(3.12) \qquad \leq \begin{cases} \frac{1}{p} (C_4 |\bar{u}_n|^\alpha + C_5)^p - (2^{-p\alpha} C_0^{p\alpha} |\bar{u}_n|^{p\alpha} - C_0'^{p\alpha}) N \delta \\ + M^{-\gamma} \int_{[0,T] \setminus E_\delta} c(t) dt + \int_{[0,T] \setminus E_\delta} d(t) dt, \text{ if } p \geq q, \\ \frac{1}{p} (C_4' |\bar{u}_n|^{q\alpha} + C_5') - (2^{-q\alpha} C_0^{q\alpha} |\bar{u}_n|^{q\alpha} - C_0'^{q\alpha}) N \delta \\ + M^{-\gamma} \int_{[0,T] \setminus E_\delta} c(t) dt + \int_{[0,T] \setminus E_\delta} d(t) dt, \text{ if } p < q \end{cases}$$

for large n. Hence, we have

$$\limsup_{n \to \infty} |\bar{u}_n|^{-\max\{p,q\}\alpha} \varphi(u_n) \leq -2^{-\max\{p,q\}\alpha} C_0^{\max\{p,q\}\alpha} \delta N + \frac{1}{p} \max\{C_4^p, C_4'^p\}.$$

By the arbitrariness of $N > 0$, one has

$$(3.13) \qquad \limsup_{n \to \infty} |\bar{u}_n|^{-\max\{p,q\}\alpha} \varphi(u_n) = -\infty$$

which contradicts with the boundedness of $\varphi(u_n)$. Therefore, $(|\bar{u}_n|)_{n \in \mathbb{N}}$ is bounded. Furthermore, by (3.5), (3.8), and

$$\|u_n\| \leq \|\bar{u}_n\| + (T+1)\|\dot{\bar{u}}_n\|_{L^p},$$

one has $(u_n)_{n \in \mathbb{N}}$ is bounded in $W_T^{1,p}$. Going if necessary to a subsequence, we can assume that $u_n \rightharpoonup u$ in $W_T^{1,p}$ and $u_n \to u$ in $C([0,T], R^N)$. Hence,

(3.14)

$$\langle \varphi'(u_n), u_n - u \rangle$$
$$= \int_0^T [(|\dot{u}_n(t)|^{p-2}\dot{u}_n(t), \dot{u}_n(t) - \dot{u}(t)) + (\nabla F(t, u_n(t)), u_n(t) - u(t))]dt \to 0$$
$$\text{as } n \to \infty,$$

and $(u_n)_{n \in \mathbb{N}}$ is bounded in $C([0,T], R^N)$. Therefore

(3.15) $$\int_0^T (|u_n|^{p-2}u_n, u_n - u)dt \to 0 \ \text{ as } n \to \infty.$$

By assumption (A),

$$\left| \int_0^T (\nabla F(t, u_n(t)), u_n(t) - u(t))dt \right| \leq \int_0^T |\nabla F(t, u_n(t))| \cdot |u_n(t) - u(t)|dt$$
$$\leq C_7 \|b\|_{L^1} \|u_n - u\|_\infty$$

holds for some positive constant C_7. Then

(3.16) $$\int_0^T (\nabla F(t, u_n(t)), u_n(t) - u(t))dt \to 0 \ \text{ as } \ n \to \infty.$$

From (3.14) and (3.16), we obtain

(3.17) $$\int_0^T (|\dot{u}_n(t)|^{p-2}\dot{u}_n(t), \dot{u}_n(t) - \dot{u}(t))dt \to 0 \ \text{ as } \ n \to \infty.$$

Set $\psi(u) = \frac{1}{p}\|u\|^p = \frac{1}{p}(\int_0^T |u|^p dt + \int_0^T |\dot{u}|^p dt)$. By (3.15) and (3.17), we have

(3.18)

$$\langle \psi'(u_n), u_n - u \rangle$$
$$= \int_0^T (|u_n|^{p-2}u_n, u_n - u)dt + \int_0^T (|\dot{u}_n|^{p-2}\dot{u}_n, \dot{u}_n - \dot{u})dt \to 0 \ \text{ as } \ n \to \infty.$$

Using the Hölder's inequality, we have

(3.19)

$$\langle \psi'(u_n) - \psi'(u), u_n - u \rangle$$
$$= \int_0^T (|u_n|^{p-2}u_n, u_n - u)dt + \int_0^T (|\dot{u}_n|^{p-2}\dot{u}_n, \dot{u}_n - \dot{u})dt$$

$$- \int_0^T (|u|^{p-2}u, u_n - u)dt - \int_0^T (|\dot{u}|^{p-2}\dot{u}, \dot{u}_n - \dot{u})dt$$

$$\geq \int_0^T |u_n|^p dt + \int_0^T |\dot{u}_n|^p dt + \int_0^T |u|^p dt + \int_0^T |\dot{u}|^p dt$$

$$- \left(\int_0^T |u_n|^p dt \right)^{\frac{1}{q}} \left(\int_0^T |u|^p dt \right)^{\frac{1}{p}} - \left(\int_0^T |\dot{u}_n|^p dt \right)^{\frac{1}{q}} \left(\int_0^T |\dot{u}|^p dt \right)^{\frac{1}{p}}$$

$$- \left(\int_0^T |u|^p dt \right)^{\frac{1}{q}} \left(\int_0^T |u_n|^p dt \right)^{\frac{1}{p}} - \left(\int_0^T |\dot{u}|^p dt \right)^{\frac{1}{q}} \left(\int_0^T |\dot{u}_n|^p dt \right)^{\frac{1}{p}}$$

$$\geq ||u_n||^p + ||u||^p - \left[\int_0^T |u_n|^p dt + \int_0^T |\dot{u}_n|^p dt \right]^{\frac{1}{q}} \cdot \left[\int_0^T |u|^p dt + \int_0^T |\dot{u}|^p dt \right]^{\frac{1}{p}}$$

$$- \left[\int_0^T |u_n|^p dt + \int_0^T |\dot{u}_n|^p dt \right]^{\frac{1}{p}} \cdot \left[\int_0^T |u|^p dt + \int_0^T |\dot{u}|^p dt \right]^{\frac{1}{q}}$$

$$= (||u_n||^{p-1} - ||u||^{p-1})(||u_n|| - ||u||) \geq 0.$$

From (3.18) and (3.19), one has $||u_n|| \to ||u||$. Then, $u_n \to u$ holds in $W_T^{1,p}$ by the uniform convexity of $W_T^{1,p}$. Therefore, the PS-condition holds. $\square$

Theorem 3.1. *Assume $F(t, u)$ satisfy (A) and (B_1)-(B_4). Then problem (1.1) has at least one solution in $W_T^{1,p}$.*

Proof. It remains to show that φ satisfies other conditions of the Saddle Point Theorem. Let $\tilde{W}_T^{1,p}$ be the subspace of $W_T^{1,p}$ given by

$$\tilde{W}_T^{1,p} = \{u \in W_T^{1,p} | \bar{u} = 0\}.$$

Then $W_T^{1,p} = R^N \oplus \tilde{W}_T^{1,p}$. By (2.5), one obtains

$$\left| \int_0^T [H(t, u(t)) - H(t, 0)]dt \right|$$

$$= \left| \int_0^T \int_0^1 (\nabla H(t, su(t)), u(t))ds dt \right|$$

$$\leq \int_0^T \int_0^1 f(t)|su(t)|^\alpha |u(t)|ds dt + \int_0^T \int_0^1 g(t)|u(t)|ds dt$$

$$\leq \int_0^T f(t)|u(t)|^{\alpha+1}dt + \int_0^T g(t)|u(t)|dt$$

$$\leq ||u(t)||_\infty^{\alpha+1} \int_0^T f(t)dt + ||u(t)||_\infty \int_0^T g(t)dt$$

$$\leq T^{\frac{\alpha+1}{q}} ||\dot{u}(t)||_{L^p}^{\alpha+1} \int_0^T f(t)dt + T^{\frac{1}{q}} ||\dot{u}(t)||_{L^p} \int_0^T g(t)dt$$

$$= C_8||\dot{u}(t)||_{L^p}^{\alpha+1} + C_9||\dot{u}(t)||_{L^p}$$

for all $u \in \tilde{W}_T^{1,p}$ and some positive constants C_8 and C_9. Moreover,

$$
\begin{aligned}
\int_0^T [G(u(t)) - G(0)]dt &= \int_0^T \int_0^1 (\nabla G(su(t)), u(t))dsdt \\
&= \int_0^T \int_0^1 (\nabla G(su(t)) - \nabla G(0), u(t))dsdt \\
&= \int_0^T \int_0^1 \frac{1}{s}(\nabla G(su(t)) - \nabla G(0), su(t))dsdt \\
&\geq \int_0^T \int_0^1 \frac{1}{s}(-r|su(t)|^p)dsdt \\
&= -r \int_0^T \int_0^1 s^{p-1}|u(t)|^p dsdt \\
&= -\frac{r}{p}||u||_{L^p}^p
\end{aligned}
$$

for all $u \in \tilde{W}_T^{1,p}$. Hence

$$
\begin{aligned}
(3.20) \quad \varphi(u) - \int_0^T F(t,0)dt &= \frac{1}{p}\int_0^T |\dot{u}(t)|^p dt + \int_0^T [F(t,u(t)) - F(t,0)]dt \\
&= \frac{1}{p}\int_0^T |\dot{u}(t)|^p dt + \int_0^T [G(u(t)) - G(0)]dt \\
&\quad + \int_0^T [H(t,u(t)) - H(t,0)]dt \\
&\geq \frac{1}{p}||\dot{u}||_{L^p}^p - \frac{r}{p}||u||_{L^p}^p - C_8||\dot{u}||_{L^p}^{\alpha+1} - C_9||\dot{u}||_{L^p} \\
&\geq \frac{1 - rT^p}{p}||\dot{u}||_{L^p}^p - C_8||\dot{u}||_{L^p}^{\alpha+1} - C_9||\dot{u}||_{L^p}.
\end{aligned}
$$

Let $||u|| \to \infty$ in $\tilde{W}_T^{1,p}$. Then $||\dot{u}||_{L^p} \to \infty$ or $||u||_{L^p} \to \infty$. From (2.6), one has $||\dot{u}||_{L^p} \to \infty$ if $||u||_{L^p} \to \infty$. Moreover, from (B_1), $\alpha + 1 < p$, hence, we have $\varphi(u) \to +\infty$ while $||u|| \to \infty$ in $\tilde{W}_T^{1,p}$. On the other hand, for $\forall u \in R^N$, we have

(3.21)

$$
\begin{aligned}
\varphi(u) &= \int_0^T F(t,u)dt \\
&\leq \int_{[0,T]\setminus E_\delta} (c(t)|u|^{-\gamma} + d(t))dt - \int_{E_\delta} N|u|^{(p+q)\alpha}dt \\
&\leq M^{-\gamma} \int_{[0,T]\setminus E_\delta} c(t)dt + \int_{[0,T]\setminus E_\delta} d(t)dt - \text{meas}(E_\delta)NM^{(p+q)\alpha}
\end{aligned}
$$

$$\leq M^{-\gamma} \int_{[0,T] \setminus E_\delta} c(t)dt + \int_{[0,T] \setminus E_\delta} d(t)dt - \text{meas}(E_\delta)N, \ \forall \ |u| \geq M.$$

It is easy to obtain that

$$(3.22) \qquad\qquad \varphi(u) \to -\infty, \ |u| \to \infty, \ u \in \mathbb{R}^N.$$

Then, from the Saddle Point Theorem, we obtain that the problem (1.1) has at least one solution. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Remark 3.1. If $\nabla F(t, \theta) \not\equiv \theta$ for $t \in [0, T]$, the Hamiltonian system (1.1) has non-trivial solutions.

Theorem 3.2. *Assume $F(t, u)$ satisfy (A) and (B_1)-(B_4) and the following conditions*:

(B_5) $F(t, \theta) = 0$, $F(t, -u) = F(t, u)$;

(B_6) *there exists $e(t) \in L^1(0, T; R^+)$ and constants β satisfying $\beta > p$ and $M' > 0$ such that $F(t, u) \geq -e(t)|u|^\beta$ holds for $u \in \tilde{W}_T^{1,p}$ and $||u|| \leq M'$;*

(B_7) *there exists $l(t)$ satisfying $\int_0^T l(t)dt > 0$ and constants $m > p$ and $M'' > M'$ such that $\int_0^T F(t, u)dt \leq -\int_0^T l(t)dt ||u||^m$ holds for $u \in V$ and $||u|| \geq M''$, where V is an arbitrary subspace of $W_T^{1,p}$ and $\dim V < \infty$.*

Then problem (1.1) has infinite periodic solutions in $W_T^{1,p}$.

Proof. From (B_5), φ is an even functional satisfying $\varphi(\theta) = 0$. For $u \in \tilde{W}_T^{1,p}$,

$$\int_0^T F(t, u)dt \geq -\int_0^T e(t)|u|^\beta dt \geq -||u||_\infty^\beta \int_0^T e(t)dt \geq -T^{\frac{\beta}{q}}||\dot{u}||_{L^p}^\beta \int_0^T e(t)dt$$

holds. Then, by (3.1),

$$(3.23) \quad \varphi(u) = \frac{1}{p}||\dot{u}||_{L^p}^p + \int_0^T F(t, u)dt \geq \frac{1}{p}||\dot{u}||_{L^p}^p - T^{\frac{\beta}{q}}||\dot{u}||_{L^p}^\beta \int_0^T e(t)dt$$

$$\geq \frac{1}{p(T+1)^p}||u||^p - T^{\frac{\beta}{q}} \int_0^T e(t)dt ||u||^\beta$$

holds. Choose $M' > \rho_1 > 0$ small enough such that $\rho_2 = \frac{1}{p(T+1)^p}\rho_1^p - T^{\frac{\beta}{q}} \int_0^T e(t)dt \rho_1^\beta > 0$. Then, one has $\varphi(u) \geq \rho_2 > 0$, that is, there exists $\rho_1 > 0$, $\rho_2 > 0$, $\varphi(u) \geq \rho_2$ holds for $u \in \{u \in \tilde{W}_T^{1,p} : ||u|| = \rho_1\}$.

Suppose (A_2) is not satisfied, that is, there exists $V_1 \subset W_T^{1,p}$ which is a subspace with finite dimension such that $\{u \in V_1 : \varphi(u) \geq 0\}$ is unbounded. Then, there exist $(u_n)_{n \in \mathbb{N}} \subset V_1$, $||u_n|| \to \infty$ as $n \to \infty$, and $\varphi(u_n) \geq 0$. Moreover, we may assume $||u_n|| \geq M''$. Then,

$$(3.24) \quad \varphi(u_n) = \frac{1}{p}||\dot{u}_n||_{L^p}^p + \int_0^T F(t, u_n)dt \leq \frac{1}{p}||\dot{u}_n||_{L^p}^p - ||u_n||^m \int_0^T l(t)dt$$

$$\leq \frac{1}{p} \left(\int_0^1 |u_n|^p dt + \int_0^1 |\dot{u}_n|^p dt \right) - ||u_n||^m \int_0^T l(t) dt$$

$$= \frac{1}{p} ||u_n||^p - ||u_n||^m \int_0^T l(t) dt.$$

Hence, we have $\overline{\lim}_{n\to\infty} \varphi(u_n) = -\infty$, which contradicts with $\varphi(u_n) \geq 0$. Then (A_2) is satisfied. Hence, φ has infinite critical points in $W_T^{1,p}$. Therefore, (1.1) has infinite solutions. $\qquad\square$

Remark 3.2. From Theorem 2.2, we have the Hamiltonian system (1.1) has infinitely non-trivial solutions.

Similarly, we have the following theorem.

Theorem 3.3. *Assume $F(t,u)$ satisfy (A) and (B_1)-(B_5) and the following conditions*:

(B_6') *there exists $e(t) \in L^1(0,T;\mathbb{R}^+)$ satisfying $\frac{1}{p} > T^{\frac{p}{q}} \int_0^T e(t)dt$ and a constant $M' > 0$ such that $F(t,u) \geq -e(t)|u|^p$ holds for $u \in \tilde{W}_T^{1,p}$ and $||u|| \leq M'$;*

(B_7') *there exists $l(t)$ satisfying $\frac{1}{p} < \int_0^T l(t)dt$ and a constant $M'' > M'$ such that $\int_0^T F(t,u)dt \leq -\int_0^T l(t)dt||u||^p$ holds for $u \in V$ and $||u|| \geq M''$, where V is an arbitrary subspace of $W_T^{1,p}$ and $\dim V < \infty$.*

Then problem (1.1) has infinite periodic solutions in $W_T^{1,p}$.

Example 3.1. Let $T = \frac{1}{3}, \gamma = \frac{11}{2}$, and $F : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ as

$$F(t,x) = \begin{cases} -2|x|^2 + |x| - 3|x|^3, & |x| < 1, \\ -2|x|^2 - 9|x| + 9 - 2|x|^{-\frac{1}{2}}, & |x| \geq 1. \end{cases}$$

Here, $G(x) = -2|x|^2$ and

$$H(t,x) = \begin{cases} |x| - 3|x|^3, & |x| < 1, \\ -9|x| + 9 - 2|x|^{-\frac{1}{2}}, & |x| \geq 1. \end{cases}$$

It is obvious that $\alpha = 0$, $F(t,0) = 0$, $F(t,-x) = F(t,x)$. Moreover, $F(t,x) \geq -4|x|^2$ if $|x| < 1$ and there exists $M'' > 1$ such that $-9|x| + 9 - 2|x|^{-\frac{1}{2}} < 0$ if $|x| \geq M''$, so, $\int_0^T F(t,x)dt \leq -\frac{2}{3}|x|^2$. Hence, $F(t,x)$ satisfies all conditions of Theorem 3.3.

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