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SCREEN CONFORMAL LIGHTLIKE REAL HYPERSURFACES OF AN INDEFINITE COMPLEX SPACE FORM

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Abstract. In this paper, we study the geometry of screen conformal lightlike real hypersurfaces of an indefinite Kaehler manifold. The main result is a characterization theorem for screen conformal lightlike real hypersurfaces of an indefinite complex space form.

1. Introduction

It is well known that the normal bundle $TM^{\perp}$ of the lightlike hypersurfaces M of a semi-Riemannian manifold $\bar{M}$ is a vector subbundle of the tangent bundle TM of rank 1. Then there exists a complementary non-degenerate vector bundle $S(TM)$ of $TM^{\perp}$ in TM, which called a *screen distribution* on M, such that

$$TM = TM^{\perp} \oplus_{\rm orth} S(TM), \tag{1.1}$$

where $\oplus_{\rm orth}$ denotes the orthogonal direct sum. We denote such a lightlike hypersurface by $(M, g, S(TM))$. Denote by $F(M)$ the algebra of smooth functions on M and by $\Gamma(E)$ the $F(M)$ module of smooth sections of a vector bundle E over M. We use the same notation for any other vector bundle.

We known [2] that, for any null section ξ of $TM^{\perp}$ on a coordinate neighborhood $\mathcal{U} \subset M$, there exists a unique null section N of a unique vector bundle $\mathrm{tr}(TM)$ of rank 1 in $S(TM)^{\perp}$ satisfying

$$\bar{g}\,(\xi, N) = 1, \quad \bar{g}(N, N) = \bar{g}(N, X) = 0 \tag{1.2}$$

for any $X \in \Gamma(S(TM))$. In this case, $T\bar{M}$ is decomposed as follows:

$$T\bar{M} = TM \oplus \mathrm{tr}(TM) = \{TM^{\perp} \oplus \mathrm{tr}(TM)\} \oplus_{\rm orth} S(TM). \tag{1.3}$$

We call $\mathrm{tr}(TM)$ and N the *transversal vector bundle* and the *null transversal vector field* of M with respect to $S(TM)$, respectively.

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The purpose of this paper is to prove a characterization theorem for lightlike real hypersurfaces M of an indefinite complex space form $\bar{M}(c)$: If M is screen conformal, then $c = 0$ (Theorem 3.7). Using this theorem, we prove several additional theorems for screen conformal lightlike real hypersurfaces M of $\bar{M}(c)$.

The local Gauss and Weingarten formulas are given by

$$\bar{\nabla}_X Y = \nabla_X Y + B(X,Y)N, \tag{1.4}$$

$$\bar{\nabla}_X N = -A_N X + \tau(X)N, \tag{1.5}$$

$$\nabla_X PY = \nabla^*_X PY + C(X,PY)\xi, \tag{1.6}$$

$$\nabla_X \xi = -A^*_\xi X - \tau(X)\xi \tag{1.7}$$

for any $X,\, Y \in \Gamma(TM)$, where $\bar{\nabla}$, ∇ and ∇^* are the Levi-Civita connection of $\bar{M}$, the liner connections on TM and $S(TM)$ respectively, P is the projection morphism of $\Gamma(TM)$ on $\Gamma(S(TM))$ with respect to the decomposition (1.1), B and C are the local second fundamental forms on TM and $S(TM)$ respectively, A_N and A^*_ξ are the shape operators on TM and $S(TM)$ respectively and τ is a 1-form on TM. Since $\bar{\nabla}$ is torsion-free, ∇ is also torsion-free and B is symmetric on TM. From the fact that $B(X,Y) = \bar{g}(\bar{\nabla}_X Y,\, \xi)$ for any $X,\, Y \in \Gamma(TM)$, we show that the local second fundamental form B is independent of the choice of a screen distribution and satisfies

$$B(X,\xi) = 0 \tag{1.8}$$

for any $X \in \Gamma(TM)$. The induced connection ∇ of M is not metric and satisfies

$$(\nabla_X g)(Y,Z) = B(X,Y)\,\eta(Z) + B(X,Z)\,\eta(Y) \tag{1.9}$$

for any $X,\, Y,\, Z \in \Gamma(TM)$, where η is a 1-form such that

$$\eta(X) = \bar{g}(X,N) \tag{1.10}$$

for any $X \in \Gamma(TM)$. But the connection ∇^* on $S(TM)$ is metric. Two local second fundamental forms B and C are related to their shape operators by

$$B(X,Y) = g(A^*_\xi X, Y), \qquad \bar{g}(A^*_\xi X, N) = 0, \tag{1.11}$$

$$C(X,PY) = g(A_N X, PY), \qquad \bar{g}(A_N X, N) = 0, \tag{1.12}$$

for any $X,\, Y \in \Gamma(TM)$. From (1.11), the operator A^*_ξ is $\Gamma(S(TM))$-valued self-adjoint on $\Gamma(TM)$ with respect to the induced metric g on M such that

$$A^*_\xi \xi = 0. \tag{1.13}$$

Thus ξ is an eigenvector of A^*_ξ corresponding to the eigenvalue 0.

We denote by $\bar{R}$, R and R^* the curvature tensors of the Levi-Civita connection $\bar{\nabla}$ of $\bar{M}$, the induced connection ∇ of M and the connection ∇^* on $S(TM)$, respectively. Using the Gauss-Weingarten equations for M and $S(TM)$, we obtain the Gauss-Codazzi equations for M and $S(TM)$ such that

$$\begin{aligned} \bar{g}(\bar{R}(X,\, Y)Z,\, PW) = {} & g(R(X,\, Y)Z,\, PW) \\ & + B(X,\, Z)C(Y,\, PW) - B(Y,\, Z)C(X,\, PW), \end{aligned} \tag{1.14}$$

$$
\begin{aligned}
(1.15)\qquad \bar{g}(\bar{R}(X, Y)Z, \xi) &= g(R(X, Y)Z, \xi) \\
&= (\nabla_X B)(Y, Z) - (\nabla_Y B)(X, Z) \\
&\quad + B(Y, Z)\tau(X) - B(X, Z)\tau(Y), \\
(1.16)\qquad \bar{g}(\bar{R}(X, Y)Z, N) &= g(R(X, Y)Z, N), \\
(1.17)\qquad g(R(X, Y)PZ, PW) &= g(R^*(X, Y)PZ, PW) \\
&\quad + C(X, PZ)B(Y, PW) \\
&\quad - C(Y, PZ)B(X, PW), \\
(1.18)\qquad g(R(X, Y)PZ, N) &= (\nabla_X C)(Y, PZ) - (\nabla_Y C)(X, PZ) \\
&\quad + C(X, PZ)\tau(Y) - C(Y, PZ)\tau(X)
\end{aligned}
$$

for any $X, Y, Z, W \in \Gamma(TM)$. The *Ricci tensor* $\bar{Ric}$ of $\bar{M}$ is defined by

$$(1.19)\qquad \bar{Ric}(X, Y) = \mathrm{trace}\{Z \to \bar{R}(Z, X)Y\}, \quad \forall\, X, Y \in \Gamma(T\bar{M}).$$

$\bar{M}$ is called *Ricci flat* if its Ricci tensor vanishes identically. If $\dim \bar{M} > 2$ and $\bar{Ric} = \bar{\gamma} g$, where $\bar{\gamma}$ is a constant, then $\bar{M}$ is called an *Einstein manifold*.

2. Hypersurfaces of indefinite Kaehler manifolds

Let $\bar{M} = (\bar{M}, J, \bar{g})$ be a real $2m$-dimensional indefinite Kaehler manifold, where $\bar{g}$ is a semi-Riemannian metric of index $q = 2v$ $(0 < v < m)$ and J is an almost complex structure on $\bar{M}$ satisfying, for all $X, Y \in \Gamma(T\bar{M})$,

$$(2.1)\qquad J^2 = -I, \quad \bar{g}(JX, JY) = \bar{g}(X, Y), \quad (\bar{\nabla}_X J)Y = 0.$$

An indefinite complex space form, denoted by $\bar{M}(c)$, is a connected indefinite Kaehler manifold of constant holomorphic sectional curvature c such that

$$
\begin{aligned}
(2.2)\qquad \bar{R}(X, Y)Z \;=\; &\frac{c}{4}\{\bar{g}(Y, Z)X - \bar{g}(X, Z)Y + \bar{g}(JY, Z)JX \\
&- \bar{g}(JX, Z)JY + 2\bar{g}(X, JY)JZ\}
\end{aligned}
$$

for all $X, Y, Z \in \Gamma(TM)$. Suppose $(M, g, S(TM))$ is a lightlike real hypersurface of $\bar{M}$, where g is the degenerate induced metric of M. Then the screen distribution $S(TM)$ splits as follows [2]:

If ξ and N are local sections of $TM^\perp$ and $\mathrm{tr}(TM)$ respectively, we have

$$(2.3)\qquad \bar{g}(J\xi, \xi) = \bar{g}(J\xi, N) = \bar{g}(JN, \xi) = \bar{g}(JN, N) = 0, \quad \bar{g}(J\xi, JN) = 1.$$

This shows that $J\xi$ and JN are vector fields tangent to M. Thus $J(TM^\perp)$ and $J(\mathrm{tr}(TM))$ are distributions on M of rank 1 such that $TM^\perp \cap J(TM^\perp) = \{0\}$ and $TM^\perp \cap J(\mathrm{tr}(TM)) = \{0\}$. Hence $J(TM^\perp) \oplus J(\mathrm{tr}(TM))$ is a vector subbundle of $S(TM)$ of rank 2. Then there exists a non-degenerate almost complex distribution D_o on M with respect to J, i.e., $J(D_o) = D_o$, such that

$$(2.4)\qquad S(TM) = \{J(TM^\perp) \oplus J(\mathrm{tr}(TM))\} \oplus_{\mathrm{orth}} D_o.$$

Therefore the general decompositions (1.1) and (1.3) become respectively

$$(2.5)\qquad TM = \{J(TM^\perp) \oplus J(\mathrm{tr}(TM))\} \oplus_{\mathrm{orth}} D_o \oplus_{\mathrm{orth}} TM^\perp,$$

$$(2.6)\qquad T\bar{M} = \{J(TM^\perp) \oplus J(\mathrm{tr}(TM))\} \oplus_{\mathrm{orth}} D_o \oplus_{\mathrm{orth}} \{\mathrm{tr}(TM) \oplus TM^\perp\}.$$

Consider the 2-lightlike almost complex distribution D such that

$$(2.7)\qquad D = \{TM^\perp \oplus_{\mathrm{orth}} J(TM^\perp)\} \oplus_{\mathrm{orth}} D_o\,; \quad TM = D \oplus J(\mathrm{tr}(TM))$$

and the local lightlike vector fields U and V such that

$$(2.8)\qquad U = -JN\,; \quad V = -J\xi.$$

Denote by S the projection morphism of TM on D with respect to the decomposition (2.7). Then any vector field X on M is expressed as follows

$$(2.9)\qquad X = SX + u(X)U\,; \quad JX = FX + u(X)N,$$

where u and v are 1-forms locally defined on M by

$$(2.10)\qquad u(X) = g(X, V), \quad v(X) = g(X, U)$$

and F is a tensor field of type $(1, 1)$ globally defined on M by

$$(2.11)\qquad FX = JSX, \quad \forall\, X \in \Gamma(TM).$$

Apply J to the second equation of (2.9) and using (2.1) and (2.8), we have

$$(2.12)\qquad F^2X = -X + u(X)U\,; \quad u(U) = 1.$$

Thus (F, u, U) defines an almost contact structure on M. But it is not an almost contact metric structure. Because, using (2.1)-2 and (2.9)-2, we have

$$(2.13)\qquad g(FX, FY) = g(X, Y) - u(X)v(Y) - u(Y)v(X)$$

for all $X, Y \in \Gamma(TM)$. By using (2.9)-2 and (2.10) and Gauss-Weingarten equations for a lightlike hypersurface, for any $X, Y \in \Gamma(TM)$, we deduce

$$(2.14)\qquad (\nabla_X u)(Y) = -u(Y)\tau(X) - B(X, FY),$$

$$(2.15)\qquad (\nabla_X v)(Y) = v(Y)\tau(X) - g(A_N X, FY),$$

$$(2.16)\qquad (\nabla_X F)(Y) = u(Y)A_N X - B(X, Y)U.$$

Differentiate (2.8) with X and use (1.5), (1.7), (2.1)-3 and (2.9)-2, we have

$$(2.17)\qquad B(X, U) = v(A^*_\xi X) = u(A_N X) = C(X, V),\ \forall\, X \in \Gamma(TM),$$

$$(2.18)\qquad \nabla_X U = F(A_N X) + \tau(X)U, \quad \nabla_X V = F(A^*_\xi X) - \tau(X)V.$$

Example 1. Let $(\mathbb{R}^6_2, \bar{g})$ be a 6-dimensional semi-Euclidean space of index 2 with signature $(-, -, +, +, +, +)$ of the canonical basis $(\partial_0, \ldots, \partial_5)$. Consider a Monge hypersurface M of $\mathbb{R}^6_2$ given by

$$x_0 = u_1 + u_2 + u_3 \quad \text{and} \quad x_i = u_i\,(1 \le i \le 5).$$

Then the tangent bundle TM is spanned by

$$\left\{\partial_{u_1} = \partial_0 + \partial_1,\ \partial_{u_2} = \partial_0 + \partial_2,\ \partial_{u_3} = \partial_0 + \partial_3,\ \partial_{u_4} = \partial_4,\ \partial_{u_5} = \partial_5\right\}.$$

It is easy to check that M is a lightlike hypersurface whose radical distribution $\mathrm{Rad}(TM)$ is spanned by

$$\xi = \partial_0 - \partial_1 + \partial_2 + \partial_3.$$

Let $V = \partial_0 - \partial_1$, then $g(V, V) = -2$ and $g(\xi, V) = -2$. Then the lightlike transversal vector bundle is given by

$$\mathrm{tr}(TM) = \mathrm{Span}\{N = -\frac{1}{4}(\partial_0 - \partial_1 - \partial_2 - \partial_3)\}.$$

It follows that the corresponding screen distribution $S(TM)$ is spanned by

$$\{W_1 = \partial_0 + \partial_1,\ W_2 = \partial_2 - \partial_3,\ W_3 = \partial_4,\ W_4 = \partial_5\}.$$

Since $\mathbb{R}^6_2$ has complex structure J, we see that $J\xi = W_1 - W_2 \in \Gamma(S(TM))$, $JN = -\frac{1}{4}\{W_1 + W_2\} \in \Gamma(S(TM))$, $JW_3 = W_4$ and $JW_4 = -W_3$. Thus the almost complex distribution D_o is given by $D_o = \mathrm{Span}\{W_3, W_4\}$.

Theorem 2.1. *Let $(M, g, S(TM))$ be a lightlike real hypersurface of an indefinite Kaehler manifold $\bar{M}$. Then we have the following assertions.*

(i) *If F and V are parallel with respect to the induced connection ∇ on M, then M is totally geodesic in $\bar{M}$ and the 1-form τ vanishes.*

(ii) *If V and U are parallel with respect to the induced connection ∇ on M, then $S(TM)$ is totally geodesic in M and the 1-form τ vanishes.*

Proof. If V is parallel with respect to the induced connection ∇ on M, then, from the second equation of (2.18), we have

$$J(A^*_\xi X) - u(A^*_\xi X)N - \tau(X)V = 0,\ \forall\, X \in \Gamma(TM).$$

Apply J to the last equation and by using (2.1) and (2.8), we obtain

$$A^*_\xi X = u(A^*_\xi X)U \quad \text{and} \quad \tau(X) = 0,\ \forall\, X \in \Gamma(TM).$$

Substituting the last equation in (2.17), we have

$$u(A_N X) = v(A^*_\xi X) = g(A^*_\xi X, U) = u(A^*_\xi X)g(U, U) = 0,\ \forall\, X \in \Gamma(TM).$$

(i) If F is parallel with respect to ∇, then, from (2.16), we have

$$B(X, Y) = u(Y)u(A_N X), \quad \forall\, X, Y \in \Gamma(TM). \tag{2.19}$$

Thus if V is also parallel, we obtain $B = 0$, that is, M is totally geodesic in $\bar{M}$.

(ii) If U is parallel with respect to ∇, then, from (2.18)-1, we have

$$J(A_N X) - u(A_N X)N + \tau(X)U = 0, \quad \forall\, X \in \Gamma(TM).$$

Apply J to this equation and by using (2.1) and (2.8), we obtain

$$A_N X = u(A_N X)U \quad \text{and} \quad \tau(X) = 0, \quad \forall\, X \in \Gamma(TM).$$

Thus if V is also parallel, we obtain $A_N X = 0$ for all $X \in \Gamma(TM)$. Thus $C = 0$ due to (1.12), that is, $S(TM)$ is totally geodesic in M. □

Theorem 2.2. *Let $(M, g, S(TM))$ be a lightlike real hypersurface of an indefinite Kaehler manifold $\bar{M}$. If F is parallel with respect to the induced connection ∇, then the almost complex distribution D is parallel with respect to the induced connection ∇ and M is locally a product manifold $L_u \times M^\sharp$, where L_u is a null curve tangent to $J(\mathrm{tr}(TM))$ and $M^\sharp$ is a leaf of D.*

Proof. In general, by using (1.4), (1.7), (1.11) and (2.1), we derive

$$g(\nabla_X\xi, J\xi) = -g(\xi, \bar{\nabla}_X J\xi) = B(X, V), \quad g(\nabla_X J\xi,\, J\xi) = 0,$$
$$g(\nabla_X Y, J\xi) = g(JY, \bar{\nabla}_X\xi) = -g(JY, A^*_\xi X) = -B(X, JY) \tag{2.20}$$

for all $X \in \Gamma(TM)$ and $Y \in \Gamma(D_o)$. If F is parallel with respect to the induced connection ∇, then, taking $Y = V$ and $Y \in \Gamma(D_o)$ in (2.19) by turns, we have $B(X, V) = 0$ and $B(X, Y) = 0$ for all $X \in \Gamma(TM)$ respectively. It follow that $g(\nabla_X\xi, J\xi) = g(\nabla_X J\xi, J\xi) = g(\nabla_X Y, J\xi) = 0$ due to $JY \in \Gamma(D_o)$. Thus D is parallel with respect to ∇ and both D and $J(\mathrm{tr}(TM))$ are integrable distributions. Thus we obtain our theorem. □

3. Screen conformal lightlike real hypersurfaces

A lightlike hypersurface $(M, g, S(TM))$ of a semi-Riemannian manifold $(\bar{M}, \bar{g})$ is *screen conformal* [1] if the shape operators A_N and A^*_ξ of M and $S(TM)$ respectively are related by $A_N = \varphi\, A^*_\xi$, or equivalently

$$C(X, PY) = \varphi\, B(X, Y), \quad \forall\, X,\, Y \in \Gamma(TM), \tag{3.1}$$

where φ is a non-vanishing smooth function on a neighborhood $\mathcal{U}$ in M. In particular, if φ is a non-zero constant, M is called *screen homothetic* [4].

Note 1. For a screen conformal M, since C is symmetric on $\Gamma(S(TM))$, $S(TM)$ is integrable. Thus M is locally a product manifold $L_\xi \times M^*$ where L_ξ is a null curve tangent to $TM^\perp$ and M^* is a leaf of $S(TM)$ [2].

From (2.17) and (3.1), we obtain

$$B(X,\, U - \varphi V) = 0, \quad \forall\, X \in \Gamma(TM). \tag{3.2}$$

Theorem 3.1. *Let $(M, g, S(TM))$ be a screen conformal lightlike real hypersurface of an indefinite Kaehler manifold $\bar{M}$. Then the non-null vector field $U - \varphi V \neq 0$ is conjugate to any vector field on M. In particular, $U - \varphi V$ is an asymptotic vector field.*

Corollary 1. *Let $(M, g, S(TM))$ be a screen conformal lightlike real hypersurface of an indefinite Kaehler manifold $\bar{M}$. Then the second fundamental form B (consequently, C) is degenerate on $\Gamma(S(TM))$.*

Proof. Since $B(X, U - \varphi V) = 0$ for all $X \in \Gamma(S(TM))$ and $U - \varphi V \in \Gamma(S(TM))$, therefore B is degenerate on $\Gamma(S(TM))$. □

Theorem 3.2. *Let $(M, g, S(TM))$ be a screen conformal lightlike real hypersurface of an indefinite Kaehler manifold $\bar{M}$. If M or $S(TM)$ is totally umbilic, then M is totally geodesic in $\bar{M}$ and the leaf M^* of $S(TM)$ is totally geodesic in both M and $\bar{M}$.*

Proof. If M is a totally umbilical lightlike real hypersurface of $\bar{M}$, then there exists a smooth function ρ such that

$$B(X, Y) = \rho g(X, Y), \quad \forall\, X,\, Y \in \Gamma(TM).$$

From this fact and the equation (3.2), we have

$$\rho g(X, U - \varphi V) = 0, \quad \forall X \in \Gamma(TM).$$

Replace X by V and U by turns in the last equation, we have $\rho = 0$ and $\varphi\rho = 0$ respectively. Thus $B = C = 0$, that is, M and $S(TM)$ are totally geodesic. By the same method for totally umbilical $S(TM)$, we have $B = C = 0$. □

Theorem 3.3. *Let $(M, g, S(TM))$ be a screen conformal lightlike real hypersurface of an indefinite Kaehler manifold $\bar{M}$. If one of the set $\{V, U, F\}$ is parallel with respect ∇ on M, then M is totally geodesic in $\bar{M}$ and $S(TM)$ is totally geodesic in both M and $\bar{M}$. Moreover, if V or U is parallel, then $\tau = 0$.*

Proof. In the proof of Theorem 2.1, if V is parallel, then $\tau = 0$, $u(A_N X) = 0$ and $A^*_\xi X = u(A^*_\xi X)U$ for any $X \in \Gamma(TM)$. Using the second equation of the above relations and the fact that $A_N = \varphi A^*_\xi$, we have

$$u(A^*_\xi X) = u(A_N X)/\varphi = 0, \ \forall X \in \Gamma(TM).$$

From this and the fact that $A^*_\xi X = u(A^*_\xi X)U$ for all $X \in \Gamma(TM)$, we have $A^*_\xi = 0$. Also $A_N = \varphi A^*_\xi = 0$. Thus M and $S(TM)$ are totally geodesic.

If U is parallel, then $\tau = 0$ and $A_N X = u(A_N X)U$ for any $X \in \Gamma(TM)$. Thus we have $v(A_N X) = 0$ for any $X \in \Gamma(TM)$. Using the equation (2.17) and the fact that $A_N = \varphi A^*_\xi$, we have

$$u(A_N X) = v(A^*_\xi X) = v(A_N X)/\varphi = 0, \ \forall X \in \Gamma(TM).$$

It follow that $A_N = 0$ and $A^*_\xi = 0$. Thus M and $S(TM)$ are totally geodesic.

If F is parallel, then we have (2.19). Replace Y by V in (2.19), we have

$$u(A_N X) = \varphi\, u(A^*_\xi X) = \varphi\, B(X, V) = 0, \quad \forall X \in \Gamma(TM).$$

Thus, from (2.19) and (3.1), we have $B = C = 0$. □

From the equation (2.20) and Theorems 3.2 and 3.3, we have:

Theorem 3.4. *Let $(M, g, S(TM))$ be a screen conformal lightlike real hypersurface of an indefinite Kaehler manifold $\bar{M}$. If* (i) *M or $S(TM)$ is a totally umbilic, or* (ii) *one of the set $\{V, U, F\}$ is parallel with respect to ∇, then D is parallel with respect to ∇ and M is locally a product manifold $L_u \times M^\sharp$, where L_u is a null curve tangent to $J(\mathrm{tr}(TM))$ and $M^\sharp$ is a leaf of D.*

As $\{U, V\}$ is a basis of $\Gamma(J(TM^\perp) \oplus J(\mathrm{tr}(TM)))$, the vector fields

$$\mu = U - \varphi V, \qquad \nu = U + \varphi V \tag{3.3}$$

form an orthogonal basis of $\Gamma(J(TM^\perp) \oplus J(\mathrm{tr}(TM)))$. From (3.2), we have

$$g(A^*_\xi \mu, X) = B(\mu, X) = 0, \qquad g(A^*_\xi \mu, N) = 0, \qquad A^*_\xi \mu = 0, \tag{3.4}$$

that is, μ is an eigenvector field of A^*_ξ on $S(TM)$ corresponding to the eigenvalue 0. Let $\mathcal{G}(\mu) = \text{Span}\{\mu\}$. Then $\mathcal{S}(\mu) = D_o \oplus_{\text{orth}} \text{Span}\{\nu\}$ is a complementary vector subbundle to $\mathcal{G}(\mu)$ in $S(TM)$ and we have the following decomposition

$$S(TM) = \mathcal{G}(\mu) \oplus_{\text{orth}} \mathcal{S}(\mu). \tag{3.5}$$

Theorem 3.5. *Let $(M, g, S(TM))$ be a screen conformal lightlike real hypersurface of an indefinite Kaehler manifold $\bar{M}$. Then the non-null vector field μ is parallel with respect to ∇ if and only if the 1-form τ vanishes and the conformal factor φ is a constant.*

Proof. From (2.18), (3.3) and the linearity of F, we have

$$\nabla_X \mu = \tau(X)\nu - X[\varphi]V, \quad \forall\, X \in \Gamma(TM), \tag{3.6}$$

due to $A_N = \varphi\, A^*_\xi$. Thus we see that μ is parallel if and only if

$$\tau(X)U - \{X[\varphi] - \varphi\tau(X)\}V = 0, \quad \forall\, X \in \Gamma(TM).$$

Taking the scalar product with V and U in turns, we get assertion. □

Note 2. From (2.18) and (3.4), we have

$$\nabla_X \nu = 2F(A_N X) + \tau(X)\mu + X[\varphi]V, \quad \forall\, X \in \Gamma(TM).$$

Thus, using the fact $g(F(A_N X), V) = g(F(A_N X), U) = 0$, we show that ν is parallel if and only if $\tau = 0$ on M, φ is a constant and both U and V are parallel. Moreover if ν is parallel, then μ is also parallel and $B = C = 0$.

Theorem 3.6. *Let $(M, g, S(TM))$ be a screen conformal lightlike real hypersurface of an indefinite Kaehler manifold $\bar{M}$. If μ is parallel with respect to ∇, then M is locally a product manifold $L_\xi \times L_\mu \times M^\natural$, where L_ξ and L_μ are null and non-null geodesic tangent to $TM^\perp$ and $\mathcal{G}(\mu)$ respectively and $M^\natural$ is a leaf of $\mathcal{S}(\mu)$. Moreover, M is screen homothetic.*

Proof. In general, using (3.6), for $X \in \Gamma(\mathcal{S}(\mu))$ and $Y \in \Gamma(D_o)$, we derive

$$g(\nabla_X Y, \mu) = g(\bar{\nabla}_X Y, \mu) = -g(Y, \nabla_X \mu) = 0, \tag{3.7}$$

$$g(\nabla_Y \nu, \mu) = -g(\nu, \nabla_Y \mu) = Y[\varphi] - 2\varphi\tau(Y). \tag{3.8}$$

If μ is parallel, then $g(\nabla_X Y, \mu) = g(\nabla_X \nu, \mu) = 0$. Thus $\mathcal{S}(\mu)$ is a integrable distribution. From this fact and Note 1, we obtain our theorem. □

Corollary 2. *Let $(M, g, S(TM))$ be a screen conformal lightlike real hypersurface of an indefinite Kaehler manifold $\bar{M}$. If μ is parallel with respect to ∇, then M is locally a product manifold $L_\mu \times M^\flat$, where L_μ is a non-null geodesic tangent to $\mathcal{G}(\mu)$ and $M^\flat$ is a leaf of $\mathcal{R}(\mu) = D_o \oplus_{\text{orth}} \text{Span}\{\xi, \nu\}$.*

Proof. From (1.1) and (3.5), we have $TM = \mathcal{G}(\mu) \oplus_{\text{orth}} \mathcal{R}(\mu)$. For any $X \in \Gamma(\mathcal{R}(\mu))$ and $Y \in \Gamma(D_o)$, we get

$$g(\nabla_Y \xi, \mu) = -g(A^*_\xi Y, \mu) = -g(Y, A^*_\xi \mu) = 0,$$
$$g(\nabla_Y \nu, \mu) = -g(\nu, \nabla_Y \mu) = Y[\varphi] - 2\varphi\tau(Y),$$

$$g(\nabla_X Y,\, \mu) = g(\bar{\nabla}_X Y,\, \mu) = -g(Y,\, \nabla_X \mu) = 0.$$

Thus the distribution $\mathcal{R}(\mu)$ is integrable. We have our assertion. □

Theorem 3.7. *Let $(M, g, S(TM))$ be a screen conformal lightlike real hypersurface of an indefinite complex space form $\bar{M}(c)$. Then we have $c = 0$. In particular, the ambient manifold $\bar{M}(c)$ is a semi-Euclidean space.*

Proof. By using (1.15) and (2.2), we have

$$\begin{aligned}(3.9)\qquad &\frac{c}{4}\{u(X)\bar{g}(JY, Z) - u(Y)\bar{g}(JX, Z) + 2u(Z)\bar{g}(X, JY)\}\\ &= (\nabla_X B)(Y, Z) - (\nabla_Y B)(X, Z) + B(Y, Z)\tau(X) - B(X, Z)\tau(Y)\end{aligned}$$

for all $X,\, Y,\, Z \in \Gamma(TM)$. Using this, (1.16), (1.18) and (3.2), we obtain

$$\begin{aligned}(3.10)\qquad &\frac{c}{4}\{g(Y, PZ)\eta(X) - g(X, PZ)\eta(Y) + v(X)\bar{g}(JY, PZ)\\ &\quad - v(Y)\bar{g}(JX, PZ) + 2v(PZ)\bar{g}(X, JY)\}\\ &= \{X[\varphi] - 2\varphi\tau(X)\}B(Y, PZ) - \{Y[\varphi] - 2\varphi\tau(Y)\}B(X, PZ)\\ &\quad + \frac{c}{4}\varphi\{u(X)\bar{g}(JY, PZ) - u(Y)\bar{g}(JX, PZ) + 2u(PZ)\bar{g}(X, JY)\}.\end{aligned}$$

Replacing Y by ξ in (3.10), we obtain

$$\begin{aligned}(3.11)\qquad &\{\xi[\varphi] - 2\varphi\tau(\xi)\}B(X, PZ)\\ &= \frac{c}{4}\{g(X, PZ) + v(X)u(PZ) + 2u(X)v(PZ) - 3\varphi\, u(X)u(PZ)\}.\end{aligned}$$

Taking $X = V;\ PZ = U$ and $X = U;\ PZ = V$, we have

$$(3.12)\qquad \{\xi[\varphi] - 2\varphi\tau(\xi)\}B(V, U) = \frac{1}{2}c, \quad \{\xi[\varphi] - 2\varphi\tau(\xi)\}B(U, V) = \frac{3}{4}c,$$

respectively. From the two equation of (3.12), we show that $c = 0$. Therefore, $\bar{M}(c)$ is a semi-Euclidean space. □

Corollary 3. *There exist no screen conformal lightlike real hypersurfaces M of indefinite complex space form $\bar{M}(c)$ with $c \neq 0$.*

The *type number* $t^*(p)$ of M at a point $p \in M$ is the rank of the shape operator A^*_ξ at p. Then, from (3.7) and (3.8), we obtain:

Theorem 3.8. *Let $(M, g, S(TM))$ be a screen conformal lightlike real hypersurface of an indefinite complex space form $\bar{M}(c)$ such that $t^*(p) > 1$ for any $p \in M$. Then M is locally a product manifold $L_\xi \times L_\mu \times M^\natural$, where L_ξ and L_μ are null and non-null curve tangent to $TM^\perp$ and $\mathcal{G}(\mu)$ respectively and $M^\natural$ is a leaf of $\mathcal{S}(\mu)$.*

Proof. First, for any $X \in \Gamma(\mathcal{S}(\mu))$ and $Y \in \Gamma(D_o)$, since $g(Y, U) = g(Y, V) = 0$ for $Y \in \Gamma(D_o)$, we show that

$$\begin{aligned}g(\nabla_X Y,\, \mu) &= g(\bar{\nabla}_X Y,\, \mu) = -g(Y,\, \bar{\nabla}_X \mu) = -g(Y,\, \nabla_X \mu)\\ &= X[\varphi]g(Y, V) - \tau(X)g(Y, \nu) = -\tau(X)\{g(Y, U) + \varphi g(Y, V)\} = 0.\end{aligned}$$

Thus (3.7) holds. Next, from the equation (3.10) with $c = 0$, we obtain

$$\{X[\varphi] - 2\varphi\tau(X)\}A^*_\xi Y = \{Y[\varphi] - 2\varphi\tau(Y)\}A^*_\xi X.$$

Suppose there exists a vector field $X_o \in \Gamma(TM)$ such that $X_o[\varphi] - 2\varphi\tau(X_o) \neq 0$. Then $A^*_\xi Y = fA^*_\xi X_o$ for any $Y \in \Gamma(TM)$, where f is a smooth function. It follows that the rank of A^*_ξ is 1. It is a contradiction as $\operatorname{rank} A^*_\xi > 1$. Consequently, we have $X[\varphi] - 2\varphi\tau(X) = 0$ for all $X \in \Gamma(TM)$ on $\mathcal{U}$. Thus (3.8) also holds. Therefore $\mathcal{S}(\mu)$ is integrable distribution by (3.7) and (3.8). Consequently, we have our theorem. □

4. Screen conformal Einstein hypersurfaces

Let $R^{(0,2)}$ denote the induced Ricci type tensor of M given by

$$R^{(0,2)}(X,Y) = \operatorname{trace}\{Z \to R(Z,X)Y\} \tag{4.1}$$

for any $X, Y \in \Gamma(TM)$. Consider the induced quasi-orthonormal frame field $\{\xi; W_a\}$ on M such that $\operatorname{Rad}(TM) = \operatorname{Span}\{\xi\}$ and $S(TM) = \operatorname{Span}\{W_a\}$. Using this quasi-orthonormal frame field and the equation (3.1), we obtain

$$R^{(0,2)}(X,Y) = \sum_{a=1}^{m} \epsilon_a\, g(R(W_a, X)Y, W_a) + \bar{g}(R(\xi, X)Y, N) \tag{4.2}$$

for any $X, Y \in \Gamma(TM)$ and $\epsilon_a = g(W_a, W_a)$ is the sign of W_a. In general, the induced Ricci type tensor $R^{(0,2)}$, defined by the method of the geometry of the non-degenerate submanifolds [8], is not symmetric [3, 5]. Therefore $R^{(0,2)}$ has no geometric or physical meaning similar to the Ricci curvature of the non-degenerate submanifolds and it is just a tensor quantity. Hence we need the following definition: A tensor field $R^{(0,2)}$ of lightlike submanifolds M is called its *induced Ricci tensor* if it is symmetric. A symmetric $R^{(0,2)}$ tensor will be denoted by *Ric*. If M is a screen conformal lightlike real hypersurface of a complex space form $\bar{M}(c)$, then $c = 0$. Using (1.14) and (1.16), we have

$$R^{(0,2)}(X,Y) = \varphi\{B(X,Y)\operatorname{tr}A^*_\xi - g(A^*_\xi X, A^*_\xi Y)\},\ \forall\, X, Y \in \Gamma(TM). \tag{4.3}$$

Theorem 4.1. *Let $(M, g, S(TM))$ be a screen conformal lightlike real hypersurface of an indefinite complex space form $\bar{M}(c)$. Then the Ricci type tensor $R^{(0,2)}$ is a symmetric Ricci tensor Ric.*

Note 3. Suppose the Ricci type tensor $R^{(0,2)}$ is symmetric. Then there exists a pair $\{\xi, N\}$ on $\mathcal{U}$ such that the corresponding 1-form τ vanishes [2]. We call such a pair a *distinguished null pair* [5] of M. Although, in general, $S(TM)$ is not unique, it is canonically isomorphic to the factor vector bundle $S(TM)^\sharp = TM/\operatorname{Rad}(TM)$ considered by Kupeli [7]. Thus all $S(TM)$ are mutually isomorphic. For this reason, in the sequel, let $(M, g, S(TM))$ be a screen homothetic lightlike real hypersurface equipped with the distinguished null pair $\{\xi, N\}$ of an indefinite complex space form $(\bar{M}(c), \bar{g})$.

Theorem 4.2. *Let $(M, g, S(TM))$ be a screen homothetic lightlike real hypersurface of an indefinite complex space form $\bar{M}(c)$. Then M is locally a product manifold $L_\xi \times L_\mu \times M^\natural$, where L_ξ and L_μ are null and non-null geodesics respectively and $M^\natural$ is a leaf of some non-degenerate distribution.*

Proof. Since M is a screen homothetic lightlike real hypersurface equipped with a distinguished null pair $\{\xi, N\}$, from (1.7), (1.13) and (3.6), we have $\nabla_\xi \xi = \nabla_\mu \mu = 0$. In particular, μ is a parallel vector field with respect to ∇ due to (3.6). Thus, by Theorem 3.6, we have our theorem. □

Theorem 4.3. *Any screen conformal Einstein lightlike real hypersurface of an indefinite complex space form $\bar{M}(c)$ is Ricci flat.*

Proof. Since M is a screen conformal lightlike real hypersurface of an indefinite complex space form $\bar{M}(c)$, we get $c = 0$. The induced tensor $R^{(0,2)}$ is a symmetric Ricci tensor Ric by (4.3). Let M be an Einstein manifold, that is, $R^{(0,2)} = \gamma g$ for some constant γ. Then the equation (4.3) reduces to

$$g(A^*_\xi X, A^*_\xi Y) - \alpha g(A^*_\xi X, Y) - \gamma\varphi^{-1} g(X, Y) = 0, \tag{4.4}$$

where $\alpha = \mathrm{tr}A^*_\xi$ is trace of A^*_ξ. Put $X = Y = \mu$ in (4.4) and using the fact that $A^*_\xi \mu = 0$ due to (3.4), we have $\gamma = 0$. Thus M is Ricci flat. □

Theorem 4.4. *Let $(M, g, S(TM))$ be a screen homothetic Einstein lightlike real hypersurface of an indefinite complex space form $\bar{M}(c)$ of index 2. Then M is locally a product manifold $L_\xi \times L_\mu \times M^\natural$ or $L_\xi \times L_\mu \times L_\alpha \times M^0$, where L_ξ, L_μ and L_α are null geodesic, timelike geodesic and spacelike curve respectively and $M^\natural$ and M^0 are Euclidean spaces.*

Proof. Let $\mu = \frac{1}{\sqrt{2\epsilon\varphi}}\{U - \varphi V\}$, where $\epsilon = \mathrm{sgn}\,\varphi$. Then μ is a unit timelike eigenvector field of A^*_ξ corresponding to the eigenvalue 0 by (3.4) and $\mathcal{S}(\mu)$ is an integrable Riemannian distribution by Theorem 4.2, due to $q = 2$. Since $g(A^*_\xi X, N) = 0$ and $g(A^*_\xi X, \mu) = 0$, A^*_ξ is $\Gamma(\mathcal{S}(\mu))$-valued real self-adjoint operator. Thus A^*_ξ have $(2m-3) \equiv n$ real orthonormal eigenvector fields in $\mathcal{S}(\mu)$ and is diagonalizable. Consider a frame field of eigenvectors $\{\mu, e_1, \ldots, e_n\}$ of A^*_ξ on $S(TM)$ such that $\{e_1, \ldots, e_n\}$ is an orthonormal frame field of A^*_ξ on $\mathcal{S}(\mu)$. Then $A^*_\xi e_i = \lambda_i e_i \,(1 \le i \le n)$. Put $X = Y = e_i$ in (4.4) with $\gamma = 0$, we show that each eigenvalue λ_i of A^*_ξ is a solution of the equation

$$x(x - \alpha) = 0. \tag{4.5}$$

The equation (4.5) has at most two distinct real solutions 0 and α on $\mathcal{U}$. Assume that there exists $p \in \{0, \ldots, n\}$ such that $\lambda_1 = \cdots = \lambda_p = 0$ and $\lambda_{p+1} = \cdots = \lambda_n = \alpha$, by renumbering if necessary. Then we have

$$\alpha = \mathrm{tr}A^*_\xi = (n - p)\alpha.$$

If $\alpha = 0$, then $A^*_\xi X = 0$ for all $X \in \Gamma(TM)$. Also we have $A_N X = 0$ for all $X \in \Gamma(TM)$. Thus M and $S(TM)$ are totally geodesic. From (1.14) and

(1.17), we have $R^*(X,Y)Z = \bar{R}(X,Y)Z = 0$ for all X, Y, $Z \in \Gamma(S(TM))$. Thus M is locally a product manifold $L_\xi \times (M^* = L_\mu \times M^\natural)$, where L_ξ and L_μ are null and timelike geodesic tangent to $TM^\perp$ and $\mathcal{G}(\mu)$ respectively and the leaf M^* of $S(TM)$ is a Minkowski space. Since $\nabla_X \mu = 0$ and

$$g(\nabla^*_X Y, \mu) = -g(Y, \nabla^*_X \mu) = -g(Y, \nabla_X \mu) = 0$$

for all X, Y, $Z \in \Gamma(S(TM))$, we have $\nabla^*_X Y \in \Gamma(\mathcal{S}(\mu))$ and $R^*(X,Y)Z \in \Gamma(\mathcal{S}(\mu))$. This imply $\nabla^*_X Y = Q(\nabla^*_X Y)$, that is, $M^\natural$ is a totally geodesic and $R^*(X,Y)Z = Q(R^*(X,Y)Z) = 0$, where Q is a projection morphism of $S(TM)$ on $\mathcal{S}(\mu)$ with respect to the decomposition (3.5). Thus $M^\natural$ is a Euclidean space.

If $\alpha \neq 0$, then $p = n - 1$. Consider the following two distributions on $\mathcal{S}(\mu)$;

$$\Gamma(E_0) = \{X \in \Gamma(\mathcal{S}(\mu)) \mid A^*_\xi X = 0\},$$
$$\Gamma(E_\alpha) = \{X \in \Gamma(\mathcal{S}(\mu)) \mid A^*_\xi X = \alpha X\}.$$

Then we know that the distributions E_0 and E_α are mutually orthogonal non-degenerate subbundle of $\mathcal{S}(\mu)$, of rank $(n-1)$ and 1 respectively, satisfy $\mathcal{S}(\mu) = E_0 \oplus_{\rm orth} E_\alpha$. From (4.4), we get $A^*_\xi(A^*_\xi - \alpha Q) = 0$. Using this equation, we have ${\rm Im}A^*_\xi \subset \Gamma(E_\alpha)$ and ${\rm Im}(A^*_\xi - \alpha Q) \subset \Gamma(E_0)$. For any X, $Y \in \Gamma(E_0)$ and $Z \in \Gamma(\mathcal{S}(\mu))$, we get $(\nabla_X B)(Y, Z) = -g(A^*_\xi \nabla_X Y, Z)$. Use this and the fact $(\nabla_X B)(Y,Z) = (\nabla_Y B)(X,Z)$, we have $g(A^*_\xi [X,Y],\, Z) = 0$. If we take $Z \in \Gamma(E_\alpha)$, since ${\rm Im}A^*_\xi \subset \Gamma(E_\alpha)$ and E_α is non-degenerate, we have $A^*_\xi[X,Y] = 0$. Thus $[X,Y] \in \Gamma(E_0)$ and E_0 is integrable. Thus M is locally a product manifold $L_\xi \times (M^* = L_\mu \times L_\alpha \times M^0)$, where L_α is a spacelike curve and M^0 is an $(n-1)$-dimensional Riemannian manifold satisfy $A^*_\xi = 0$. From (1.14) and (1.18), we have $R^*(X,Y)Z = \bar{R}(X,Y)Z = 0$ for all X, Y, $Z \in \Gamma(E_0)$. Since $g(\nabla^*_X Y, \mu) = 0$ and $g(\nabla^*_X Y, e_n) = -g(Y, \nabla_X e_n) = 0$ for all X, $Y \in \Gamma(E_0)$ because $\nabla_X W \in \Gamma(E_\alpha)$ for $X \in \Gamma(E_0)$ and $W \in \Gamma(E_\alpha)$. In fact, from (1.15) such that $c = \tau = 0$, we get

$$g(\{(A^*_\xi - \alpha Q)\nabla_X W - A^*_\xi \nabla_W X\}, Z) = 0$$

for all $X \in \Gamma(E_0)$, $W \in \Gamma(E_\alpha)$ and $Z \in \Gamma(\mathcal{S}(\mu))$. Using the fact that $\mathcal{S}(\mu)$ is non-degenerate distribution, we have $(A^*_\xi - \alpha Q)\nabla_X W = A^*_\xi \nabla_W X$. Since the left term of this equation is in $\Gamma(E_0)$ and the right term is in $\Gamma(E_\alpha)$ and $E_0 \cap E_\alpha = \{0\}$, we have $(A^*_\xi - \alpha Q)\nabla_X W = 0$ and $A^*_\xi \nabla_W X = -X[\varphi]W$. This imply that $\nabla_X W \in \Gamma(E_\alpha)$. Thus $\nabla^*_X Y = \pi \nabla^*_X Y$ for all X, $Y \in \Gamma(E_0)$, where π is the projection morphism of $\Gamma(S(TM))$ on $\Gamma(E_0)$ and $\pi\nabla^*$ is the induced connection on E_0. This imply that the leaf M^0 of E_0 is totally geodesic. As $g(R^*(X,Y)Z, \mu) = 0$ and $g(R^*(X,Y)Z, e_n) = 0$ for all X, Y, $Z \in \Gamma(E_0)$, we have $R^*(X,Y)Z = \pi R^*(X,Y)Z \in \Gamma(E_0)$ and the curvature tensor πR^* of E_0 is flat. Thus M^0 is a Euclidean space. □

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