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MEROMORPHIC FUNCTIONS SHARING A NONZERO POLYNOMIAL CM

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Abstract. In this paper, we prove that if $f^n f' - P$ and $g^n g' - P$ share 0 CM, where f and g are two distinct transcendental meromorphic functions, $n \geq 11$ is a positive integer, and P is a nonzero polynomial such that its degree $\gamma_P \leq 11$, then either $f = c_1 e^{cQ}$ and $g = c_2 e^{-cQ}$, where c_1, c_2 and c are three nonzero complex numbers satisfying $(c_1 c_2)^{n+1} c^2 = -1$, Q is a polynomial such that $Q = \int_0^z P(\eta) d\eta$, or $f = tg$ for a complex number t such that $t^{n+1} = 1$. The results in this paper improve those given by M. L. Fang and H. L. Qiu, C. C. Yang and X. H. Hua, and other authors.

1. Introduction and main results

In this paper, by meromorphic functions we will always mean meromorphic functions in the complex plane. We adopt the standard notations in the Nevanlinna theory of meromorphic functions as explained in [7], [10], and [17]. It will be convenient to let E denote any set of positive real numbers of finite linear measure, not necessarily the same at each occurrence. For a nonconstant meromorphic function h, we denote by $T(r, h)$ the Nevanlinna characteristic of h and by $S(r, h)$ any quantity satisfying $S(r, h) = o\{T(r, h)\}$ as $r \to \infty$ and $r \notin E$.

Let f and g be two nonconstant meromorphic functions and let a be a finite complex number. We say that f and g share a CM, provided that $f - a$ and $g - a$ have the same zeros with the same multiplicities. Similarly, we say that f and g share a IM, provided that $f - a$ and $g - a$ have the same zeros ignoring multiplicities. In addition, we say that f and g share ∞ CM, if $1/f$ and $1/g$ share 0 CM, and we say that f and g share ∞ IM, if $1/f$ and $1/g$ share 0 IM (see [16]). We say that a is a small function of f, if a is a meromorphic function

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satisfying $T(r,a)=S(r,f)$ (see [16]). In addition, we need the following two definitions.

Definition 1.1 (see [1, Definition 1]). Let p be a positive integer and $a \in C \cup \{\infty\}$. Then by $N_{p)}(r,1/(f-a))$ we denote the counting function of those a-points of f (counted with proper multiplicities) whose multiplicities are not greater than p, by $\overline{N}_{p)}(r,1/(f-a))$ we denote the corresponding reduced counting function (ignoring multiplicities). By $N_{(p}(r,1/(f-a))$ we denote the counting function of those a-points of f (counted with proper multiplicities) whose multiplicities are not less than p, by $\overline{N}_{(p}(r,1/(f-a))$ we denote the corresponding reduced counting function (ignoring multiplicities).

Definition 1.2. Let a be an any value in the extended complex plane, and let k be an arbitrary nonnegative integer. We define

$$\delta_k(a,f)=1-\limsup_{r\to\infty}\frac{N_k\left(r,\frac{1}{f-a}\right)}{T(r,f)}, \tag{1.1}$$

where

$$N_k(r,1/(f-a))=\overline{N}\left(r,\frac{1}{f-a}\right)+\overline{N}_{(2}\left(r,\frac{1}{f-a}\right)+\cdots+\overline{N}_{(k}\left(r,\frac{1}{f-a}\right). \tag{1.2}$$

Remark 1.1. From (1.1) and (1.2) we have $0 \le \delta_k(a,f) \le \delta_{k-1}(a,f) \le \delta_1(a,f) \le \Theta(a,f) \le 1$.

In 1959, W. K. Hayman proved that if f is a transcendental meromorphic function and $n \ge 3$ is a positive integer, then $f^n f' = 1$ has infinitely many solutions (see [8, Corollary of Theorem 9]). In 1995, W. Bergweiler and A. Eremenko, H. H. Chen and M. L. Fang, L. Zalcman respectively proved the following result:

Theorem A (see [2, Theorem 2], [4, Theorem 1] and [19]). *Let f be a transcendental meromorphic function, and let n be a positive integer. Then $f^n f' = 1$ has infinitely many solutions.*

In 2000, M. L. Fang proved the following result:

Theorem B (see [5, Theorem 2]). *Let f be a transcendental meromorphic function, and let n be a positive integer. Then $f^n f' - z = 0$ has infinitely many solutions.*

In 2003, W. Bergweiler and X. C. Pang proved the following result:

Theorem C (see [3, Theorem 1.1]). *Let f be a transcendental meromorphic function, and let $R \not\equiv 0$ be a rational function. If all zeros and poles of f are multiple, except possibly finitely many, then $f' - R = 0$ has infinitely many solutions.*

From Theorem B we get the following result:

Theorem D. *Let f be a transcendental meromorphic function, and let $P \not\equiv 0$ be a polynomial, and let n be a positive integer. Then $f^n f' - P = 0$ has infinitely many solutions.*

In 1997, C. C. Yang and X. H. Hua proved the following result, which corresponded to Theorem A.

Theorem E (see [15, Theorem 1]). *Let f and g be two nonconstant meromorphic functions, and let $n \geq 11$ be a positive integer. If $f^n f'$ and $g^n g'$ share 1 CM, then either $f = c_1 e^{cz}$ and $g = c_2 e^{-cz}$, where c_1, c_2 and c are three nonzero complex numbers satisfying $(c_1 c_2)^{n+1} c^2 = -1$, or $f = tg$ for a complex number t such that $t^{n+1} = 1$.*

In 2000, M. L. Fang and H. L. Qiu proved the following result, which corresponded to Theorem B.

Theorem F (see [6, Theorem 1]). *Let f and g be two nonconstant meromorphic functions, and let $n \geq 11$ be a positive integer. If $f^n f' - z$ and $g^n g' - z$ share 0 CM, then either $f = c_1 e^{cz^2}$ and $g = c_2 e^{-cz^2}$, where c_1, c_2 and c are three nonzero complex numbers satisfying $4(c_1 c_2)^{n+1} c^2 = -1$, or $f = tg$ for a complex number t such that $t^{n+1} = 1$.*

Regarding Theorem D and Theorem F, it is natural to ask the following question:

Question 1.1. *Is there a corresponding uniqueness theorem to Theorem D ?*

In this paper, we will prove the following two theorems, which correspond to Theorem D, improve Theorems A-C and Theorem F, and deal with Question 1.1.

Theorem 1.1. *Let f and g be two transcendental meromorphic functions, let $n \geq 11$ be a positive integer, and let $P \not\equiv 0$ be a polynomial with its degree $\gamma_P \leq 11$. If $f^n f' - P$ and $g^n g' - P$ share 0 CM, then either $f = tg$ for a complex number t satisfying $t^{n+1} = 1$, or $f = c_1 e^{cQ}$ and $g = c_2 e^{-cQ}$, where c_1, c_2 and c are three nonzero complex numbers satisfying $(c_1 c_2)^{n+1} c^2 = -1$, Q is a polynomial satisfying $Q = \int_0^z P(\eta) d\eta$.*

Theorem 1.2. *Let f and g be two transcendental meromorphic functions, let $n \geq 15$ be a positive integer, and let $P \not\equiv 0$ be a polynomial. If $(f^n(f-1))' - P$ and $(g^n(g-1))' - P$ share 0 CM and $\Theta(\infty, f) > 2/n$, then $f = g$.*

2. Some lemmas

Lemma 2.1 (see [18, Proof of Lemma 1]). *Let f be a nonconstant meromorphic function, let k be a positive integer, and let φ be a small function of f such that $\varphi \not\equiv 0, \infty$. Then*

$$T(r,f) \leq \overline{N}(r,f) + N\left(r, \frac{1}{f}\right) + N\left(r, \frac{1}{f^{(k)} - \varphi}\right) - N\left(r, \frac{1}{(\frac{f^{(k)}}{\varphi})'}\right) + S(r,f).$$

Lemma 2.2 (see [9, Proof of Lemma 2.3])**.** *Let f be a nonconstant meromorphic function, and let k and p be two positive integers. Then*

$$N_p\left(r, \frac{1}{f^{(k)}}\right) \le N_{p+k}\left(r, \frac{1}{f}\right) + k\overline{N}(r,f) + S(r,f).$$

Lemma 2.3. *Let f and g be two transcendental meromorphic functions such that $f^{(k)} - P$ and $g^{(k)} - P$ share 0 CM, where k is a positive integer, $P \not\equiv 0$ is a polynomial. If*

$$\begin{aligned}\Delta_1 = {} & (k+2)\Theta(\infty,f) + 2\Theta(\infty,g) + \Theta(0,f) + \Theta(0,g) \\ & + \delta_{k+1}(0,f) + \delta_{k+1}(0,g) > k+7\end{aligned} \tag{2.1}$$

and

$$\begin{aligned}\Delta_2 = {} & (k+2)\Theta(\infty,g) + 2\Theta(\infty,f) + \Theta(0,g) + \Theta(0,f) \\ & + \delta_{k+1}(0,g) + \delta_{k+1}(0,f) > k+7,\end{aligned} \tag{2.2}$$

then either $f^{(k)}g^{(k)} = P^2$ or $f = g$.

Proof. From the condition that f and g are transcendental meromorphic functions we know that $f^{(k)}$ and $g^{(k)}$ are transcendental meromorphic functions. Let

$$F = \frac{f^{(k)}}{P} \quad \text{and} \quad G = \frac{g^{(k)}}{P}, \tag{2.3}$$

and let

$$h = \left(\frac{F''}{F'} - \frac{2F'}{F-1}\right) - \left(\frac{G''}{G'} - \frac{2G'}{G-1}\right). \tag{2.4}$$

Let $z_0 \notin \{z : P(z) = 0\}$ be a common simple zero of $f^{(k)} - P$ and $g^{(k)} - P$. Then it follows from (2.3) that z_0 is a common simple zero of $F-1$ and $G-1$. Moreover, from (2.3) and by calculating we get $h(z_0) = 0$.

Let $z_1 \notin \{z : P(z) = 0\}$ be a simple pole of F. Then by calculating we see that $F''/F' - 2F'/(F-1)$ is analytic at z_1. Similarly, if $z_2 \notin \{z : P(z) = 0\}$ is a simple pole of G, then by calculating we see that $G''/G' - 2G'/(G-1)$ is analytic at z_2.

Let $z_3 \notin \{z : P(z) = 0\}$ be a pole of h. Then from (2.3)–(2.4) and the above analysis we see that z_3 is possible to be an element of one of the following sets:

(i) $S_1 = \{z : f(z) = \infty\} \cup \{z : g(z) = \infty\}$;

(ii) $S_2 = \{z : f(z) = 0 \text{ and } f^{(k+1)}(z) \neq 0, \infty\}$;

(iii) $S_3 = \{z : g(z) = 0 \text{ and } g^{(k+1)}(z) \neq 0, \infty\}$;

(iv) $S_4 = \{z : F'(z) = 0 \text{ and } f(z)(F(z)-1) \neq 0, \infty\}$;

(iv) $S_5 = \{z : G'(z) = 0 \text{ and } g(z)(G(z)-1) \neq 0, \infty\}$.

Next we denote by $N_0(r, 1/F')$ the counting function of those zeros of F' that are not the zeros of $f(F-1)$, denote by $\overline{N}_0(r, 1/F')$ the reduced form of $N_0(r, 1/F')$, and denote by $\overline{N}_{(1,1)}(r, 1/F)$ the reduced counting function of the common simple zeros of $F-1$ and $G-1$. Similarly, $N_0(r,1/G')$, $\overline{N}_0(r,1/G')$ and $\overline{N}_{(1,1)}(r,1/G)$ have the same meanings. From above analysis and (2.4) we get

$$\begin{aligned}
N(r,h) &\leq \overline{N}_{(2}(r,F)+\overline{N}_{(2}(r,G)+\overline{N}\left(r,\frac{1}{f}\right)+\overline{N}\left(r,\frac{1}{g}\right)+\overline{N}_0\left(r,\frac{1}{F'}\right)\\
&\quad+\overline{N}_0\left(r,\frac{1}{G'}\right)+O(\log r)\\
&=\overline{N}(r,f)+\overline{N}(r,g)+\overline{N}\left(r,\frac{1}{f}\right)+\overline{N}\left(r,\frac{1}{g}\right)+N_0\left(r,\frac{1}{F'}\right)\\
&\quad+N_0\left(r,\frac{1}{G'}\right)+O(\log r).
\end{aligned}\tag{2.5}$$

From the condition that f is a transcendental meromorphic function we get

$$T(r,P)=o\{T(r,f)\}.\tag{2.6}$$

Suppose that $z_4\notin\{z:P(z)=0\}$ is a zero of f with its multiplicity $l\geq k+2$. Then it follows from (2.3) that z_0 is a zero of F_1' with its multiplicity $l-k-1\geq 1$. Thus it follows from (2.3) and Lemma 2.1 that

$$\begin{aligned}
T(r,f)&\leq\overline{N}(r,f)+N\left(r,\frac{1}{f}\right)+N\left(r,\frac{1}{F-1}\right)-N\left(r,\frac{1}{F'}\right)+S(r,f)\\
&\leq\overline{N}(r,f)+N_{k+1}\left(r,\frac{1}{f}\right)+\overline{N}\left(r,\frac{1}{F-1}\right)-N_0\left(r,\frac{1}{F'}\right)+S(r,f).
\end{aligned}\tag{2.7}$$

Similarly

$$T(r,g)\leq\overline{N}(r,g)+N_{k+1}\left(r,\frac{1}{g}\right)+\overline{N}\left(r,\frac{1}{G-1}\right)-N_0\left(r,\frac{1}{G'}\right)+S(r,g).\tag{2.8}$$

From (2.3)-(2.5) and the condition that $f^{(k)}-P$ and $g^{(k)}-P$ share 0 CM we get

$$\begin{aligned}
&\overline{N}\left(r,\frac{1}{F-1}\right)+\overline{N}\left(r,\frac{1}{G-1}\right)\\
&\leq\overline{N}_{(1,1)}\left(r,\frac{1}{F-1}\right)+N\left(r,\frac{1}{F-1}\right)+O(\log r)\\
&\leq\overline{N}_{(1,1)}\left(r,\frac{1}{F-1}\right)+T(r,F)+O(\log r)\\
&\leq N\left(r,\frac{1}{h}\right)+T(r,f^{(k)})+O(\log r)\\
&\leq T(r,h)+T(r,f)+k\overline{N}(r,f)+O(\log r)+S(r,f)\\
&\leq N(r,h)+T(r,f)+k\overline{N}(r,f)+S(r,f)
\end{aligned}\tag{2.9}$$

$$\leq (k+1)\overline{N}(r,f) + \overline{N}(r,g) + \overline{N}\left(r,\frac{1}{f}\right) + \overline{N}\left(r,\frac{1}{g}\right) + T(r,f)$$
$$+ N_0\left(r,\frac{1}{F'}\right) + N_0\left(r,\frac{1}{G'}\right) + S(r,f).$$

From (2.7)-(2.9) we get

$$T(r,g) \leq (k+2)\overline{N}\,(r,f) + 2\overline{N}(r,g) + N_{k+1}\left(r,\frac{1}{f}\right) + N_{k+1}\left(r,\frac{1}{g}\right) + \overline{N}\left(r,\frac{1}{f}\right) + \overline{N}\left(r,\frac{1}{g}\right) + S(r,f) + S(r,g). \tag{2.10}$$

Similarly

$$T(r,f) \leq (k+2)\overline{N}(r,g) + 2\overline{N}(r,f) + N_{k+1}\left(r,\frac{1}{g}\right) + N_{k+1}\left(r,\frac{1}{f}\right) + \overline{N}\left(r,\frac{1}{g}\right) + \overline{N}\left(r,\frac{1}{f}\right) + S(r,f) + S(r,g). \tag{2.11}$$

Suppose that there exists a subset $I \subseteq \mathbb{R}^+$ satisfying $\operatorname{mes} I = \infty$ such that

$$T(r,f) \leq T(r,g)\,(r \in I). \tag{2.12}$$

Then it follows from (2.10) and (2.12) that

$$\Delta_1 = (k+2)\Theta(\infty,f) + 2\Theta(\infty,g) + \Theta(0,f) + \Theta(0,g) + \delta_{k+1}(0,f) + \delta_{k+1}(0,g) \leq k+7,$$

which contradicts (2.1). Similarly, if there exists a subset $I \subseteq \mathbb{R}^+$ satisfying $\operatorname{mes} I = \infty$ such that

$$T(r,g) \leq T(r,f)\,(r \in I), \tag{2.13}$$

from (2.11) and (2.13) we get $\Delta_2 \leq k+7$, which contradicts (2.2). Thus $h = 0$, and so it follows from (2.4) that

$$\frac{F''}{F'} - \frac{2F'}{F-1} = \frac{G''}{G'} - \frac{2G'}{G-1}. \tag{2.14}$$

From (2.14) we get

$$\frac{1}{F-1} = \frac{bG+a-b}{G-1}, \tag{2.15}$$

where and in what follows, a and b are two finite complex numbers.

We discuss the following two cases.

Case 1. Suppose that $b \neq 0$ and $a = b$. If $b = -1$, from (2.3) and (2.15) we get $f^{(k)}g^{(k)} = P^2$, which reveals the conclusion of Lemma 2.3. If $b \neq -1$, then (2.15) can be rewritten as

$$\frac{1}{F} = \frac{bG}{(1+b)G-1} \tag{2.16}$$

or

$$G=\frac{-1}{b}\cdot\frac{1}{F-(1+b)/b}. \tag{2.17}$$

From (2.3), (2.16), and (2.17) we get

$$\overline{N}\left(r,\frac{1}{G-1/(b+1)}\right)=\overline{N}\left(r,\frac{1}{f^{(k)}}\right)+O(\log r) \tag{2.18}$$

and

$$\overline{N}\left(r,\frac{1}{F-(1+b)/b}\right)=\overline{N}(r,g)+O(\log r). \tag{2.19}$$

From (2.3), (2.18), (2.19), Lemma 2.1, Lemma 2.2 and in the same manner as in the proof of (2.7) we get

(2.20)

$$\begin{aligned}T(r,g)&\le\overline{N}(r,g)+N\left(r,\frac{1}{g}\right)+N\left(r,\frac{1}{G-1/(1+b)}\right)-N\left(r,\frac{1}{G'}\right)+S(r,g)\\&\le\overline{N}(r,g)+N_{k+1}\left(r,\frac{1}{g}\right)+\overline{N}\left(r,\frac{1}{f^{(k)}}\right)+O(\log r)+S(r,g)\\&\le\overline{N}(r,g)+N_{k+1}\left(r,\frac{1}{g}\right)+N_1\left(r,\frac{1}{f^{(k)}}\right)+O(\log r)+S(r,g)\\&\le\overline{N}(r,g)+N_{k+1}\left(r,\frac{1}{g}\right)+N_{k+1}\left(r,\frac{1}{f}\right)+k\overline{N}(r,f)+S(r,f)+S(r,g)\end{aligned}$$

and

(2.21)

$$\begin{aligned}T(r,f)&\le\overline{N}(r,f)+N\left(r,\frac{1}{f}\right)+N\left(r,\frac{1}{F-(1+b)/b}\right)-N\left(r,\frac{1}{F'}\right)+S(r,f)\\&\le\overline{N}(r,f)+N_{k+1}\left(r,\frac{1}{f}\right)+\overline{N}(r,g)+O(\log r)+S(r,f)\\&=\overline{N}(r,f)+N_{k+1}\left(r,\frac{1}{f}\right)+\overline{N}(r,g)+S(r,f).\end{aligned}$$

Suppose that there exists a subset $I\subseteq\mathbb{R}^+$ satisfying $\operatorname{mes} I=\infty$ such that (2.12) holds. Then from (2.12) and (2.20) we get

$$\Theta(\infty,g)+\delta_{k+1}(0,g)+\delta_{k+1}(0,f)+k\Theta(\infty,f)\le k+2. \tag{2.22}$$

From (2.1) and (2.22) we get

$$2\Theta(\infty,f)+\Theta(\infty,g)+\Theta(0,f)+\Theta(0,g)>5. \tag{2.23}$$

From (2.23) and Remark 1.1 we get a contradiction. Suppose that there exists a subset $I\subseteq\mathbb{R}^+$ satisfying $\operatorname{mes} I=\infty$ such that (2.13) holds. Then from (2.13) and (2.21) we get

$$\Theta(\infty,f)+\Theta(\infty,g)+\delta_{k+1}(0,f)\le 2. \tag{2.24}$$

From (2.1) and (2.24) we get

$$(k+1)\Theta(\infty,f)+\Theta(\infty,g)+\Theta(0,f)+\Theta(0,g)+\delta_{k+1}(0,g)>k+5. \tag{2.25}$$

From (2.25) and Remark 1.1 we get a contradiction.

Case 2. Suppose that $b\neq 0$ and $a\neq b$. We discuss the following two subcases.

Subcase 2.1. Suppose that $b=-1$. Then $a\neq 0$ and (2.15) can be rewritten as

$$F=\frac{a}{a+1-G}. \tag{2.26}$$

From (2.3) and (2.26) we get

$$\overline{N}\left(r,\frac{1}{a+1-G}\right)=\overline{N}\left(r,\frac{a}{a+1-G}\right)=\overline{N}(r,f^{(k)})=\overline{N}(r,f). \tag{2.27}$$

From (2.3), (2.27), Lemma 2.1 and in the same manner as in the proof of (2.7) we get

(2.28)

$$\begin{aligned}T(r,g)&\leq \overline{N}(r,g)+N\left(r,\frac{1}{g}\right)+N\left(r,\frac{1}{a+1-G}\right)-N\left(r,\frac{1}{G'}\right)+S(r,g)\\&\leq \overline{N}(r,g)+N_{k+1}\left(r,\frac{1}{g}\right)+\overline{N}\left(r,\frac{1}{a+1-G}\right)+O(\log r)+S(r,g)\\&=\overline{N}(r,g)+N_{k+1}\left(r,\frac{1}{g}\right)+\overline{N}(r,f)+S(r,g).\end{aligned}$$

From (2.2), (2.28) and in the same manner as in Case 1 we get contradictions.

Subcase 2.2. Suppose that $b\neq -1$. Then (2.15) can be rewritten as

$$F-\frac{b+1}{b}=\frac{-a}{b^2}\cdot\frac{1}{G+(a-b)/b}. \tag{2.29}$$

From (2.3) and (2.29) we get

$$\overline{N}\left(r,\frac{1}{G+(a-b)/b}\right)=\overline{N}(r,f^{(k)})+O(\log r)=\overline{N}(r,f)+O(\log r). \tag{2.30}$$

From (2.3), (2.30), Lemma 2.1 and in the same manner as in the proof of (2.7) we get

(2.31)

$$\begin{aligned}T(r,g)&\leq \overline{N}(r,g)+N\left(r,\frac{1}{g}\right)+N\left(r,\frac{1}{G+(a-b)/b}\right)-N\left(r,\frac{1}{G'}\right)+S(r,g)\\&\leq \overline{N}(r,g)+N_{k+1}\left(r,\frac{1}{g}\right)+\overline{N}\left(r,\frac{1}{G+(a-b)/b}\right)+O(\log r)+S(r,g)\\&=\overline{N}(r,g)+N_{k+1}\left(r,\frac{1}{g}\right)+\overline{N}(r,f)+S(r,g).\end{aligned}$$

From (2.2), (2.31) and in the same manner as in Case 1 we get contradictions.

Case 3. Suppose that $b = 0$. Then $a \neq 0$ and we get from (2.15) that

$$g = af + (1-a)P_1, \tag{2.32}$$

where P_1 is a polynomial with its degree $\gamma_{P_1} \geq k$. If $a \neq 1$, then $(1-a)P_1 \not\equiv 0$. This together with (2.32) and Nevanlinna's three small functions theorem (see [16, Theorem 1.36]) implies

$$\begin{aligned} T(r,g) &\leq \overline{N}(r,g) + \overline{N}\left(r, \frac{1}{g}\right) + \overline{N}\left(r, \frac{1}{g-(1-a)P_1}\right) + S(r,g) \\ &= \overline{N}(r,g) + \overline{N}\left(r, \frac{1}{g}\right) + \overline{N}\left(r, \frac{1}{f}\right) + S(r,g). \end{aligned} \tag{2.33}$$

From (2.32) we get $T(r,f) = T(r,g) + O(\log r)$. From this and (2.33) we get

$$\Theta(0,f) + \Theta(0,g) + \Theta(\infty,g) \leq 2. \tag{2.34}$$

From (2.34) and (2.1) we get

$$(k+2)\Theta(\infty,f) + \Theta(\infty,g) + \delta_{k+1}(0,f) + \delta_{k+1}(0,g) > k+5. \tag{2.35}$$

From (2.35) and Remark 1.1 we get a contradiction. Thus $a = 1$, and so it follows from (2.32) that $f = g$, which reveals the conclusion of Lemma 2.3. Lemma 2.3 is thus completely proved. □

Lemma 2.4 (see [16, Theorem 1.24]). *Suppose that f is a nonconstant meromorphic function in the complex plane and k is a positive integer. Then*

$$N\left(r, \frac{1}{f^{(k)}}\right) \leq N\left(r, \frac{1}{f}\right) + k\overline{N}(r,f) + S(r,f).$$

Lemma 2.5 (see [14, Proof of Lemma 2]). *Let f be a transcendental meromorphic function, and let $P_n(f)$ be a differential polynomial in f of the form*

$$P_n(f) = a_n f^n(z) + a_{n-1}f^{n-1}(z) + a_{n-2}f^{n-2} + \cdots + a_1 f(z) + a_0,$$

where $a_n(\neq 0)$, a_{n-1}, a_{n-2}, …, a_1, a_0 are $n+1$ complex numbers. Then

$$T(r, P_n(f)) = nT(r,f) + O(1).$$

Lemma 2.6 (see [16, Lemma 1.10]). *Let f_1 and f_2 be two nonconstant meromorphic functions in the complex plane, and let c_1, c_2, c_3 be three nonzero complex numbers. If $c_1 f_1 + c_2 f_2 = c_3$, then*

$$T(r,f_1) \leq \overline{N}(r,f_1) + \overline{N}\left(r, \frac{1}{f_1}\right) + \overline{N}\left(r, \frac{1}{f_2}\right) + S(r,f_1).$$

Lemma 2.7 (see [7, Theorem 3.5]). *Suppose that f is a nonconstant meromorphic function in the complex plane and k is a positive integer. Then*

$$T(r,f) \leq \left(2 + \frac{1}{k}\right) N\left(r, \frac{1}{f}\right) + \left(2 + \frac{2}{k}\right) \overline{N}\left(r, \frac{1}{f^{(k)}-1}\right) + S(r,f).$$

Lemma 2.8 (see [16, Proof of Theorem 4.8])**.** *Let F and G be two distinct nonconstant meromorphic functions, and let c be a complex number such that $c \neq 0, 1$. If F and G share 1 and c IM, and if $\overline{N}(r, 1/F) + \overline{N}(r, F) = S(r, F)$ and $\overline{N}(r, 1/G) + \overline{N}(r, G) = S(r, G)$, then F and G share 0, 1, c, ∞ CM.*

Lemma 2.9 (see [13])**.** *If f and g are distinct nonconstant meromorphic functions that share four values a_1, a_2, a_3, a_4 CM, then f is a Möbius transformation of g, two of the shared values, say a_1 and a_2 are Picard exceptional values, and the cross ratio $(a_1, a_2, a_3, a_4) = -1$.*

Lemma 2.10. *Let f and g be two transcendental meromorphic functions, let $n \geq 2$ be a positive integers, and let P be a nonconstant polynomial with its degree $\gamma_P \leq n$. If $f^n f' g^n g' = P^2$, then f and g are expressed as $f = c_1 e^{cQ}$ and $g = c_2 e^{-cQ}$ respectively, where c_1, c_2 and c are three nonzero complex numbers satisfying $(c_1c_2)^{n+1}c^2 = -1$, Q is a polynomial satisfying $Q = \int_0^z P(\eta)d\eta$.*

Proof. First, we will prove

$$N\left(r, \frac{1}{f}\right) + N\left(r, \frac{1}{g}\right) = O(\log r). \tag{2.36}$$

In fact, suppose that z_0 is a zero of f with multiplicity m, such that $z_0 \notin \{z : P(z) = 0\}$. Then from $f^n f' g^n g' = P^2$ we see that z_0 is a pole of g with multiplicity p, such that $nm + m - 1 = np + p + 1$, and so $(m - p)(n + 1) = 2$, which contradicts the fact that $n \geq 2$ and m, p are positive integers. Thus if z_0 is a zero of f, then $z_0 \in \{z : P(z) = 0\}$, and so we have $N(r, 1/f) = O(\log r)$. Similarly, we get $N(r, 1/g) = O(\log r)$, and so we get (2.36). Next we prove

$$\overline{N}(r, f) + \overline{N}(r, g) = S(r, f) + S(r, g). \tag{2.37}$$

In fact, from (2.36), Lemma 2.4 and $f^n f' g^n g' = P^2$ we have

$$\begin{aligned}(n+2)\overline{N}(r, f) &\leq N(r, f^n f') \\ &= N\left(r, \frac{P^2}{g^n g'}\right) \\ &\leq N\left(r, \frac{1}{g^n g'}\right) \\ &= N\left(r, \frac{1}{(g^{n+1})'}\right) \\ &\leq N\left(r, \frac{1}{g^{n+1}}\right) + \overline{N}(r, g^{n+1}) + S(r, g^{n+1}) \\ &= (n+1)N\left(r, \frac{1}{g}\right) + \overline{N}(r, g) + S(r, g^{n+1}) \\ &= \overline{N}(r, g) + O(\log r) + S(r, g^{n+1}) \\ &= \overline{N}(r, g) + S(r, g^{n+1}).\end{aligned} \tag{2.38}$$

From Lemma 2.5 we have $T(r,g^{n+1}) = (n+1)T(r,g) + O(1)$. This together with (2.38) implies $S(r,g^{n+1}) = S(r,g)$. Combining (2.38), we get

$$(n+2)\overline{N}(r,f) \le \overline{N}(r,g) + S(r,g). \tag{2.39}$$

Similarly

$$(n+2)\overline{N}(r,g) \le \overline{N}(r,f) + S(r,f). \tag{2.40}$$

From (2.39) and (2.40) we get

$$(n+1)\overline{N}(r,f) + (n+1)\overline{N}(r,g) \le S(r,f) + S(r,g),$$

which reveals (2.37). Let

$$F_1 = \frac{f^n f'}{P} \quad \text{and} \quad G_1 = \frac{g^n g'}{P}. \tag{2.41}$$

From (2.41) and $f^n f' g^n g' = P^2$ we get

$$F_1 G_1 = 1. \tag{2.42}$$

If $F_1 = G_1$, then it follows from (2.41) that

$$f^{n+1} - g^{n+1} = c_3, \tag{2.43}$$

where c_3 is a finite complex number. If $c_3 = 0$, from (2.43) we get

$$f = tg, \tag{2.44}$$

where t is a complex number satisfying $t^{n+1} = 1$. If $c_3 \ne 0$, from (2.43) and Lemma 2.6 we get

$$\begin{aligned} T(r,f^{n+1}) &\le \overline{N}\left(r, \frac{1}{f^{n+1}}\right) + \overline{N}\left(r, \frac{1}{g^{n+1}}\right) + S(r,f^{n+1}) \\ &= \overline{N}\left(r, \frac{1}{f}\right) + \overline{N}\left(r, \frac{1}{g}\right) + S(r,f^{n+1}) \\ &\le T(r,f) + T(r,g) + S(r,f^{n+1}). \end{aligned} \tag{2.45}$$

From (2.43) and Lemma 2.5 we get

$$T(r,f^{n+1}) = (n+1)T(r,f) + O(1) \tag{2.46}$$

and

$$T(r,f) = T(r,g) + O(1). \tag{2.47}$$

From (2.46) we have

$$S(r,f^{n+1}) = S(r,f). \tag{2.48}$$

From (2.45)-(2.48) we get $(n-1)T(r,f) = S(r,f)$. From this and $n \ge 2$ we get a contradiction. Next we suppose that $F_1 \not\equiv G_1$. From (2.36), (2.37), the left equality of (2.41) and Lemma 2.4 we get

$$
\begin{aligned}
N\left(r,\frac{1}{F_1}\right) &\le N\left(r,\frac{1}{f^n}\right)+N\left(r,\frac{1}{f'}\right)\\
&= \ N\left(r,\frac{1}{f'}\right)+nN\left(r,\frac{1}{f}\right)\\
&\le (n+1)N\left(r,\frac{1}{f}\right)+\overline{N}(r,f)+S(r,f)\\
&= O(\log r)+S(r,f)+S(r,g)\\
&= S(r,f)+S(r,g).
\end{aligned}
\tag{2.49}
$$

Similarly

$$N\big(r,\frac{1}{G_1}\big)=S(r,f)+S(r,g). \tag{2.50}$$

Since

$$S(r,g)+O(\log r)=S(r,g), \tag{2.51}$$

from Lemma 2.7 we get

$$T(r,g^{n+1})\le 3N\left(r,\frac{1}{g^{n+1}}\right)+4\overline{N}\left(r,\frac{1}{(g^{n+1})'-1}\right)+S(r,g^{n+1}). \tag{2.52}$$

From (2.36), (2.52), the right equality of (2.41) and Lemma 2.5 we get

$$
\begin{aligned}
(n+1)T(r,g) &\le 4T(r,g^ng')+S(r,g)\\
&\le 4T(r,G_1)+S(r,g)+O(\log r)\\
&= 4T(r,G_1)+S(r,g).
\end{aligned}
\tag{2.53}
$$

Since

$$
\begin{aligned}
T(r,G_1) &= T\left(r,\frac{g^ng'}{P}\right)\\
&\le (n+2)T(r,g)+S(r,g)+O(\log r)\\
&= (n+2)T(r,g)+S(r,g),
\end{aligned}
\tag{2.54}
$$

from (2.53), (2.54) and the condition that g is a transcendental meromorphic function we know that $G_1=g^ng'/P$ is a transcendental meromorphic function and

$$S(r,g)=S(r,G_1). \tag{2.55}$$

Similarly, $F_1=f^nf'/P$ is a transcendental meromorphic function and

$$S(r,f)=S(r,F_1). \tag{2.56}$$

From (2.42) we get

$$T(r,F_1)=T(r,G_1)+O(1). \tag{2.57}$$

From (2.57) we get

$$S(r, F_1) = S(r, G_1). \tag{2.58}$$

From (2.55), (2.56) and (2.58) we get

$$S(r, f) + S(r, g) = S(r, F_1), \quad S(r, f) + S(r, g) = S(r, G_1). \tag{2.59}$$

From (2.49), (2.50) and (2.59) we get

$$N\left(r, \frac{1}{F_1}\right) = S(r, F_1), \quad N\left(r, \frac{1}{G_1}\right) = S(r, G_1). \tag{2.60}$$

From (2.42), (2.58) and (2.60) we get
(2.61)

$$N(r, F_1) = N\left(r, \frac{1}{G_1}\right) = S(r, F_1), \quad N(r, G_1) = N\left(r, \frac{1}{F_1}\right) = S(r, G_1).$$

From (2.42) we know that F_1 and G_1 share 1 and -1 IM. This together with (2.58), (2.60), (2.61) and Lemma 2.8 implies that F_1 and G_1 share 0, 1, c, ∞ CM, and so it follows from Lemma 2.9 that 0 and ∞ are Picard exceptional values of F_1 and G_1. This together with (2.41) implies that f and g are two transcendental entire functions. Let

$$f = P_1 e^{\alpha_1}, \quad g = P_2 e^{\alpha_2}, \tag{2.62}$$

where P_1 and P_2 are two nonzero polynomials, α_1 and α_2 are two nonconstant entire functions. Substituting (2.62) into $f^n f' g^n g' = P^2$, we get

$$P_1^n(P_1' + P_1\alpha_1')P_2^n(P_2' + P_2\alpha_2') \cdot e^{(n+1)\alpha_1 + (n+1)\alpha_2} = P^2. \tag{2.63}$$

From (2.63) we get

$$(n+1)\alpha_1 + (n+1)\alpha_2 = 0 \tag{2.64}$$

and

$$P_1^n(P_1' + P_1\alpha_1')P_2^n(P_2' + P_2\alpha_2') = P^2. \tag{2.65}$$

From (2.64) we get

$$\alpha_1 = -\alpha_2 =: \alpha \tag{2.66}$$

and

$$\alpha_1' = -\alpha_2' = \alpha'. \tag{2.67}$$

Substituting (2.67) into (2.65), we get

$$P_1^n(P_1' + P_1\alpha')P_2^n(P_2' - P_2\alpha') = P^2. \tag{2.68}$$

From (2.68) we see that α' is a polynomial such that $\alpha' \not\equiv 0$. From (2.62), (2.66) and (2.67) we get

$$F_1 = \frac{f^n f'}{P} = \frac{(P_1' + P_1\alpha')P_1^n}{P} \cdot e^{(n+1)\alpha} \tag{2.69}$$

and

$$G_1 = \frac{g^n g'}{P} = \frac{(P_2' - P_2\alpha')P_2^n}{P} \cdot e^{-(n+1)\alpha}. \tag{2.70}$$

Since $F_1 \neq 0, \infty$, from (2.69) we have $((P_1' + P_1\alpha')P_1^n)/P \neq 0, \infty$, and so

$$\frac{(P_1' + P_1\alpha')P_1^n}{P} = c_4, \tag{2.71}$$

where $c_4 \neq 0$ is a complex number. Similarly

$$\frac{(P_2' - P_2\alpha')P_2^n}{P} = c_5, \tag{2.72}$$

where $c_5 \neq 0$ is a complex number. From (2.71) and (2.72) we get

$$P_1^n P_1' + P_1^{n+1}\alpha' = c_4 P \tag{2.73}$$

and

$$P_2' P_2^n - P_2^{n+1}\alpha' = c_5 P. \tag{2.74}$$

If one of P_1 and P_2 is not a constant, from (2.73), (2.74) and the fact that $\alpha' \not\equiv 0$ is a polynomial we get $\gamma_P \geq n+1$, which contradicts the condition $\gamma_P \leq n$. Thus P_1 and P_2 are nonzero constants, and so (2.65) can be rewritten as

$$(P_1 P_2)^{n+1}\alpha_1'\alpha_2' = P^2. \tag{2.75}$$

From (2.67) and (2.75) we get

$$-(P_1 P_2)^{n+1}\alpha'^2 = P^2. \tag{2.76}$$

From (2.76) we get

$$\alpha = c_6 \int_0^z P(\eta)d\eta + c_7, \tag{2.77}$$

where c_6 and c_7 are complex numbers, and c_6 satisfies $c_6^2 = -(P_1P_2)^{-n-1}$. From (2.57), (2.66) and (2.77) we get

$$f(z) = c_8 \cdot e^{c_6 Q}, \quad g(z) = c_9 \cdot e^{-c_6 Q}, \tag{2.78}$$

where c_8 and c_9 are nonzero complex numbers satisfying $c_8c_9 = P_1P_2$, Q is a nonconstant polynomial such that

$$Q = \int_0^z P(\eta)d\eta. \tag{2.79}$$

From $c_6^2 = -(P_1P_2)^{-n-1}$ and $c_8c_9 = P_1P_2$ we get

$$c_6^2(c_8c_9)^{n+1} = -1. \tag{2.80}$$

From (2.78), (2.79) and (2.80) we get the conclusion of Lemma 2.10. Lemma 2.10 is thus completely proved. □

Lemma 2.11 (see [20]). *Let $s\,(>0)$ and t be two relatively prime integers, and let c be a finite complex number satisfying $c^s=1$. Then there exists one and only one common zero of ω^s-1 and ω^t-c.*

Lemma 2.12 (see [11]). *Let f be a nonconstant meromorphic function, and let $F=\sum_{k=0}^{p}a_kf^k\Big/\sum_{j=0}^{q}b_jf^j$ be an irreducible rational function in f with constant coefficients $\{a_k\}$ and $\{b_j\}$, where $a_p\neq 0$ and $b_q\neq 0$. Then $T(r,F)=d\,T(r,f)+O(1)$, where $d=\max\{p,q\}$.*

3. Proof of theorems

Proof of Theorem 1.1. Let

$$F_1=\frac{f^{n+1}}{n+1}\quad\text{and}\quad G_1=\frac{g^{n+1}}{n+1}. \tag{3.1}$$

Then from (3.1) and the condition that $f^nf'-P$ and $g^ng'-P$ share 0 CM we see that $F_1'-P$ and $G_1'-P$ share 0 CM. Let

$$\Delta_1=3\Theta(\infty,F_1)+2\Theta(\infty,G_1)+\Theta(0,F_1)+\Theta(0,G_1)+\delta_2(0,F_1)+\delta_2(0,G_1) \tag{3.2}$$

and

$$\Delta_2=3\Theta(\infty,G_1)+2\Theta(\infty,F_1)+\Theta(0,G_1)+\Theta(0,F_1)+\delta_2(0,G_1)+\delta_2(0,F_1). \tag{3.3}$$

From (3.1) we have

$$\begin{aligned}\Theta(0,F_1)&=1-\limsup_{r\to\infty}\frac{\overline{N}\left(r,\frac{1}{F_1}\right)}{T(r,F_1)}\\&=1-\limsup_{r\to\infty}\frac{\overline{N}\left(r,\frac{1}{f}\right)}{(n+1)T(r,f)+O(1)}\\&\geq 1-\limsup_{r\to\infty}\frac{T(r,f)}{(n+1)T(r,f)}=\frac{n}{n+1},\end{aligned} \tag{3.4}$$

$$\begin{aligned}\Theta(\infty,F_1)&=1-\limsup_{r\to\infty}\frac{\overline{N}(r,F_1)}{T(r,F_1)}\\&=1-\limsup_{r\to\infty}\frac{\overline{N}(r,f)}{(n+1)T(r,f)+O(1)}\\&\geq 1-\limsup_{r\to\infty}\frac{T(r,f)}{(n+1)T(r,f)}=\frac{n}{n+1}.\end{aligned} \tag{3.5}$$

Similarly

$$\Theta(0,G_1)\geq\frac{n}{n+1}\quad\text{and}\quad\Theta(\infty,G_1)\geq\frac{n}{n+1}. \tag{3.6}$$

Again from (3.1) and Definition 1.1 we get

$$\begin{aligned}\delta_2(0,F_1) &= 1-\limsup_{r\to\infty}\frac{N_2\left(r,\frac{1}{F_1}\right)}{T(r,F_1)}\\ &= 1-\limsup_{r\to\infty}\frac{N_2\left(r,\frac{n+1}{f^{n+1}}\right)}{T\left(r,\frac{f^{n+1}}{n+1}\right)}\\ &= 1-\limsup_{r\to\infty}\frac{2\overline{N}\left(r,\frac{1}{f}\right)}{(n+1)T(r,f)+O(1)}\ge\frac{n-1}{n+1}.\end{aligned}\tag{3.7}$$

Similarly

$$\delta_2(0,G_1)\ge\frac{n-1}{n+1}.\tag{3.8}$$

From (3.2)-(3.8) and $n\ge 11$ we get

$$\Delta_1\ge\frac{9n-2}{n+1}>8.\tag{3.9}$$

From (3.3) and in the same manner as above, we get

$$\Delta_2\ge\frac{9n-2}{n+1}>8\tag{3.10}$$

if $n\ge 11$.

From the condition that f and g are transcendental meromorphic functions we know that F_1 and G_1 are transcendental meromorphic functions. This together with (3.9), (3.10), Lemma 2.3 and the condition that $F_1'-P$ and $G_1'-P$ share 0 CM gives $F_1'G_1'=P^2$ or $F_1=G_1$. We discuss the following two cases.

Case 1. Suppose that $F_1'G_1'=P^2$. Then it follows from (3.1) that $f^nf'g^ng'=P^2$. This together with Lemma 2.10 reveals the conclusion of Theorem 1.1.

Case 2. Suppose that $F_1=G_1$. Then it follows from (3.1) that $f^{n+1}=g^{n+1}$ which implies that $f=tg$, where t is a complex number satisfying $t^{n+1}=1$. This reveals the conclusion of Theorem 1.1. Theorem 1.1 is thus completely proved. □

Proof of Theorem 1.2. Let

$$F_2=f^n(f-1),\quad G_2=g^n(g-1),\tag{3.11}$$

and let

$$\Delta_1=3\Theta(\infty,F_2)+2\Theta(\infty,G_2)+\Theta(0,F_2)+\Theta(0,G_2)+\delta_2(0,F_2)+\delta_2(0,G_2)\tag{3.12}$$

and

$$\Delta_2=3\Theta(\infty,G_2)+2\Theta(\infty,F_2)+\Theta(0,G_2)+\Theta(0,F_2)+\delta_2(0,G_2)+\delta_2(0,F_2).\tag{3.13}$$

From the left equality of (3.11) and Lemma 2.5 we get

$$
\begin{aligned}
\Theta(0,F_2) &= 1-\limsup_{r\to\infty}\frac{\overline{N}\left(r,\frac{1}{F_2}\right)}{T(r,F_2)}\\
&= 1-\limsup_{r\to\infty}\frac{\overline{N}\left(r,\frac{1}{f}\right)+\overline{N}\left(r,\frac{1}{f-1}\right)}{(n+1)T(r,f)+O(1)}\\
&\geq 1-\limsup_{r\to\infty}\frac{2T(r,f)}{(n+1)T(r,f)}=\frac{n-1}{n+1}
\end{aligned}
\tag{3.14}
$$

and

$$
\begin{aligned}
\Theta(\infty,F_2) &= 1-\limsup_{r\to\infty}\frac{\overline{N}(r,F_2)}{T(r,F_2)}\\
&= 1-\limsup_{r\to\infty}\frac{\overline{N}(r,f)}{(n+1)T(r,f)+O(1)}\\
&\geq 1-\limsup_{r\to\infty}\frac{T(r,f)}{(n+1)T(r,f)}=\frac{n}{n+1}.
\end{aligned}
\tag{3.15}
$$

Similarly, from the right equality of (3.11) we get

$$
\Theta(0,G_2)\geq\frac{n-1}{n+1},\quad \Theta(\infty,G_2)\geq\frac{n}{n+1}.
\tag{3.16}
$$

From the left equality of (3.11) and Lemma 2.5 we get

$$
\begin{aligned}
\delta_2(0,F_2) &= 1-\limsup_{r\to\infty}\frac{N_2(r,\frac{1}{F_2})}{T(r,F_2)}\\
&= 1-\limsup_{r\to\infty}\frac{N_2\left(r,\frac{1}{f^n(f-1)}\right)}{T(r,f^n(f-1))}\\
&\geq 1-\limsup_{r\to\infty}\frac{2\overline{N}\left(r,\frac{1}{f}\right)+N\left(r,\frac{1}{f-1}\right)}{(n+1)T(r,f)+O(1)}\\
&\geq 1-\limsup_{r\to\infty}\frac{3T(r,f)+O(1)}{(n+1)T(r,f)+O(1)}=\frac{n-2}{n+1}.
\end{aligned}
\tag{3.17}
$$

Similarly, from the right equality of (3.11) and Lemma 2.5 we get

$$
\delta_2(0,G_2)\geq\frac{n-2}{n+1}.
\tag{3.18}
$$

From (3.12)-(3.17) and $n\geq 15$ we get

$$
\Delta_1\geq\frac{9n-6}{n+1}>8.
\tag{3.19}
$$

Similarly, from (3.13) we get

$$
\Delta_2\geq\frac{9n-6}{n+1}>8\quad\text{if }\ n\geq 15.
\tag{3.20}
$$

From the condition that f and g are transcendental meromorphic functions we know that F_2 and G_2 are transcendental meromorphic functions. This together with (3.19), (3.20), Lemma 2.3 and the condition that $F_2' - P$ and $G_2' - P$ share 0 CM gives $F_2'G_2' = P^2$ or $F_2 = G_2$. We discuss the following two cases.

Case 1. Suppose that $F_2'G_2' = P^2$. Then it follows from (3.11) that

$$(f^n(f-1))'(g^n(g-1))' = P^2. \tag{3.21}$$

Since

$$(f^n(f-1))' = (f^{n+1} - f^n)' = (n+1)f^{n-1}(f - \frac{n}{n+1})f' \tag{3.22}$$

and

$$(g^n(g-1))' = (g^{n+1} - g^n)' = (n+1)g^{n-1}(g - \frac{n}{n+1})g', \tag{3.23}$$

from (3.21)-(3.23) we get

$$f^{n-1}(f - \frac{n}{n+1})f'g^{n-1}(g - \frac{n}{n+1})g' = \frac{P^2}{(n+1)^2}. \tag{3.24}$$

Let $z_0 \notin \{z : P(z) = 0\}$ be a zero of f of order p. Then it follows from (3.24) that z_0 is a pole of g. Suppose that z_0 is a pole of g of order q. Then we have $np - 1 = (n+1)q + 1$, i.e., $n(p-q) = q+2$, which implies that $p \geq q+1$ and $q + 2 \geq n$, and so

$$p \geq n - 1. \tag{3.25}$$

Let $z_1 \notin \{z : P(z) = 0\}$ be a zero of $f - n/(n+1)$ of order p_1. Then it follows from (3.24) that z_1 is a pole of g. Suppose that z_1 is a pole of g of order q_1. Then from (3.24) we have $2p_1 - 1 = (n+1)q_1 + 1$. From this we get

$$p_1 \geq 1 + \frac{n+1}{2} = \frac{n+3}{2}. \tag{3.26}$$

Let $z_2 \notin \{z : P(z) = 0\}$ be a zero of f' of order p_2 that is not a zero of $f(f - n/(n+1))$. Then from (3.24) we see that z_2 is a pole of g. Suppose that z_2 is a pole of g of order q_2. Then it follows from (3.24) that $p_2 = (n+1)q_2 + 1$. From this we get

$$p_2 \geq n + 2. \tag{3.27}$$

Let $z_3 \notin \{z : P(z) = 0\}$ be a pole of f. Then it follows from (3.24) that z_3 is a zero of $g(g - n/(n+1))g'$. Combining (3.24)-(3.27) and $n \geq 15$, we get (3.28)

$$\begin{aligned}\overline{N}(r, f) &\leq \overline{N}\left(r, \frac{1}{g}\right) + \overline{N}\left(r, \frac{1}{g - n/(n+1)}\right) + \overline{N}\left(r, \frac{1}{g'}\right) + O(\log r)\\ &\leq \frac{1}{n-1}N\left(r, \frac{1}{g}\right) + \frac{2}{n+3}N\left(r, \frac{1}{g - n/(n+1)}\right) + \frac{1}{n+2}N\left(r, \frac{1}{g'}\right)\\ &\quad + O(\log r) + S(r, g)\\ &\leq \left(\frac{1}{14} + \frac{1}{9} + \frac{2}{17}\right)T(r, g) + S(r, g).\end{aligned}$$

From (3.25)-(3.28), the condition $n \geq 15$ and the second fundamental theorem we get (3.29)

$$\begin{aligned}T(r, f) &\leq \overline{N}\left(r, \frac{1}{f}\right) + \overline{N}\left(r, \frac{1}{f - n/(n+1)}\right) + \overline{N}(r, f) + S(r, f)\\ &\leq \left(\frac{1}{9} + \frac{1}{14}\right)T(r, f) + \left(\frac{1}{14} + \frac{1}{9} + \frac{2}{17}\right)T(r, g) + O(\log r) + S(r, f)\\ &\leq \left(\frac{1}{9} + \frac{1}{14}\right)T(r, f) + \left(\frac{1}{14} + \frac{1}{9} + \frac{2}{17}\right)T(r, g) + S(r, f).\end{aligned}$$

Similarly

$$T(r, g) \leq \left(\frac{1}{9} + \frac{1}{14}\right)T(r, g) + \left(\frac{1}{14} + \frac{1}{9} + \frac{2}{17}\right)T(r, f) + S(r, g). \tag{3.30}$$

From (3.29) and (3.30) we get $T(r, f) + T(r, g) = S(r, f) + S(r, g)$, which is impossible.

Case 2. Suppose that $F_2 = G_2$. Then from (3.11) we get

$$f^n(f - 1) = g^n(g - 1). \tag{3.31}$$

Suppose that $f \not\equiv g$. Let

$$H = f/g. \tag{3.32}$$

We discuss the following two subcases.

Subcase 2.1. Suppose that H is a constant. Then $H \neq 1$. From (3.31) and (3.32) we get

$$(H^{n+1} - 1)g = H^n - 1. \tag{3.33}$$

If $H^{n+1} - 1 \neq 0$, from (3.33) we have $g = (H^n - 1)/(H^{n+1} - 1)$. Thus g is a constant, which is impossible. If $H^{n+1} - 1 = 0$, from (3.33) we get $H^n - 1 = 0$, and so $H = 1$, which is impossible.

Subcase 2.2. Suppose that H is not a constant. First of all, from (3.31) and (3.32) we get (3.33). This together with $H^{n+1}-1\not\equiv 0$ gives

$$g=\frac{1-H^n}{1-H^{n+1}}. \tag{3.34}$$

Since n and $n+1$ are two relatively prime integers, from (3.32), (3.34), Lemma 2.11 and Lemma 2.12 we get

$$T(r,f)=T(r,Hg)=nT(r,H)+O(1). \tag{3.35}$$

On the other hand, from (3.32), (3.34) and the second fundamental theorem we get

$$\overline{N}(r,f)=\sum_{j=1}^{n}\overline{N}\left(r,\frac{1}{H-\lambda_j}\right)\geq (n-2)T(r,H)+S(r,H), \tag{3.36}$$

where λ_1, λ_2, ..., λ_n are finite complex numbers satisfying $\lambda_j\neq 1$ and $\lambda_j^{n+1}=1\,(1\leq j\leq n)$. From (3.35) and (3.36) we get

$$\begin{aligned}\Theta(\infty,f)&=1-\limsup_{r\to\infty}\frac{\overline{N}(r,f)}{T(r,f)}\\ &\leq 1-\limsup_{r\longrightarrow\infty}\frac{(n-2)T(r,H)+S(r,H)}{nT(r,H)}\\ &\leq 1-\frac{n-2}{n}=\frac{2}{n},\end{aligned} \tag{3.37}$$

which contradicts the condition $\Theta(\infty,f)>2/n$. Theorem 1.2 is thus completely proved. □

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