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THE CONVERGENCE THEOREMS FOR COMMON FIXED POINTS OF UNIFORMLY L-LIPSCHITZIAN ASYMPTOTICALLY Φ-PSEUDOCONTRACTIVE MAPPINGS

Zhiqun Xue

Abstract. In this paper, we show that the modified Mann iteration with errors converges strongly to fixed point for uniformly L-Lipschitzian asymptotically Φ-pseudocontractive mappings in real Banach spaces.

Meanwhile, it is proved that the convergence of Mann and Ishikawa iterations is equivalent for uniformly L-Lipschitzian asymptotically Φ-pseudocontractive mappings in real Banach spaces. Finally, we obtain the convergence theorems of Ishikawa iterative sequence and the modified Ishikawa iterative process with errors.

1. Introduction

In this paper, we assume that E is a real Banach space, E^* is the dual space of E, D is a nonempty closed convex subset of E and $J : E \to 2^{E^*}$ is the normalized duality mapping defined by

$$J(x) = \{f \in E^* : \langle x, f\rangle = \|x\|^2 = \|f\|^2\}, \quad \forall x \in E,$$

where $\langle\cdot,\cdot\rangle$ denotes the generalized duality pairing. The single-valued normalized duality mapping is denoted by j.

Definition 1.1. Let $T : D \to D$ be a mapping.

(1) T is called uniformly L-Lipschitzian if there is a constant $L > 0$ such that for any $x, y \in D$,

$$\|T^n x - T^n y\| \le L\|x - y\|, \quad \forall n \ge 1.$$

(2) T is called asymptotically pseudocontractive with a sequence $\{k_n\} \subset [1, +\infty)$ and $\lim_{n\to\infty} k_n = 1$ if for each $x, y \in D$, there exists $j(x-y) \in J(x-y)$ such that

$$\langle T^n x - T^n y, j(x - y)\rangle \le k_n\|x - y\|^2, \quad \forall n \ge 1.$$

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(3) T is called asymptotically Φ-pseudocontractive with a sequence $\{k_n\} \subset [1,+\infty)$ and $\lim_{n\to\infty} k_n = 1$ if for all $x, y \in D$, there exist $j(x-y) \in J(x-y)$ and a strictly increasing continuous function $\Phi : [0,+\infty) \to [0,+\infty)$ with $\Phi(0) = 0$ such that

$$\langle T^n x - T^n y, j(x-y)\rangle \leq k_n \|x-y\|^2 - \Phi(\|x-y\|), \quad \forall n \geq 1.$$

It is obvious that if T is an asymptotically Φ-pseudocontractive mapping, then T is an asymptotically pseudocontractive mapping. Conversely, it is not true. The convergence of Mann-type and Ishikawa-type iteration processes for uniformly L-Lipschitzian and asymptotically Φ-pseudocontractive mappings in Banach spaces have been studied extensively by many authors, see for example [1, 2, 3, 6].

Recently, Ofoedu [3] gave iterative approximation problem of fixed points for uniformly L-Lipschitzian asymptotically pseudocontractive mappings in Banach spaces. The results are as following.

Theorem 1.2 ([3, Theorem 3.1]). *Let E be a real Banach space. Let K be a nonempty closed and convex subset of E, $T : K \to K$ a uniformly L-Lipschitzian asymptotically pseudocontractive mapping with sequence $\{k_n\}_{n\geq 0} \subset [1,+\infty)$, $\lim_{n\to\infty} k_n = 1$ such that $x^* \in F(T) = \{x \in K : Tx = x\}$. Let $\{\alpha_n\}_{n\geq 0} \subset [0,1]$ be such that $\sum_{n\geq 0} \alpha_n = \infty, \sum_{n\geq 0} \alpha_n^2 < \infty$ and $\sum_{n\geq 0} \alpha_n(k_n - 1) < \infty$. For arbitrary $x_0 \in K$ let $\{x_n\}_{n\geq 0}$ be iteratively defined by*

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T^n x_n, \quad n \geq 0.$$

Suppose there exists a strictly increasing function $\Phi : [0,+\infty) \to [0,+\infty), \Phi(0) = 0$ such that

$$\langle T^n x - x^*, j(x-x^*)\rangle \leq k_n \|x-x^*\|^2 - \Phi(\|x-x^*\|), \quad \forall x \in K.$$

Then, $\{x_n\}_{n\geq 0}$ is bounded.

Theorem 1.3 ([3, Theorem 3.2]). *Let E be a real Banach space. Let K be a nonempty closed and convex subset of E, $T : K \to K$ a uniformly L-Lipschitz asymptotically pseudocontractive mapping with sequence $\{k_n\}_{n\geq 0} \subset [1,+\infty)$, $\lim_{n\to\infty} k_n = 1$. Let $x^* \in F(T) = \{x \in K : Tx = x\}$. Let $\{\alpha_n\}_{n\geq 0} \subset [0,1]$ be such that $\sum_{n\geq 0} \alpha_n = \infty, \sum_{n\geq 0} \alpha_n^2 < \infty$ and $\sum_{n\geq 0} \alpha_n(k_n - 1) < \infty$. For arbitrary $x_0 \in K$ let $\{x_n\}_{n\geq 0}$ be iteratively defined by*

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T^n x_n, \quad n \geq 0.$$

Suppose there exists a strictly increasing continuous function $\Phi : [0,+\infty) \to [0,+\infty), \Phi(0) = 0$ such that

$$\langle T^n x - x^*, j(x-x^*)\rangle \leq k_n \|x-x^*\|^2 - \Phi(\|x-x^*\|), \quad \forall x \in K.$$

Then, $\{x_n\}_{n\geq 0}$ converges strongly to $x^* \in F(T)$.

Remark 1.4. As mentioned above Theorem 1.2 and Theorem 1.3, it is possible that $\Phi^{-1}(a_0)$ may be not sense while $\lim_{r\to+\infty}\Phi(r) = A < a_0$. For example, define $\Phi : [0,+\infty) \to [0,+\infty)$ by $\Phi(r) = \frac{r}{r+1}$, then $\Phi^{-1}(2)$ is senseless. Therefore, the proof of Theorem 3.1 and Theorem 3.2 in [3] is not reasonable. In order to avoid this problem, we will provide a significant improvement. Meanwhile, we obtain the convergence results of Ishikawa iterative sequence by the equivalence of the strong convergence results for uniformly L-Lipschitzian mappings in real Banach spaces. For this, we need the following concepts and lemmas.

Definition 1.5. Let $T_1, T_2 : D \to D$ be two mappings.

(i) For any given $x_1 \in D$, define the sequence $\{x_n\}_{n=1}^{\infty} \subset D$ by the iterative schemes

$$\begin{cases} y_n = (1-b_n-d_n)x_n + b_nT_2^nx_n + d_nw_n, & n \ge 1,\\ x_{n+1} = (1-a_n-c_n)x_n + a_nT_1^ny_n + c_nz_n, & n \ge 1, \end{cases} \tag{1.1}$$

which is called the modified Ishikawa iterative process with errors, where $\{a_n\}_{n=1}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$, $\{c_n\}_{n=1}^{\infty}$, $\{d_n\}_{n=1}^{\infty}$ are real sequences in $[0,1]$ satisfying conditions $a_n + c_n \le 1, b_n + d_n \le 1$, and $\{z_n\}, \{w_n\}$ are the bounded sequences in D. If $b_n = d_n = 0$, then the modified Ishikawa iterative process with errors reduces to:

(ii) For any given $u_1 \in D$, define the sequence $\{u_n\}_{n=1}^{\infty} \subset D$ by the iterative schemes

$$u_{n+1} = (1-a_n-c_n)u_n + a_nT_1^nu_n + c_nv_n, \quad n \ge 1, \tag{1.2}$$

which is called the modified Mann iterative process with errors, where $\{v_n\}$ is the bounded sequence in D.

If $c_n = d_n = 0$ in (1.1) and (1.2), then the corresponding iterations are called the modified Ishikawa and Mann iterations respectively, that is,

$$\begin{cases} y_n = (1-b_n)x_n + b_nT_2^nx_n, & n \ge 1,\\ x_{n+1} = (1-a_n)x_n + a_nT_1^ny_n, & n \ge 1; \end{cases} \tag{1.3}$$

$$u_{n+1} = (1-a_n)u_n + a_nT_1^nu_n, \quad n \ge 1. \tag{1.4}$$

Lemma 1.6 ([3])**.** *Let E be a real Banach space and let $J : E \to 2^{E^*}$ be a normalized duality mapping. Then*

$$\|x+y\|^2 \le \|x\|^2 + 2\langle y, j(x+y)\rangle$$

for all $x, y \in E$ and each $j(x+y) \in J(x+y)$.

Lemma 1.7 ([4])**.** *Let $\{\rho_n\}_{n=1}^{\infty}$ be a nonnegative real numbers sequence satisfying the inequality*

$$\rho_{n+1} \le (1-\theta_n)\rho_n + o(\theta_n),$$

where $\theta_n \in (0,1)$ with $\sum_{n=1}^{\infty}\theta_n = \infty$. Then $\rho_n \to 0$ as $n \to \infty$.

2. Main results

Theorem 2.1. *Let E be a real Banach space, D be a nonempty closed convex subset of E and $T_1 : D \to D$ be a uniformly L_1-Lipschitzian asymptotically Φ-pseudocontractive mapping with sequence $\{k_{1n}\}_{n\geq 1} \subset [1, +\infty)$, $\lim_{n\to\infty} k_{1n} = 1$. Let $q \in F(T_1) = \{x \in D : T_1 x = x\}$. Let $\{a_n\}_{n\geq 1}, \{c_n\}_{n\geq 1} \subset [0, 1]$ be such that $\sum_{n\geq 1} \alpha_n = \infty, \lim_{n\to\infty} a_n = 0$ and $c_n = o(a_n)$ with $a_n + c_n \leq 1$, for any $n \geq 1$. The sequence $\{u_n\}_{n=1}^{\infty}$ is defined by (1.2). Then, $\{u_n\}_{n\geq 1}$ converges strongly to q.*

Proof. Applying Lemma 1.6 and (1.2), we have

$$
\begin{aligned}
&\|u_{n+1} - q\|^2 \\
&= \|(1 - a_n - c_n)(u_n - q) + a_n(T_1^n u_n - q) + c_n(v_n - q)\|^2 \\
&\leq (1 - a_n - c_n)^2\|u_n - q\|^2 + 2a_n\langle T_1^n u_n - q, j(u_{n+1} - q)\rangle \\
&\quad + 2c_n\langle v_n - q, j(u_{n+1} - q)\rangle \\
&\leq (1 - a_n - c_n)^2\|u_n - q\|^2 + 2a_n\langle T_1^n u_n - T_1^n u_{n+1}, j(u_{n+1} - q)\rangle \\
&\quad + 2a_n\langle T_1^n u_{n+1} - T_1^n q, j(u_{n+1} - q)\rangle + 2c_n\|v_n - q\| \cdot \|u_{n+1} - q\| \\
&\leq (1 - a_n - c_n)^2\|u_n - q\|^2 + 2a_n L_1\|u_n - u_{n+1}\| \cdot \|u_{n+1} - q\| \\
&\quad + 2a_n(k_{1n}\|u_{n+1} - q\|^2 - \Phi(\|u_{n+1} - q\|)) \\
&\quad + c_n\|v_n - q\| + c_n\|v_n - q\| \cdot \|u_{n+1} - q\|^2.
\end{aligned} \tag{2.1}
$$

From (1.2), we observe that

$$
\begin{aligned}
&\|u_n - u_{n+1}\| \\
&\leq a_n\|u_n - T_1^n u_n\| + c_n\|u_n - v_n\| \\
&\leq (a_n(1 + L_1) + c_n)\|u_n - q\| + c_n\|v_n - q\|.
\end{aligned} \tag{2.2}
$$

Taking (2.2) into (2.1), we obtain

$$
\begin{aligned}
&\|u_{n+1} - q\|^2 \\
&\leq (1 - a_n - c_n)^2\|u_n - q\|^2 + 2a_n L_1[(a_n(1 + L_1) + c_n)\|u_n - q\| \\
&\quad + c_n\|v_n - q\|] \cdot \|u_{n+1} - q\| \\
&\quad + 2a_n k_{1n}\|u_{n+1} - q\|^2 - 2a_n\Phi(\|u_{n+1} - q\|) + c_n\|v_n - q\| \\
&\quad + c_n\|v_n - q\| \cdot \|u_{n+1} - q\|^2 \\
&\leq (1 - a_n)^2\|u_n - q\|^2 \\
&\quad + a_n L_1[a_n(1 + L_1) + c_n](\|u_n - q\|^2 + \|u_{n+1} - q\|^2) \\
&\quad + a_n L_1 c_n\|v_n - q\|(1 + \|u_{n+1} - q\|^2) + 2a_n k_{1n}\|u_{n+1} - q\|^2 \\
&\quad - 2a_n\Phi(\|u_{n+1} - q\|) \\
&\quad + c_n\|v_n - q\| + c_n\|v_n - q\| \cdot \|u_{n+1} - q\|^2.
\end{aligned} \tag{2.3}
$$

It then follows from (2.3) that

(2.4)

$$\begin{aligned}&\|u_{n+1}-q\|^2\\ \le\ &\frac{(1-a_n)^2+a_nL_1[a_n(1+L_1)+c_n]}{1-2a_nk_{1n}-a_nL_1[a_n(1+L_1)+c_n]-(a_nL_1+1)c_n\|v_n-q\|}\|u_n-q\|^2\\ &-\frac{2a_n}{1-2a_nk_{1n}-a_nL_1[a_n(1+L_1)+c_n]-(a_nL_1+1)c_n\|v_n-q\|}\Phi(\|u_{n+1}-q\|)\\ &+\frac{(a_nL_1+1)c_n\|v_n-q\|}{1-2a_nk_{1n}-a_nL_1[a_n(1+L_1)+c_n]-(a_nL_1+1)c_n\|v_n-q\|}.\end{aligned}$$

Since $2a_nk_{1n}+a_nL_1[a_n(1+L_1)+c_n]+(a_nL_1+1)c_n\|v_n-q\|\to 0$ as $n\to\infty$, then there exists N such that $2a_nk_{1n}+a_nL_1[a_n(1+L_1)+c_n]+(a_nL_1+1)c_n\|v_n-q\|<\frac{1}{2}$, $\forall n>N$, i.e., $1>1-2a_nk_{1n}-a_nL_1[a_n(1+L_1)+c_n]-(a_nL_1+1)c_n\|v_n-q\|>\frac{1}{2}(n>N)$. Thus, we have

(2.5)

$$\begin{aligned}&\|u_{n+1}-q\|^2\\ \le\ &\|u_n-q\|^2\\ &+a_n\frac{a_n+2(k_{1n}-1)+2L_1[a_n(1+L_1)+c_n]+(a_nL_1+1)c_n/a_n\|v_n-q\|}{1-2a_nk_{1n}-a_nL_1[a_n(1+L_1)+c_n]-(a_nL_1+1)c_n\|v_n-q\|}\|u_n-q\|^2\\ &-\frac{2a_n}{1-2a_nk_{1n}-a_nL_1[a_n(1+L_1)+c_n]-(a_nL_1+1)c_n\|v_n-q\|}\Phi(\|u_{n+1}-q\|)\\ &+\frac{(a_nL_1+1)c_n\|v_n-q\|}{1-2a_nk_{1n}-a_nL_1[a_n(1+L_1)+c_n]-(a_nL_1+1)c_n\|v_n-q\|}\\ \le\ &\|u_n-q\|^2+2a_nA_n\|u_n-q\|^2-2a_n\Phi(\|u_{n+1}-q\|)+B_n,\end{aligned}$$

where $A_n=a_n+2(k_{1n}-1)+2L_1[a_n(1+L_1)+c_n]+(a_nL_1+1)c_n/a_n\|v_n-q\|\to 0$ as $n\to\infty$, $B_n=2(a_nL_1+1)c_n\|v_n-q\|=o(a_n)$.

Let $\inf_{n\ge N}\frac{\Phi(\|u_{n+1}-q\|)}{1+\|u_{n+1}-q\|^2}=\lambda$. Then $\lambda=0$. Suppose this is not the case, i.e., suppose $\lambda>0$, and choose a $\gamma>0$ such that $\gamma<\min\{1,\lambda\}$. Then $\frac{\Phi(\|u_{n+1}-q\|)}{1+\|u_{n+1}-q\|^2}\ge\gamma$, i.e., $\Phi(\|u_{n+1}-q\|)\ge\gamma+\gamma\|u_{n+1}-q\|^2\ge\gamma\|u_{n+1}-q\|^2$. And it results that

$$\begin{aligned}&\|u_{n+1}-q\|^2\\ \le\ &\frac{1+2a_nA_n}{1+2a_n\gamma}\|u_n-q\|^2+\frac{B_n}{1+2a_n\gamma}\\ =\ &(1-a_n\frac{2\gamma-2A_n}{1+2a_n\gamma})\|u_n-q\|^2+\frac{B_n}{1+2a_n\gamma}.\end{aligned}\tag{2.6}$$

Since $a_n,A_n\to 0$ as $n\to\infty$, there exists $N_1>N$ such that $\frac{2\gamma-2A_n}{1+2a_n\gamma}>\gamma$ for all $n>N_1$. Hence, from (2.6)

$$\|u_{n+1}-q\|^2\le(1-a_n\gamma)\|u_n-q\|^2+\frac{B_n}{1+2a_n\gamma}$$

for all $n > N_1$. Applying Lemma 1.7, we obtain that $\|u_{n+1} - q\| \to 0$ as $n \to \infty$. By the continuity of Φ, then $\frac{\Phi(\|u_{n+1}-q\|)}{1+\|u_{n+1}-q\|^2} \to 0$ as $n \to \infty$. This is a contradiction and so $\lambda = 0$. Consequently, there exists an infinite subsequence such that $\|u_{n_j+1} - q\| \to 0$ as $j \to \infty$. Next we want to prove that $\|u_{n_j+m} - q\| \to 0$ as $j \to \infty$ by induction. Let $\forall \varepsilon \in (0,1)$, choose $n_j > N$ such that $\|u_{n_j+1} - q\| < \epsilon$, $A_{n_j+1} < \frac{\Phi(\epsilon)}{2(1+\epsilon^2)}$, $\frac{B_{n_j+1}}{2a_{n_j+1}} < \frac{\Phi(\epsilon)}{4}$. First, we want to prove $\|u_{n_j+2} - q\| < \epsilon$. Suppose it is not this case. Then $\|u_{n_j+2} - q\| \geq \epsilon$, this implies $\Phi(\|u_{n_j+2} - q\|) \geq \Phi(\epsilon)$. Using the formula (2.5), then we may obtain the following estimates

$$
\begin{aligned}
&\|u_{n_j+2} - q\|^2 \\
&\leq \|u_{n_j+1} - q\|^2 + 2a_{n_j+1}A_{n_j+1}\|u_{n_j+1} - q\|^2 \\
&\quad - 2a_{n_j+1}\Phi(\|u_{n_j+2} - q\|) + 2a_{n_j+1}\frac{B_{n_j+1}}{2a_{n_j+1}} \\
&< \epsilon^2 - 2a_{n_j+1}\frac{\Phi(\epsilon)}{4} < \epsilon^2
\end{aligned}
\tag{2.7}
$$

is a contradiction. Hence $\|u_{n_j+2} - q\| < \epsilon$. Assume that it holds for $m = k$. Then by the argument above, we easily prove that it holds for $m = k + 1$. Hence for $\forall m > 1$, we obtain $\|u_{n_j+m} - q\| < \epsilon$. This completes the proof. □

Remark 2.2. Theorem 2.1 improves and extends Theorem 3.1 and Theorem 3.2 in Ofoedu [3] in the following sense:

1. Theorem 2.1 differs greatly from Theorem 3.1 and Theorem 3.2 of Ofoedu [3] in the proof method.
2. The conditions $\sum_{n\geq 0} \alpha_n^2 < \infty$ and $\sum_{n\geq 0} \alpha_n(k_n - 1) < \infty$ in [3, Theorem 3.1 and Theorem 3.2] are replaced by the more general condition $\lim_{n\to\infty} \alpha_n = 0$, the conclusion still holds.
3. The Mann iteration method in [3] is extended to the modified Mann iteration method with errors introduced by Xu [5]. Therefore, while $c_n = 0$, for any $n \geq 1$ in Theorem 2.1, then the following result holds.

Corollary 2.3. *Let E be a real Banach space, D be a nonempty closed convex subset of E and $T_1 : D \to D$ be a uniformly L_1-Lipschitzian asymptotically Φ-pseudocontractive mapping with sequence $\{k_{1n}\}_{n\geq 1} \subset [1, +\infty)$ and $\lim_{n\to\infty} k_{1n} = 1$. Let $q \in F(T_1) = \{x \in D : T_1x = x\}$. Let $\{a_n\}_{n\geq 1} \subset [0,1]$ be such that $\sum_{n\geq 1} \alpha_n = \infty$ and $\lim_{n\to\infty} a_n = 0$. The sequence $\{u_n\}_{n=1}^{\infty}$ is defined by* (1.4)*. Then, $\{u_n\}_{n\geq 1}$ converges strongly to q.*

Theorem 2.4. *Let E be a real Banach space, D be a nonempty closed convex subset of E and $T_i : D \to D$ $(i = 1, 2)$ be two uniformly L-Lipschitzian asymptotically Φ-pseudocontractive mappings with the sequences $\{k_{1n}\}, \{k_{2n}\} \subset [1, +\infty)$ and $\lim_{n\to\infty} k_{1n} = \lim_{n\to\infty} k_{2n} = 1$. Let $q \in F(T_1) \cap F(T_2) \neq \emptyset$. Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be two sequences in $[0,1]$ satisfying the following conditions*: (i) *$a_n, b_n \to 0$ as $n \to \infty$*; (ii) *$\sum_{n=1}^{\infty} a_n = \infty$. The sequences*

$\{u_n\}_{n=1}^{\infty}$ *and* $\{x_n\}_{n=1}^{\infty}$ *are defined by* (1.3) *and* (1.4), *respectively. Then the following two assertions are equivalent*:

(i) *The modified Mann iteration* (1.4) *converges strongly to the fixed point* q *of* T_1;

(ii) *The modified Ishikawa iteration* (1.3) *converges strongly to the common fixed point* q *of* $T_1 \cap T_2$.

Proof. If the Ishikawa iteration (1.3) converges to the fixed point q, then by putting $b_n = 0$, we can get the convergence of the Mann iteration (1.4). Conversely, we only need to prove (i) $\Rightarrow$ (ii), i.e., $\|u_n - q\| \to 0$ as $n \to \infty \Rightarrow \|x_n - q\| \to 0$ as $n \to \infty$. Denote $L = \max\{L_1, L_2\}$, where L_1 and L_2 satisfy the inequality: $\|T_i^n x - T_i^n y\| \leq L_i \|x - y\|$ for $\forall x, y \in D$, $i = 1, 2$; $k_n = \max\{k_{1n}, k_{2n}\}$, where $\{k_{1n}\}$ and $\{k_{2n}\}$ satisfy the inequality: $\langle T_i^n x - T_i^n y, j(x - y)\rangle \leq k_{in}\|x - y\|^2 - \Phi(\|x - y\|), \forall x, y \in D, \forall n \geq 1, i = 1, 2.$

Applying (1.3), (1.4) and Lemma 1.6, we have

$$
\begin{aligned}
&\|x_{n+1} - u_{n+1}\|^2 \\
&= \|(1 - a_n)(x_n - u_n) + a_n(T_1^n y_n - T_1^n u_n)\|^2 \\
&\leq (1 - a_n)^2\|x_n - u_n\|^2 + 2a_n\langle T_1^n y_n - T_1^n u_n, j(x_{n+1} - u_{n+1})\rangle \\
&\leq (1 - a_n)^2\|x_n - u_n\|^2 + 2a_n\langle T_1^n y_n - T_1^n x_{n+1}, j(x_{n+1} - u_{n+1})\rangle \\
&\quad + 2a_n\langle T_1^n x_{n+1} - T_1^n u_{n+1}, j(x_{n+1} - u_{n+1})\rangle \\
&\quad + 2a_n\langle T_1^n u_{n+1} - T_1^n u_n, j(x_{n+1} - u_{n+1})\rangle \\
&\leq (1 - a_n)^2\|x_n - u_n\|^2 + 2a_n L\|y_n - x_{n+1}\| \cdot \|x_{n+1} - u_{n+1}\| \\
&\quad + 2a_n(k_n\|x_{n+1} - u_{n+1}\|^2 - \Phi(\|x_{n+1} - u_{n+1}\|)) \\
&\quad + 2a_n L\|u_{n+1} - u_n\| \cdot \|x_{n+1} - u_{n+1}\|.
\end{aligned} \tag{2.8}
$$

Observe that

$$
\begin{aligned}
&\|y_n - x_{n+1}\| \\
&= \|a_n(x_n - T_1^n y_n) - b_n(x_n - T_2^n x_n)\| \\
&\leq (a_n + b_n + b_n L_2)\|x_n - q\| + a_n L_1\|y_n - q\| \\
&\leq (a_n + b_n + b_n L_2)\|x_n - q\| + a_n L_1(1 + b_n L_2)\|x_n - q\| \\
&\leq (a_n + b_n + b_n L + a_n L + a_n b_n L^2)\|x_n - q\| \\
&\leq \gamma_n(\|x_n - u_n\| + \|u_n - q\|),
\end{aligned} \tag{2.9}
$$

where $\gamma_n = a_n + b_n + b_n L + a_n L + a_n b_n L^2$.

Substituting (2.9) into (2.8), we obtain

(2.10)

$$
\begin{aligned}
&\|x_{n+1} - u_{n+1}\|^2 \\
&\leq (1 - a_n)^2\|x_n - u_n\|^2 + 2a_n L\gamma_n(\|x_n - u_n\| + \|u_n - q\|) \cdot \|x_{n+1} - u_{n+1}\|
\end{aligned}
$$

$$
\begin{aligned}
&+ 2a_n(k_n\|x_{n+1} - u_{n+1}\|^2 - \Phi(\|x_{n+1} - u_{n+1}\|))\\
&+ 2a_nL\|u_{n+1} - u_n\| \cdot \|x_{n+1} - u_{n+1}\|\\
\le{}& (1 - a_n)^2\|x_n - u_n\|^2 + a_nL\gamma_n(\|x_n - u_n\|^2 + \|u_n - q\|^2 + 2\|x_{n+1} - u_{n+1}\|^2)\\
&+ 2a_n(k_n\|x_{n+1} - u_{n+1}\|^2 - \Phi(\|x_{n+1} - u_{n+1}\|))\\
&+ a_nL\|u_{n+1} - u_n\|(1 + \|x_{n+1} - u_{n+1}\|^2).
\end{aligned}
$$

Without loss of generality, we assume that

$$0 < 1 - 2a_nk_n - 2a_n\gamma_nL - a_nL\|u_{n+1} - u_n\| < 1$$

for any $n \ge 1$. Then (2.10) implies that

$$
\begin{aligned}
&\|x_{n+1} - u_{n+1}\|^2\\
\le{}& \frac{(1 - a_n)^2 + a_n\gamma_nL}{1 - 2a_nk_n - 2a_n\gamma_nL - a_nL\|u_{n+1} - u_n\|}\|x_n - u_n\|^2\\
&+ \frac{a_nL\gamma_n\|u_n - q\|^2 + a_nL\|u_{n+1} - u_n\|}{1 - 2a_nk_n - 2a_n\gamma_nL - a_nL\|u_{n+1} - u_n\|}\\
&- \frac{2a_n}{1 - 2a_nk_n - 2a_n\gamma_nL - a_nL\|u_{n+1} - u_n\|}\Phi(\|x_{n+1} - u_{n+1}\|).
\end{aligned}
\tag{2.11}
$$

Since $2a_nk_n + 2a_n\gamma_nL + a_nL\|u_{n+1} - u_n\| \to 0$ as $n \to \infty$, then there exists N such that $2a_nk_n + 2a_n\gamma_nL + a_nL\|u_{n+1} - u_n\| < \frac{1}{2}$, $\forall n > N$, i.e., $1 > 1 - 2a_nk_n - 2a_n\gamma_nL - a_nL\|u_{n+1} - u_n\| > \frac{1}{2}(n > N)$. Thus, we have

$$
\begin{aligned}
&\|x_{n+1} - u_{n+1}\|^2\\
\le{}& \|x_n - u_n\|^2 + 2a_n\frac{a_n + (k_n - 1) + \gamma_nL + L\|u_{n+1} - u_n\|}{1 - 2a_nk_n - 2a_n\gamma_nL - a_nL\|u_{n+1} - u_n\|}\|x_n - u_n\|^2\\
&+ 2a_n\frac{L\gamma_n\|u_n - q\|^2 + L\|u_{n+1} - u_n\|}{1 - 2a_nk_n - 2a_n\gamma_nL - a_nL\|u_{n+1} - u_n\|}\\
&- \frac{2a_n}{1 - 2a_nk_n - 2a_n\gamma_nL - a_nL\|u_{n+1} - u_n\|}\Phi(\|x_{n+1} - u_{n+1}\|)\\
\le{}& \|x_n - u_n\|^2 + \frac{2a_nB_n}{1 - 2a_nk_n - 2a_n\gamma_nL - a_nL\|u_{n+1} - u_n\|}\|x_n - u_n\|^2\\
&+ \frac{2a_nC_n}{1 - 2a_nk_n - 2a_n\gamma_nL - a_nL\|u_{n+1} - u_n\|}\\
&- \frac{2a_n}{1 - 2a_nk_n - 2a_n\gamma_nL - a_nL\|u_{n+1} - u_n\|}\Phi(\|x_{n+1} - u_{n+1}\|)\\
\le{}& \|x_n - u_n\|^2 + 4a_nB_n\|x_n - u_n\|^2 + 4a_nC_n - 2a_n\Phi(\|x_{n+1} - u_{n+1}\|),
\end{aligned}
\tag{2.12}
$$

where $B_n = a_n + (k_n - 1) + \gamma_nL + L\|u_{n+1} - u_n\|$, $C_n = L\gamma_n\|u_n - q\|^2 + L\|u_{n+1} - u_n\|$.

Set $\inf_{n\geq N} \frac{\Phi(\|x_{n+1}-u_{n+1}\|)}{1+\|x_{n+1}-u_{n+1}\|^2} = \lambda$. Then $\lambda = 0$. If it is not the case, assume that $\lambda > 0$. Let $0 < \gamma < \min\{1, \lambda\}$. Then $\frac{\Phi(\|x_{n+1}-u_{n+1}\|)}{1+\|x_{n+1}-u_{n+1}\|^2} \geq \gamma$, i.e.,

$$\Phi(\|x_{n+1} - u_{n+1}\|) \geq \gamma + \gamma\|x_{n+1} - u_{n+1}\|^2 \geq \gamma\|x_{n+1} - u_{n+1}\|^2.$$

Thus

$$\begin{aligned} &\|x_{n+1} - u_{n+1}\|^2 \\ &\leq \frac{1 + 4a_nB_n}{1 + 2a_n\gamma}\|x_n - u_n\|^2 + \frac{4a_nC_n}{1 + 2a_n\gamma} \\ &= (1 - a_n\frac{2\gamma - 4B_n}{1 + 2a_n\gamma})\|x_n - u_n\|^2 + \frac{4a_nC_n}{1 + 2a_n\gamma}. \end{aligned} \tag{2.13}$$

By $a_n, B_n \to 0$ as $n \to \infty$, we choose $N_1 > N$ such that $\frac{2\gamma-4B_n}{1+2\alpha_n\gamma} > \gamma$ for all $n > N_1$. It follows from (2.13) that

$$\|x_{n+1} - u_{n+1}\|^2 \leq (1 - a_n\gamma)\|x_n - u_n\|^2 + \frac{4a_nC_n}{1 + 2a_n\gamma}$$

for all $n > N_1$. It follows from Lemma 1.7 that $\|x_{n+1} - u_{n+1}\| \to 0$ as $n \to \infty$, this is a contradiction and so $\lambda = 0$. Consequently, there exists an infinite subsequence such that $\|x_{n_j+1} - u_{n_j+1}\| \to 0$ as $j \to \infty$. Next we want to prove that $\|x_{n_j+m} - u_{n_j+m}\| \to 0$ as $j \to \infty$ by induction. Let $\forall\varepsilon \in (0, 1)$, choose $n_j > N$ such that $\|x_{n_j+1} - u_{n_j+1}\| < \epsilon$, $B_{n_j+1} < \frac{\Phi(\epsilon)}{8(1+\epsilon^2)}$, $C_{n_j+1} < \frac{\Phi(\epsilon)}{8}$. First we want to prove $\|x_{n_j+2} - u_{n_j+2}\| < \epsilon$. Suppose it is not this case. Then $\|x_{n_j+2} - u_{n_j+2}\| \geq \epsilon$, this implies $\Phi(\|x_{n_j+2} - u_{n_j+2}\|) \geq \Phi(\epsilon)$. Using the formula (2.12), we now obtain the following estimates:

$$\begin{aligned} &\|x_{n_j+2} - u_{n_j+2}\|^2 \\ &\leq \|x_{n_j+1} - u_{n_j+1}\|^2 + 4a_{n_j+1}B_{n_j+1}\|x_{n_j+1} - u_{n_j+1}\|^2 \\ &\quad + 4a_{n_j+1}C_{n_j+1} - 2a_{n_j+1}\Phi(\|x_{n_j+2} - u_{n_j+2}\|) \\ &< \epsilon^2 - a_{n_j+1}\Phi(\epsilon) \leq \epsilon^2 \end{aligned} \tag{2.14}$$

is a contradiction. Hence $\|x_{n_j+2} - u_{n_j+2}\| < \epsilon$. Assume that it holds for $m = k$. Then by the argument above, we easily prove that it holds for $m = k+1$. Hence, we obtain $\|x_n - u_n\| \to 0$ as $n \to \infty$. Since $\|u_n - q\| \to 0$ as $n \to \infty$. From the inequality $0 \leq \|x_n - q\| \leq \|x_n - u_n\| + \|u_n - q\|$, we get $\|x_n - q\| \to 0$ as $n \to \infty$. □

Theorem 2.5. *Let E be a real Banach space, D be a nonempty closed and convex subset of E and $T_i : D \to D$ $(i = 1, 2)$ be two uniformly L-Lipschitzian asymptotically Φ-pseudocontractive mappings with the sequences $\{k_{1n}\}, \{k_{2n}\} \subset [1, +\infty)$ and $\lim_{n\to\infty} k_{1n} = \lim_{n\to\infty} k_{2n} = 1$. Let $q \in F(T_1) \cap F(T_2) \neq \emptyset$. Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be two sequences in $[0, 1]$ satisfying the following conditions*: (i) $a_n, b_n \to 0$ *as* $n \to \infty$; (ii) $\sum_{n=1}^{\infty} a_n = \infty$. *Then the modified Ishikawa iteration sequence* (1.3) $\{x_n\}_{n=1}^{\infty}$ *converges strongly to the common fixed point q of $T_1 \cap T_2$.*

Proof. Using Corollary 2.3 and Theorem 2.4, we obtain the conclusion of Theorem 2.5. □

Similarly, we also obtain the following results.

Theorem 2.6. *Let E be a real Banach space, D be a nonempty closed convex subset of E and $T_i : D \to D$ $(i = 1, 2)$ be two uniformly L-Lipschitzian asymptotically Φ-pseudocontractive mappings with the sequences $\{k_{1n}\}, \{k_{2n}\} \subset [1, +\infty)$ and $\lim_{n\to\infty} k_{1n} = \lim_{n\to\infty} k_{2n} = 1$. Let $q \in F(T_1) \cap F(T_2) \neq \emptyset$. Let $\{a_n\}_{n=1}^{\infty}, \{b_n\}_{n=1}^{\infty}, \{c_n\}_{n=1}^{\infty}$ and $\{d_n\}_{n=1}^{\infty}$ be the sequences in $[0, 1]$ satisfying the following conditions*: (i) $a_n + c_n \leq 1, b_n + d_n \leq 1$ *for any* $n \geq 1$; (ii) $a_n, b_n, d_n \to 0$ *as* $n \to \infty$; (iii) $\sum_{n=1}^{\infty} a_n = \infty$; (iv) $c_n = o(a_n)$. *And let $\{z_n\}$ and $\{w_n\}$ be the bounded sequences in D. Suppose that the sequences $\{u_n\}_{n=1}^{\infty}$ and $\{x_n\}_{n=1}^{\infty}$ are defined by* (1.2) *and* (1.1) *respectively. Then the following two assertions are equivalent*:

(i) *The modified Mann iteration with errors* (1.2) *converges strongly to the fixed point q of T_1*;

(ii) *The modified Ishikawa iteration with errors* (1.1) *converges strongly to the common fixed point q of $T_1 \cap T_2$.*

Theorem 2.7. *Let E be a real Banach space, D be a nonempty closed convex subset of E and $T_i : D \to D$ $(i = 1, 2)$ be two uniformly L-Lipschitzian asymptotically Φ-pseudocontractive mappings with the sequences $\{k_{1n}\}, \{k_{2n}\} \subset [1, +\infty)$ and $\lim_{n\to\infty} k_{1n} = \lim_{n\to\infty} k_{2n} = 1$. Let $q \in F(T_1) \cap F(T_2) \neq \emptyset$. Let $\{a_n\}_{n=1}^{\infty}, \{b_n\}_{n=1}^{\infty}, \{c_n\}_{n=1}^{\infty}$ and $\{d_n\}_{n=1}^{\infty}$ be the sequences in $[0, 1]$ satisfying the following conditions*: (i) $a_n + c_n \leq 1, b_n + d_n \leq 1$ *for any* n; (ii) $a_n, b_n, d_n \to 0$ *as* $n \to \infty$; (iii) $\sum_{n=1}^{\infty} a_n = \infty$; (iv) $c_n = o(a_n)$. *And let $\{z_n\}$ and $\{w_n\}$ be the bounded sequences in D. Suppose that the sequence $\{x_n\}_{n=1}^{\infty}$ is defined by* (1.1)*. Then the modified Ishikawa iteration with errors* (1.1) *converges strongly to the common fixed point q of $T_1 \cap T_2$.*

Proof. Using Theorem 2.1 and Theorem 2.6, we get the results of Theorem 2.7. □

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Department of Mathematics and Physics
Shijiazhuang Railway Institute
Shijiazhuang 050043, P. R. China
E-mail address: xuezhiqun@126.com