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GENERAL NONLINEAR VARIATIONAL INCLUSIONS WITH H-MONOTONE OPERATOR IN HILBERT SPACES

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Abstract. In this paper, a new class of general nonlinear variational inclusions involving H-monotone is introduced and studied in Hilbert spaces. By applying the resolvent operator associated with H-monotone, we prove the existence and uniqueness theorems of solution for the general nonlinear variational inclusion, construct an iterative algorithm for computing approximation solution of the general nonlinear variational inclusion and discuss the convergence of the iterative sequence generated by the algorithm. The results presented in this paper improve and extend many known results in recent literatures.

1. Introduction

Variational inequalities have wide applications in many fields including mechanics, physics, optimization and control, nonlinear programming, economics and transportation equilibrium and engineering sciences. For details, we refer to [1-11] and the references therein. It is well known that one of the most interesting and important problems in the variational inequality theory is the development of H-monotonicity. In [3], Fang and Huang introduced the concept of H-monotone operators and defined an associated resolvent operator. Nga [9] introduced and studied set-valued nonlinear variational inequalities for H-monotone mappings in nonreflexive Banach spaces, especially, Zeng, Guu, and Yao [11] introduced an iterative method for generalized nonlinear set-valued mixed quasi-variational inequalities with H-monotone mappings.

Inspired and motivated by the recent research works, in this paper, we introduce and study a new class of general nonlinear variational inclusions, which includes the variational inclusions in [1], [3] and [10] as special cases. By applying the resolvent operator technique and fixed point theorem, we suggest a new iterative process with error for solving the general nonlinear variational

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inclusion. Several existence and uniqueness results of solutions for the general nonlinear variational inclusion involving H-monotone are given. The results presented in this paper extend, improve and unify a host of results in recent literatures.

2. Preliminaries

Throughout this paper, we assume that X is a Hilbert space endowed with a norm $\|\cdot\|$ and an inner product $\langle \cdot, \cdot \rangle$, respectively, 2^X stands for the family of all the nonempty subsets of X.

We now recall and introduce the following definitions and results.

Definition 2.1. Let $T, H : X \to X$ be two mappings. T is called

(1) *monotone* if

$$\langle Tx - Ty, x - y \rangle \geq 0, \quad \forall x, y \in X;$$

(2) *strictly monotone* if T is monotone and

$$\langle Tx - Ty, x - y \rangle = 0$$

if and only if $x = y$;

(3) *strongly monotone* if there exists a constant $r > 0$ such that

$$\langle Tx - Ty, x - y \rangle \geq r\|x - y\|^2, \quad \forall x, y \in X;$$

(4) *Lipschitz continuous* if there exists a constant $s > 0$ such that

$$\|Tx - Ty\| \leq s\|x - y\|, \quad \forall x, y \in X;$$

(5) *relaxed monotone* with respect to H if there exists a constant $\alpha > 0$ such that

$$\langle Tx - Ty, Hx - Hy \rangle \geq -\alpha\|x - y\|^2, \quad \forall x, y \in X;$$

(6) *anti-monotone* if

$$\langle Tx - Ty, x - y \rangle \leq 0, \quad \forall x, y \in X.$$

Definition 2.2. Let $T, g : X \to X$ be two mappings. T is called *g-k-strongly monotone* if there exists a constant $k > 0$ such that

$$\langle Tx - Ty, gx - gy \rangle \geq k\|gx - gy\|^2, \quad \forall x, y \in X.$$

Definition 2.3. Let $N : X \times X \to X$ be a mapping, $H, A, B : X \to X$ be mappings. N is said to be

(1) *strongly monotone* in the first argument if there exists a constant $\xi > 0$ such that

$$\langle N(x, u) - N(y, u), x - y \rangle \geq \xi\|x - y\|^2, \quad \forall x, y, u \in X;$$

(2) *strongly monotone* with respect to A in the first argument if there exists a constant $\beta > 0$ such that

$$\langle N(Ax, u) - N(Ay, u), x - y \rangle \geq \beta\|x - y\|^2, \quad \forall x, y, u \in X;$$

(3) *Lipschitz continuous* with respect to the first argument if there exists a constant $\delta > 0$ such that

$$\|N(x,u) - N(y,u)\| \leq \delta\|x-y\|, \quad \forall x,y,u \in X.$$

Definition 2.4. Let $N : X \times X \to X$ and $A, B, g : X \to X$ be mappings. $N(A,B)$ is said to be *strongly monotone* with respect to g if there exists a constant $\rho > 0$ such that

$$\langle N(Au,Bu) - N(Av,Bv), gu - gv\rangle \geq \rho\|u-v\|^2, \quad \forall u,v \in X,$$

where $N(A,B)u = N(Au,Bu)$, $\forall u \in X$.

Definition 2.5. A multi-valued mapping $M : X \to 2^X$ is called

(1) *monotone* if

$$\langle x-y, u-v\rangle \geq 0, \quad \forall u,v \in X,\ x \in Mu,\ y \in Mv;$$

(2) *maximal monotone* if W is monotone and $(I+\lambda W)(X) = X$ for any $\lambda > 0$, where I denotes the identity mapping on X.

Definition 2.6 ([3])**.** Let $H : X \to X$ be a mapping and $M : X \to 2^X$ be a mapping. M is said to be H-*monotone* if M is monotone and $(H+\lambda M)X = X$ for any $\lambda > 0$.

Let $g, h, A, B : X \to X$ and $N : X \times X \to X$ be mappings and $W : X \to 2^X$ be an H-monotone mapping. Given $f \in X$, we consider the following problem:

Find $u \in X$ such that

$$f \in N(Au,Bu) - M((g-h)u), \tag{2.1}$$

which is called *a general nonlinear variational inclusion*, where $(g-h)x = gx - hx$ for all $x \in X$.

Some special cases of the problem (2.1) are as follows:

(A) If $f = 0$ and $N(Ax,Bx) = Ax - Bx$ for any $x \in X$, then the problem (2.1) collapses to seeking $u \in X$ such that

$$0 \in Au - Bu + M((g-h)u),$$

which is called the *generalized equation* by Uko [10].

(B) If $f = 0$, h=0 and $N(Ax,Bx) = Ax - Bx$ for any $x \in X$, then the problem (2.1) is equivalent to finding $u \in X$ such that

$$0 \in Au - Bu + M(gu),$$

which was introduced and studied by Adly [1].

(C) If $f = 0$, $g - h = I$ and $N(x,y) = x$ for any $x, y \in X$, then the problem (2.1) is equivalent to finding $u \in X$ such that

$$0 \in Au + M(u),$$

which was introduced and studied by Fang and Huang [3].

Definition 2.7 ([3])**.** Let $H : X \to X$ be a strictly monotone mapping and $M : X \to 2^X$ be an H-monotone mapping. For any given $\lambda > 0$, the resolvent operator $R^H_{M,\lambda} : X \to X$ is defined by

$$R^H_{M,\lambda}(x) = (H + \lambda M)^{-1}(x), \quad \forall x \in X.$$

Lemma 2.1 ([3])**.** *Let $H : X \to X$ be a strongly monotone mapping with constant $r > 0$ and $M : X \to 2^X$ be an H-monotone mapping. Then the resolvent operator $R^H_{M,\lambda} : X \to X$ is Lipschitz continuous with constant r^{-1}.*

Lemma 2.2 ([6])**.** *Let $\{a_n\}_{n\geq 0}$, $\{b_n\}_{n\geq 0}$ and $\{c_n\}_{n\geq 0}$ be nonnegative sequences satisfying*

$$a_{n+1} \leq (1 - t_n)a_n + t_n b_n + c_n, \quad \forall n \geq 0,$$

where $\{t_n\}_{n\geq 0} \subset [0, 1]$, $\sum_{n=0}^{\infty} t_n = +\infty$, $\lim_{n\to\infty} b_n = 0$ and $\sum_{n=0}^{\infty} c_n < +\infty$. Then $\lim_{n\to\infty} a_n = 0$.

3. Existence and uniqueness of solution for the general nonlinear variational inclusion

Now we use the resolvent operator technique due to Fang and Huang [3] to establish the equivalence between the general nonlinear variational inclusion (2.1) and the fixed point problem.

Lemma 3.1. *Let $H : X \to X$ be a strongly monotone mapping, $M : X \to 2^X$ be an H-monotone mapping, $g, h, A, B : X \to X$ be mappings and λ be a positive constant. Then the following statements are equivalent*:

(a) *The general nonlinear variational inclusion* (2.1) *has a solution $u \in X$;*

(b) *There exists $u \in X$ satisfying*

$$(g - h)u = R^H_{M,\lambda}[H(g - h)u - \lambda N(Au, Bu) + \lambda f];$$

(c) *The mapping $F : X \to X$ defined by*

$$Fx = x - (g - h)x + R^H_{M,\lambda}[H(g - h)x - \lambda N(Ax, Bx) + \lambda f], \quad \forall x \in X \tag{3.1}$$

has a fixed point $u \in X$.

Proof. It is clear that $u \in X$ is a solution of the general nonlinear variational inclusion (2.1) if and only if

$$\begin{aligned}
&\lambda f \in \lambda N(Au, Bu) - \lambda M((g - h)u) \\
\Leftrightarrow\ & H(g - h)u + \lambda f - \lambda N(Au, Bu) \in (H + \lambda M)((g - h)u) \\
\Leftrightarrow\ & (g - h)u = R^H_{M,\lambda}[H(g - h)u - \lambda N(Au, Bu) + \lambda f] \\
\Leftrightarrow\ & u = u - (g - h)u + R^H_{M,\lambda}[H(g - h)u - \lambda N(Au, Bu) + \lambda f].
\end{aligned}$$

This completes the proof. □

Based on Lemma 3.1, we suggest the following iterative process with error for the general nonlinear variational inclusion (2.1).

Algorithm 3.1. *Let $A, B, g, h, H : X \to X$ be mappings, $M : X \to 2^X$ be H-monotone and $N : X \times X \to X$ be mappings. For given $u_0 \in X$, compute the iterative sequence $\{u_n\}_{n\geq 0}$ by*

$$\begin{aligned} u_{n+1} = (1-a_n)u_n &+ a_n[u_n - (g-h)u_n \\ &+ R^H_{M,\lambda}(H(g-h)u_n - \lambda N(Au_n, Bu_n) + \lambda f)] + w_n, \quad n \geq 0, \end{aligned}$$

where $\{a_n\}_{n\geq 0}$ is a sequence in $[0,1]$ such that $\sum_{n=0}^{\infty} a_n = +\infty$, $\{w_n\}_{n\geq 0} \subset X$, $\sum_{n=0}^{\infty} \|w_n\| < +\infty$ and $\lambda > 0$ is a constant.

Next we study those conditions under which the approximate solutions u_n obtained from Algorithm 3.1 converge strongly to the unique solution $u \in X$ of the general nonlinear implicit variational inclusion (2.1).

Theorem 3.1. *Let $H, A, B, g, h : X \to X$ be Lipschitz continuous with constants l, t, s, α, β, respectively, $g-h$ be strongly monotone with constant ρ and g be relaxed monotone with respect to h with constant τ. Let $N : X \times X \to X$ be Lipschitz continuous in the first and second arguments with constants a and b, respectively, and $N(A,B)$ be strongly monotone with respect to $H(g-h)$ with constant δ. Let $M : X \to 2^X$ be an H-monotone mapping. Assume that $K = \alpha^2 + 2\tau + \beta^2$, $D = (bs+at)^2$ and $E = \delta^2 - D[l^2K - r^2(1-\sqrt{1-2\rho+K})^2] > 0$. If there exists a constant $\lambda > 0$ such that*

$$\left|\lambda - \frac{\delta}{D}\right| < \frac{\sqrt{E}}{D}, \tag{3.2}$$

then for any given $f \in X$, the general nonlinear variational inclusion (2.1) *has a unique solution $u^* \in X$ and the sequence $\{u_n\}_{n\geq 0}$ defined by Algorithm 3.1 strongly converges to u^*.*

Proof. It follows from Lemma 2.1 and (3.1) that for any $u, v \in X$,

$$\begin{aligned} &\|Fu - Fv\| \\ &= \|u - (g-h)u + R^H_{M,\lambda}[H(g-h)u - \lambda N(Au,Bu) + \lambda f] \\ &\quad - [v - (g-h)v + R^H_{M,\lambda}[H(g-h)v - \lambda N(Av,Bv) + \lambda f]]\| \\ &\leq \|u - v - ((g-h)u - (g-h)v)\| \\ &\quad + \|R^H_{M,\lambda}[H(g-h)u - \lambda N(Au,Bu) + \lambda f] \\ &\quad - R^H_{M,\lambda}[H(g-h)v - \lambda N(Av,Bv) + \lambda f]\| \\ &\leq \|u - v - ((g-h)u - (g-h)v)\| \\ &\quad + \frac{1}{r}\|H(g-h)u - H(g-h)v - \lambda[N(Au,Bu) - N(Av,Bv)]\|. \end{aligned} \tag{3.3}$$

By the Lipschitz continuity of g, h, H, A, B, N and strong monotonicity of $g-h$, and relaxed monotonicity of g, we obtain that

$$\begin{aligned} &\|u-v-((g-h)u-(g-h)v)\|^2 \\ &= \|u-v\|^2 - 2\langle u-v,(g-h)u-(g-h)v\rangle + \|gu-gv-(hu-hv)\|^2 \\ &\le \|u-v\|^2 - 2\rho\|u-v\|^2 + \|gu-gv\|^2 - 2\langle gu-gv, hu-hv\rangle \\ &\quad + \|hu-hv\|^2 \\ &\le (1-2\rho+\alpha^2+2\tau+\beta^2)\|u-v\|^2 \end{aligned} \tag{3.4}$$

and

$$\begin{aligned} &\|H(g-h)u-H(g-h)v-\lambda[N(Au,Bu)-N(Av,Bv)]\|^2 \\ &= \|H(g-h)u-H(g-h)v\|^2 \\ &\quad - 2\lambda\langle H(g-h)u-H(g-h)v, N(Au,Bu)-N(Av,Bv)\rangle \\ &\quad + \lambda^2\|N(Au,Bu)-N(Av,Bv)\|^2 \\ &\le l^2\|(g-h)u-(g-h)v\|^2 - 2\lambda\delta\|u-v\|^2 \\ &\quad + \lambda^2(\|N(Au,Bu)-N(Au,Bv)\| + \|N(Au,Bv)-N(Av,Bv)\|)^2 \\ &\le [l^2(\alpha^2+2\tau+\beta^2)-2\lambda\delta]\|u-v\|^2 \\ &\quad + \lambda^2(b\|Bu-Bv\| + a\|Au-Av\|)^2 \\ &\le [l^2(\alpha^2+2\tau+\beta^2)-2\lambda\delta+\lambda^2(bs+at)^2]\|u-v\|^2, \end{aligned} \tag{3.5}$$

which yields that

$$\|Fu-Fv\| \le \theta\|u-v\|, \quad \forall u,v\in H, \tag{3.6}$$

where

$$\theta = \sqrt{1-2\rho+K} + \frac{1}{r}\sqrt{l^2K-2\lambda\delta+\lambda^2 D} \ge 0.$$

By (3.2) we get that $\theta < 1$. Hence the contraction mapping F has a unique fixed point $u^* \in X$. In light of Lemma 3.1, we obtain that u^* is a unique solution of the general nonlinear variational inclusion (2.1).

It follows from Algorithm 3.1 that

$$u_{n+1} = (1-a_n)u_n + a_nF(u_n) + w_n, \quad n \ge 0 \tag{3.7}$$

and

$$u^* = (1-a_n)u^* + a_nF(u^*), \quad n \ge 0. \tag{3.8}$$

Using (3.6), (3.7) and (3.8), we have

$$\begin{aligned} &\|u_{n+1}-u^*\| \\ &\le \|(1-a_n)u_n + a_nF(u_n) + w_n - [(1-a_n)u^* + a_nF(u^*)]\| + \|w_n\| \\ &\le (1-a_n)\|u_n-u^*\| + \theta a_n\|u_n-u^*\| + \|w_n\| \\ &= (1-(1-\theta)\alpha_n)\|u_n-u^*\| + \|w_n\|, \quad \forall n\ge 0. \end{aligned} \tag{3.9}$$

In terms of Lemma 2.2 and (3.9), we know that $u_n \to u^*$ as $n \to \infty$. Therefore, $\{u_n\}_{n\geq 0}$ strongly converges to the unique solution u^* of the general nonlinear variational inclusion (2.1). This completes the proof. $\square$

It follows from Theorem 3.1 that:

Theorem 3.2. *Let $H, A, B : X \to X$ be Lipschitz continuous with constants l, t, s, respectively. Let $g, h : X \to X$ satisfy that $g-h$ is Lipschitz continuous with constant τ, g is strongly monotone with constant α and h is anti-monotone. Let $N : X \times X \to X$ be Lipschitz continuous in the first and second arguments with constants a and b, respectively, $N(A,B)$ be strongly monotone with respect to $H(g-h)$ with constant δ. Let $M : X \to 2^X$ be an H-monotone mapping. Assume that $D = (bs+at)^2$ and $E = \delta^2 - D[l^2\tau^2 - r^2(1-\sqrt{1-2\alpha+\tau^2})^2] > 0$. If there exists a constant $\lambda > 0$ satisfying* (3.2)*, then for any given $f \in X$, the general nonlinear variational inclusion* (2.1) *has a unique solution $u^* \in X$ and the sequence $\{u_n\}_{n\geq 0}$ defined by Algorithm 3.1 strongly converges to u^*.*

Theorem 3.3. *Let $H, A, B, g, h : X \to X$ be Lipschitz continuous with constants l, t, s, α, β, respectively, H be strongly monotone with constant δ, $g-h$ be strongly monotone with constant ρ and g be relaxed monotone with respect to h with constant τ. Let $N : X \times X \to X$ be Lipschitz continuous in the first and second arguments with constants a and b, respectively, and N be also strongly monotone with respect to A in the first argument with constant ξ. Let $M : X \to 2^X$ be an H-monotone mapping. Assume that $T = r - [(1+r)\sqrt{1-2\rho+\alpha^2+2\tau+\beta^2} + \sqrt{(l^2-2\delta+1)(\alpha^2+2\tau+\beta^2)}]$, $D = a^2t^2 - b^2s^2$, $W = \xi - bsT$ and $E = W^2 - D(1-T^2) > 0$. If there exists a constant $\lambda > 0$ such that*

$$T > \lambda bs, \tag{3.10}$$

and one of the following conditions:

$$at > bs, \quad \left|\lambda - \frac{W}{D}\right| < \frac{\sqrt{E}}{D}; \tag{3.11}$$

$$at < bs, \quad \left|\lambda - \frac{W}{D}\right| > -\frac{\sqrt{E}}{D}, \tag{3.12}$$

then for any given $f \in X$, the general nonlinear variational inclusion (2.1) *has a unique solution $u^* \in X$ and the sequence $\{u_n\}_{n\geq 0}$ defined by Algorithm 3.1 strongly converges to u^*.*

Proof. Let $u, v \in X$. It follows from Lemma 2.1 and (3.1) that

$$\begin{aligned} &\|Fu - Fv\| \\ &\leq \|u - v - ((g-h)u - (g-h)v)\| \\ &\quad + \frac{1}{r}\|H(g-h)u - H(g-h)v - \lambda[N(Au,Bu) - N(Av,Bv)]\| \end{aligned} \tag{3.13}$$

$$
\begin{aligned}
&\leq \Big(1+\frac{1}{r}\Big)\|u-v-((g-h)u-(g-h)v)\| \\
&\quad+\frac{1}{r}\|H(g-h)u-H(g-h)v-[(g-h)u-(g-h)v]\| \\
&\quad+\frac{1}{r}\|u-v-\lambda[N(Au,Bu)-N(Av,Bu)]\| \\
&\quad+\frac{\lambda}{r}\|N(Av,Bu)-N(Av,Bv)\|.
\end{aligned}
$$

Note that

$$
\begin{aligned}
&\|H(g-h)u-H(g-h)v-[(g-h)u-(g-h)v]\|^2 \\
&\leq (l^2-2\delta+1)\|(g-h)u-(g-h)v\|^2 \\
&\leq (l^2-2\delta+1)(\alpha^2+2\tau+\beta^2)\|u-v\|^2,
\end{aligned}
\tag{3.14}
$$

$$
\|u-v-\lambda[N(Au,Bu)-N(Av,Bu)]\|^2 \leq (1-2\lambda\xi+\lambda^2a^2t^2)\|u-v\|^2 \tag{3.15}
$$

and

$$
\|N(Av,Bu)-N(Av,Bv)\| \leq bs\|u-v\|. \tag{3.16}
$$

It follows from (3.13)-(3.16) that (3.6) holds, where

$$
\begin{aligned}
\theta &= \Big(1+\frac{1}{r}\Big)\sqrt{1-2\rho+\alpha^2+2\tau+\beta^2}+\frac{1}{r}\sqrt{(l^2-2\delta+1)(\alpha^2+2\tau+\beta^2)} \\
&\quad+\frac{1}{r}\sqrt{1-2\lambda\xi+\lambda^2a^2t^2}+\frac{\lambda bs}{r}.
\end{aligned}
$$

In view of (3.10) and one of (3.11) and (3.12), we conclude that $0<\theta<1$. That is, the contraction mapping F has a unique fixed point $u^*\in X$. By Lemma 3.1 and (3.6), we infer that u^* is a unique solution of the general nonlinear variational inclusion (2.1). The proof of convergence of $\{u_n\}_{n\geq 0}$ is similar to that of Theorem 3.1 and is omitted. This completes the proof. □

Theorem 3.4. *Let $H, A, B : X \to X$ be Lipschitz continuous with constants l, t, s, respectively. Let $g, h : X \to X$ satisfy that $g-h$ is Lipschitz continuous with constant α, g is strongly monotone with constant τ, h is anti-monotone and $H(g-h)$ is strongly monotone with constant δ. Let $N : X\times X \to X$ be Lipschitz continuous in the first and second arguments with constants a and b, respectively, and N be also strongly monotone with respect to A in the first argument with constant ξ. Let $M : X \to 2^X$ be an H-monotone mapping. Assume that $D=a^2t^2-b^2s^2$, $T=r-r\sqrt{1-2\tau+\alpha^2}-\sqrt{l^2\alpha^2-2\delta+1}$, $W=\xi-Tbs$ and $E=W^2-D(1-T^2)>0$. If there exists a constant $\lambda>0$ satisfying (3.10) and one of (3.11) and (3.12), then for any given $f\in X$, the general nonlinear variational inclusion (2.1) has a unique solution $u^*\in X$ and the sequence $\{u_n\}_{n\geq 0}$ defined by Algorithm 3.1 strongly converges to u^*.*

Proof. Let $u, v \in X$. It follows from Lemma 2.1 and (3.1) that

$$\begin{aligned} \|Fu - Fv\| \le{}& \|u - v - ((g-h)u - (g-h)v)\| \\ &+ \frac{1}{r}\|H(g-h)u - H(g-h)v - (u-v)\| \\ &+ \frac{1}{r}\|u - v - \lambda[N(Au,Bu) - N(Av,Bu)]\| \\ &+ \frac{\lambda}{r}\|N(Av,Bu) - N(Av,Bv)\|, \end{aligned} \tag{3.17}$$

$$\|u - v - ((g-h)u - (g-h)v)\|^2 \le (1 - 2\tau + \alpha^2)\|u - v\|^2 \tag{3.18}$$

and

$$\|H(g-h)u - H(g-h)v - (u-v)\|^2 \le (l^2\alpha^2 - 2\delta + 1)\|u - v\|^2. \tag{3.19}$$

It follows from (3.15), (3.16), (3.17)-(3.19) that (3.6) holds, where

$$\theta = \sqrt{1 - 2\tau + \alpha^2} + \frac{1}{r}\sqrt{l^2\alpha^2 - 2\delta + 1} + \frac{1}{r}\sqrt{1 - 2\lambda\xi + \lambda^2 a^2 t^2} + \frac{\lambda b s}{r} \ge 0.$$

In light of (3.10) and one of (3.11) and (3.12), we derive that $\theta < 1$. Consequently, the contraction mapping F has a unique fixed point $u^* \in X$. By Lemma 3.1 and (3.6), we infer that u^* is a unique solution of the general nonlinear variational inclusion (2.1). The proof of convergence of $\{u_n\}_{n\ge 0}$ is similar to that of Theorem 3.1 and is omitted. This completes the proof. □

Theorem 3.5. *Let $H, A, B, g, h : X \to X$ be Lipschitz continuous with constants l, t, s, α, β, respectively, g be relaxed monotone with respect to h with constant τ, $g-h$ be strongly monotone with constant ρ and $H(g-h)$ be A-δ-strongly monotone. Let $N : X \times X \to X$ be Lipschitz continuous in the first and second arguments with constants a and b, respectively, and N be also strongly monotone in the first argument with constant ξ. Let $M : X \to 2^X$ be an H-monotone mapping. Assume that $D = a^2t^2 - b^2s^2$, $T = r - r\sqrt{1 - 2\rho + \alpha^2 + 2\tau + \beta^2} - \sqrt{l^2(\alpha^2 + 2\tau + \beta^2) - 2\delta t^2 + t^2}$, $W = t^2\xi - Tbs$ and $E = W^2 - D(t^2 - T^2) > 0$. If there exists a constant $\lambda > 0$ satisfying* (3.10) *and one of* (3.11) *and* (3.12), *then for any given $f \in X$, the general nonlinear variational inclusion* (2.1) *has a unique solution $u^* \in X$ and the sequence $\{u_n\}_{n\ge 0}$ defined by Algorithm 3.1 strongly converges to u^*.*

Proof. Let $u, v \in X$. It follows from Lemma 2.1 and (3.1) that

$$\begin{aligned} \|Fu - Fv\| \le{}& \|u - v - ((g-h)u - (g-h)v)\| \\ &+ \frac{1}{r}\|H(g-h)u - H(g-h)v - (Au - Av)\| \\ &+ \frac{1}{r}\|Au - Av - \lambda[N(Au,Bu) - N(Av,Bu)]\| \\ &+ \frac{1}{r}\|N(Av,Bu) - N(Av,Bv)\|. \end{aligned} \tag{3.20}$$

Note that

$$\begin{aligned}
&\|H(g-h)u - H(g-h)v - (Au - Av)\|^2 \\
&= \|H(g-h)u - H(g-h)v\|^2 - 2\langle H(g-h)u - H(g-h)v, Au - Av\rangle \\
&\quad + \|Au - Av\|^2 \\
&\le l^2(\alpha^2 + 2\tau + \beta^2)\|u - v\|^2 - 2\delta\|Au - Av\|^2 + \|Au - Av\|^2 \\
&= (l^2(\alpha^2 + 2\tau + \beta^2) - 2\delta t^2 + t^2)\|u - v\|^2
\end{aligned} \tag{3.21}$$

and

$$\begin{aligned}
&\|Au - Av - \lambda[N(Au, Bu) - N(Av, Bu)]\|^2 \\
&= \|Au - Av\|^2 - 2\langle Au - Av, N(Au, Bu) - N(Av, Bu)\rangle \\
&\quad + \lambda^2\|N(Au, Bu) - N(Av, Bu)\|^2 \\
&\le \|Au - Av\|^2 - 2\lambda\xi\|Au - Av\|^2 + \lambda^2 a^2\|Au - Av\|^2 \\
&\le (1 - 2\lambda\xi + \lambda^2 a^2)t^2\|u - v\|^2.
\end{aligned} \tag{3.22}$$

It follows from (3.4), (3.16), (3.20), (3.21) and (3.22) that (3.6) holds, where

$$\begin{aligned}
\theta = {}& \sqrt{1 - 2\rho + \alpha^2 + 2\tau + \beta^2} + \frac{1}{r}\sqrt{l^2(\alpha^2 + 2\tau + \beta^2) - 2\delta t^2 + t^2} \\
&+ \frac{1}{r}\sqrt{(1 - 2\lambda\xi + \lambda^2 a^2)t^2} + \frac{\lambda bs}{r}.
\end{aligned}$$

The rest of the proof is similar to that of Theorem 3.1 and is omitted. This completes the proof. □

Theorem 3.6. *Let $H, A, B, g, h : X \to X$ be Lipschitz continuous with constants l, t, s, α, β, respectively, H be strongly monotone with constant δ, g be relaxed monotone with respect to h with constant τ, $g - h$ be strongly monotone with constant ρ. Let $N : X \times X \to X$ be Lipschitz continuous in the first and second arguments with constants a and b, respectively, and $N(A, B)$ be strongly monotone with respect to $g - h$ with constant ξ. Let $M : X \to 2^X$ be an H-monotone mapping. Assume that $K = \alpha^2 + 2\tau + \beta^2$, $D = (at + bs)^2$, $T = (r - r\sqrt{1 - 2\rho + K} - \sqrt{(l^2 - 2\delta + 1)K})^2$ and $E = \xi^2 - D(K - T) > 0$. If there exists a constant $\lambda > 0$ such that*

$$\left|\lambda - \frac{\xi}{D}\right| < \frac{\sqrt{E}}{D}, \tag{3.23}$$

then for any given $f \in X$, the general nonlinear variational inclusion (2.1) *has a unique solution $u^* \in X$ and the sequence $\{u_n\}_{n\ge 0}$ defined by Algorithm 3.1 strongly converges to u^*.*

Proof. Let $u, v \in X$. It follows from Lemma 2.1 and (3.1) that

$$\begin{aligned} & \|Fu - Fv\| \\ \le\ & \|u - v - ((g-h)u - (g-h)v)\| \\ & + \frac{1}{r}\|H(g-h)u - H(g-h)v - ((g-h)u - (g-h)v)\| \\ & + \frac{1}{r}\|(g-h)u - (g-h)v - \lambda[N(Au, Bu) - N(Av, Bv)]\|, \end{aligned} \tag{3.24}$$

and

$$\begin{aligned} & \|(g-h)u - (g-h)v - \lambda[N(Au, Bu) - N(Av, Bv)]\|^2 \\ \le\ & [\alpha^2 + 2\tau + \beta^2 - 2\lambda\xi + \lambda^2(at + bs)^2]\|u - v\|^2. \end{aligned} \tag{3.25}$$

It follows from (3.4), (3.14), (3.24) and (3.25) that (3.6) holds, where

$$\theta = \sqrt{l - 2\rho + K} + \frac{1}{r}\sqrt{(l^2 - 2\delta + 1)K} + \frac{1}{r}\sqrt{K - 2\lambda\xi + \lambda^2 D}.$$

The rest of the proof is similar to that of Theorem 3.1 and is omitted. This completes the proof. □

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