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STATIONARY PATTERNS FOR A PREDATOR-PREY MODEL WITH HOLLING TYPE III RESPONSE FUNCTION AND CROSS-DIFFUSION

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Abstract. This paper deals with a predator-prey model with Holling type III response function and cross-diffusion subject to the homogeneous Neumann boundary condition. We first give a priori estimates (positive upper and lower bounds) of positive steady states. Then the non-existence and existence results of non-constant positive steady states are given as the cross-diffusion coefficient is varied, which means that stationary patterns arise from cross-diffusion.

1. Introduction

From last century, many kinds of biological models have received extensive concerns, and in particular, the predator-prey models have been of great interest to both applied mathematicians and ecologists. Many excellent works have been done for the Lotka-Volterra type predator-prey system. In [3], Holling proposed that there exist three functional responses of the predator which usually called Holling type I, Holling type II and Holling type III. He proposed the form

$$p(u) = \frac{mu}{a+u}$$

as a Holling type II response function, it usually describes the uptake of substrate by the microorganisms in microbial dynamics. If the predator is the invertebrate, it is always the case. He also proposed the Holling type III response function in the following form:

$$p(u) = \frac{mu^2}{a+u^2}.$$

This case suits the vertebral predator. Similar types of response functions can be found in [2].

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Hsu developed a class of predator-prey system in which he incorporated Holling's rate [4], and in particular, the model with Holling type III is

$$
(1.1) \qquad \begin{cases} \frac{du}{dt} = ru(1-\frac{u}{k}) - \frac{mu^2 v}{(A+u)(B+u)}, \\ \frac{dv}{dt} = v[s(1-\frac{hv}{u})], \end{cases}
$$

where $u(t), v(t)$ represent the densities of the prey and predator, the parameters r, k, m, A, B, s, h are positive constants. For the detailed background on the ODE system (1.1), we refer the readers to [4].

In [4], the linear stability of nonnegative constant solutions and the existence of limits cycle solutions for the model (1.1) were studied. If the densities of the prey and predator are spatially inhomogeneous, by taking into account the effect of diffusion, instead of the ODE system (1.1), we consider the following reaction-diffusion system:

$$
(1.2) \qquad \begin{cases} \frac{\partial u}{\partial t} - d_1 \Delta u = ru(1-\frac{u}{k}) - \frac{mu^2 v}{(A+u)(B+u)}, & x \in \Omega, t > 0, \\ \frac{\partial v}{\partial t} - d_2 \Delta v = v[s(1-\frac{hv}{u})], & x \in \Omega, t > 0, \\ \frac{\partial u}{\partial \eta} = \frac{\partial v}{\partial \eta} = 0, & x \in \partial\Omega, t > 0, \end{cases}
$$

where Ω is a bounded domain in $\mathcal{R}^N (N \geq 2)$ with smooth boundary $\partial\Omega$, η denotes the outward normal derivative on $\partial\Omega$ and $\partial_\eta = \frac{\partial}{\partial \eta}$. d_1, d_2 are the diffusion coefficients corresponding to u, v, and all the parameters appearing in model (1.2) are assumed to be positive constants. The homogeneous Neumann boundary condition means that (1.2) is self-contained and no population can flux across the boundary of Ω.

Using the non-dimensional variables, the problem of (1.2) satisfies

$$
(1.3) \qquad \begin{cases} \frac{\partial u}{\partial t} - d_1 \Delta u = \lambda u - \alpha u^2 - \frac{\beta u^2 v}{(a+u)(b+u)}, & x \in \Omega, t > 0, \\ \frac{\partial v}{\partial t} - d_2 \Delta v = \mu v(1-\frac{v}{u}), & x \in \Omega, t > 0, \\ \frac{\partial u}{\partial \eta} = \frac{\partial v}{\partial \eta} = 0, & x \in \partial\Omega, t > 0, \end{cases}
$$

and the stead-state system of (1.3) satisfies

$$
(1.4) \qquad \begin{cases} -d_1 \Delta u = \lambda u - \alpha u^2 - \frac{\beta u^2 v}{(a+u)(b+u)}, & x \in \Omega, \\ -d_2 \Delta v = \mu v(1-\frac{v}{u}), & x \in \Omega, \\ \frac{\partial u}{\partial \eta} = \frac{\partial v}{\partial \eta} = 0, & x \in \partial\Omega, \end{cases}
$$

where $d_1, d_2, \lambda, \alpha, \beta, a, b$ and μ are positive constants.

For a partial differential equation (PDE) model such as (1.2), an important issue is to find the non-constant steady states, referred to as stationary patterns. In the present paper, in order to obtain patterns, we will introduce the cross-diffusion to (1.4) and consider the following elliptic equations:

$$
(1.5) \qquad \begin{cases} -d_1 \Delta u = \lambda u - \alpha u^2 - \frac{\beta u^2 v}{(a+u)(b+u)}, & x \in \Omega, \\ -d_2 \Delta[(1+d_3 u)v] = \mu v(1-\frac{v}{u}), & x \in \Omega, \\ \frac{\partial u}{\partial \eta} = \frac{\partial v}{\partial \eta} = 0, & x \in \partial\Omega. \end{cases}
$$

Here, d_3 is a non-negative constant. In this model, the predator v diffuses with the flux

$$J = -\nabla(d_2 v + d_2 d_3 uv) = -d_2 d_3 v \nabla u - (d_2 + d_2 d_3 u)\nabla v.$$

If $d_3 > 0$, the term $-d_2 d_3 v \nabla u$ of the flux is directed towards the decreasing population density of u, which represents that the prey congregates and forms a huge group to protect themselves from the attack of the predator. The constant d_3 is usually referred as cross-diffusion coefficient, which has been introduced by many authors, see for example [1, 6, 12] and references therein.

The main aim of this paper is to study the effects of the cross-diffusion pressures on the non-existence and existence of non-constant positive steady states of (1.5). We will show that there is no pattern if d_3 is small, while pattern occurs when d_3 is suitably chosen. The employed method is Leray-Schauder degree theory, which has been used by many authors to create spatially non-constant positive solutions and establish stationary patterns, the interested readers can read [5, 10, 11, 13, 14] and references therein.

The organization of this paper is as follows: Section 2 deals with a priori estimate of upper and lower bounds for positive solutions of (1.5). Section 3 is devoted to the non-existence of positive non-constant solution of (1.5) by using the energy method. The existence of positive non-constant solutions of (1.5) is given in Section 4 by using Leray-Schauder degree theory. We end the paper with concluding remarks.

2. A priori estimate

We first state two propositions.

Proposition 2.1 (Maximum Principle (Lou and Ni [8])). *Suppose that $g \in C(\bar{\Omega} \times \mathcal{R}^1)$.*

(i) *Assume that $\omega \in C^2(\Omega) \cap C^1(\bar{\Omega})$ and satisfies*

$$\Delta\omega(x) + g(x, \omega(x)) \geq 0, \quad x \in \Omega; \qquad \partial_\eta \omega \leq 0, \quad x \in \partial\Omega.$$

If $\omega(x_0) = \max_{\bar{\Omega}} \omega$, then $g(x_0, \omega(x_0)) \geq 0$.

(ii) *Assume that $\omega \in C^2(\Omega) \cap C^1(\bar{\Omega})$ and satisfies*

$$\Delta\omega(x) + g(x, \omega(x)) \leq 0, \quad x \in \Omega; \qquad \partial_\eta \omega \geq 0, \quad x \in \partial\Omega.$$

If $\omega(x_0) = \min_{\bar{\Omega}} \omega$, then $g(x_0, \omega(x_0)) \leq 0$.

Proposition 2.2 (Harnack Inequality (Lin et al. [7])). *Assume that $c \in C(\bar{\Omega})$ and let $\omega \in C^2(\Omega) \cap C^1(\bar{\Omega})$ be a positive solution to*

$$\Delta\omega(x) + c(x)\omega(x) = 0, \quad x \in \Omega; \qquad \partial_\eta \omega = 0, \quad x \in \partial\Omega.$$

Then there exists a positive constant $C_* = C_*(\| c \|_\propto, \Omega)$ such that

$$\max_{\bar{\Omega}} \omega \leq C_* \min_{\bar{\Omega}} \omega.$$

In the following, the generic constants $\bar{C}, \underline{C}, C_1$ will depend on the domain Ω. However, as Ω is fixed, we will not mention this dependence explicitly. We denote the constants $(\lambda, \alpha, \beta, a, b, \mu)$ by Λ.

Theorem 2.1. *Let D_1, D be arbitrary fixed positive numbers. Then there exist positive constants $\overline{C}(\Lambda, D_1, D)$ and $\underline{C}(\Lambda, D_1, D)$ such that when $d_1, d_2 \geq D_1$ and $d_3 \leq D$, every positive solution (u, v) of* (1.5) *in* $[C^2(\Omega) \cap C^1(\overline{\Omega})]^2$ *satisfies*

$$\underline{C}(\Lambda, D_1, D) < u(x), v(x) < \overline{C}(\Lambda, D_1, D), \qquad \forall x \in \overline{\Omega}.$$

Proof. Assume that (u, v) be a positive solution of (1.5), a direct application of (i) of Proposition 2.1 to the first equation of (1.5) yields

$$\max_{\bar{\Omega}} u \leq \lambda/\alpha \triangleq \bar{C}_1.$$

Define $\varphi(x) = d_2 v(1 + d_3 u)$, set $\varphi(x_0) = \max_{\bar{\Omega}} \varphi(x)$. Then, by (i) of Proposition 2.1 to the second equation of (1.5), we have that $v(x_0) \leq u(x_0) \leq \lambda/\alpha$. Thus

$$\max_{\bar{\Omega}} v \leq \varphi(x_0)/d_2 \leq (1 + d_3 u(x_0))v(x_0) \leq (1 + D\lambda/\alpha)\lambda/\alpha \triangleq \bar{C}_2,$$

and applying Proposition 2.2 to the first equation of (1.5) shows that there exists C_1 such that

$$\max_{\bar{\Omega}} u \leq C_1 \min_{\bar{\Omega}} u.$$

Set $u(x_1) = \min_{\bar{\Omega}} u$, by (ii) of Proposition 2.1 to the first equation of (1.5), we then have

$$\lambda \leq \alpha u(x_1) + \beta u(x_1)v(x_1)/ab \leq (\alpha + \beta\bar{C}_2/ab)u(x_1),$$

so

$$u(x_1) = \min_{\bar{\Omega}} u \geq \lambda(\alpha + \beta\bar{C}_2/ab)^{-1} \triangleq \underline{C}_1 > 0.$$

Set $\varphi(x_2) = \min_{\bar{\Omega}} \varphi(x)$, applying (ii) of Proposition 2.1 to the second equation of (1.5) gives

$$v(x_2) \geq u(x_2),$$

since

$$\frac{v(x)}{v(x_2)} = \frac{\varphi(x)d_2(1 + d_3u(x_2))}{\varphi(x_2)d_2(1 + d_3u(x))} \geq \min\{1, \frac{u(x_2)}{u(x)}\} \geq \min\{1, 1/C_1\},$$

so

$$\min_{\bar{\Omega}} v \geq \min_{\bar{\Omega}} u \min\{1, 1/C_1\}.$$

□

3. Non-existence of non-constant positive solution

We use the energy method to obtain the results of non-existence of non-constant positive solution of (1.5). Let $0 = \mu_0 < \mu_1 < \mu_2 < \cdots$ be the eigenvalues of the operator $-\Delta$ with the homogeneous Neumann condition.

Theorem 3.1. *Let ε be an arbitrary positive constant. There exists $C = C(\varepsilon, \Lambda, D)$ such that (1.5) has no non-constant positive solution when $d_1 > C(1 + d_2^2 d_3^2)/\mu_1$, $d_2 > \mu/\mu_1 + \varepsilon$ and $d_3 \le D$.*

Proof. Let (u, v) be a positive solution of (1.5) and $(\bar{u}, \bar{v})$ be the average of (u, v) over Ω. Multiplying the equations of (1.5) by $(u-\bar{u}), (v-\bar{v})$ respectively, and integrating over Ω, we have that

$$
\begin{aligned}
&\int_\Omega \{d_1 \mid \nabla u \mid^2 + d_2(1 + d_3 u) \mid \nabla v \mid^2 + d_2 d_3 v \nabla u \cdot \nabla v\} \mathrm{d}x \\
= &\int_\Omega (u - \bar{u})^2 \{\lambda - \alpha(u + \bar{u}) - \frac{\beta[ab\bar{v}(u + \bar{u}) + (a + b)u\bar{u}\bar{v}]}{(a + u)(b + u)(a + \bar{u})(b + \bar{u})}\} \mathrm{d}x \\
&+ \int_\Omega (u - \bar{u})(v - \bar{v})\{-\frac{\beta[abu^2 + (a + b)u^2\bar{u} + u^2\bar{u}^2]}{(a + u)(b + u)(a + \bar{u})(b + \bar{u})}\} \mathrm{d}x \\
&+ \int_\Omega \{(v - \bar{v})^2[\mu - \frac{\bar{u}(v + \bar{v})}{u\bar{u}}] + (v - \bar{v})(u - \bar{u})\frac{\bar{v}^2}{u\bar{u}}\} \mathrm{d}x.
\end{aligned}
$$

Then using Theorem 2.1 and the ε-Young Inequality yield

$$
\begin{aligned}
&\int_\Omega \{d_1 \mid \nabla u \mid^2 + d_2(1 + d_3 u) \mid \nabla v \mid^2\} \mathrm{d}x \\
\le &\int_\Omega \{(\lambda + C(\varepsilon))(u - \bar{u})^2 + (\mu + \varepsilon)(v - \bar{v})^2 + \varepsilon \mid \nabla v \mid^2 + \frac{d_2^2 d_3^2 v^2}{4\varepsilon} \mid \nabla u \mid^2\} \mathrm{d}x.
\end{aligned}
$$

Here, $C(\varepsilon)$ depends only on Λ, Ω, D, and ε. Hence, combing Theorem 2.1 and the Poincaré Inequality

$$
\mu_1 \int_\Omega (\varphi - \bar{\varphi})^2 \mathrm{d}x \le \int_\Omega \mid \nabla(\varphi - \bar{\varphi}) \mid^2
$$

give that

$$
\begin{aligned}
&\int_\Omega \{d_1 \mid \nabla u \mid^2 + d_2(1 + d_3 u) \mid \nabla v \mid^2\} \mathrm{d}x \\
\le &\int_\Omega \{C(1 + d_2^2 d_3^2)/\mu_1 \mid \nabla u \mid^2 + (\mu/\mu_1 + \varepsilon) \mid \nabla v \mid^2\} \mathrm{d}x.
\end{aligned}
$$

Letting ε be small enough such that $d_2 > \mu/\mu_1 + \varepsilon$, since $d_1 > C(1 + d_2^2 d_3^2)/\mu_1$, we have that $(u, v) \equiv (\bar{u}, \bar{v})$, which asserts our result. □

Similarly we can derive a priori estimate for positive solutions of (1.4) and use the energy method to obtain the following results:

Remark 3.1. If $d_3 = 0$,

(i) there exists $\bar{D}_2(\Lambda, d_1)$ such that (1.4) has no positive non-constant solution when $d_1 > \lambda/\mu_1$, $d_2 > \bar{D}_2$.

(ii) there exists $\bar{D}_1(\Lambda, d_2)$ such that (1.4) has no positive non-constant solution when $d_1 > \bar{D}_1$, $d_2 > \mu/\mu_1$.

4. Existence of positive non-constant solutions

In the following, we denote $\mathbf{u} = (u, v)^T$ and $\tilde{\mathbf{u}}$ is the unique positive constant solution, and define

$$\begin{aligned} X &= \{\mathbf{u} \in [C^1(\bar{\Omega})]^2 \mid \partial_\eta \mathbf{u} = 0, x \in \partial\Omega\}, \\ X^+ &= \{\mathbf{u} \in X \mid u, v > 0, x \in \bar{\Omega}\}, \\ B(C) &= \{\mathbf{u} \in X \mid C^{-1} < u, v < C, x \in \bar{\Omega}\}, C > 0, \\ \Phi(\mathbf{u}) &= (d_1 u, d_2(1 + d_3 u)v)^T, \\ g(u) &= \lambda - \alpha u, \quad p(u) = \frac{\beta u^2}{(a+u)(b+u)}, \quad h(u) = \frac{ug(u)}{p(u)}, \\ G(\mathbf{u}) &= \begin{pmatrix} G_1(\mathbf{u}) \\ G_2(\mathbf{u}) \end{pmatrix} = \begin{pmatrix} \lambda u - \alpha u^2 - \frac{\beta u^2 v}{(a+u)(b+u)} \\ \mu v(1 - \frac{v}{u}) \end{pmatrix}. \end{aligned}$$

Then (1.5) can be written as

$$\begin{cases} -\Delta \Phi(\mathbf{u}) = G(\mathbf{u}), & x \in \Omega, \\ \partial_\eta \mathbf{u} = 0, & x \in \partial\Omega. \end{cases} \tag{4.1}$$

Applying the fixed point index method, we see that finding positive solutions of (4.1) is equivalent to finding positive solutions of the equation

$$F(\mathbf{u}) \triangleq \mathbf{u} - (I - \Delta)^{-1}\{\Phi_{\mathbf{u}}^{-1}(\mathbf{u})[G(\mathbf{u}) + \bigtriangledown \mathbf{u} \Phi_{\mathbf{uu}}(\mathbf{u}) \bigtriangledown \mathbf{u}] + \mathbf{u}\} = 0, \mathbf{u} \in X^+,$$

where $(I - \Delta)^{-1}$ is the inverse of $I - \Delta$ in X with the homogeneous Neumann boundary condition. As $F(\cdot)$ is a compact perturbation of the identity operator for any $B = B(C)$, the Leray-Schauder degree $\deg(F(\cdot), 0, B)$ is well-defined if $F(\tilde{\mathbf{u}}) \neq 0$ on ∂B. Now we will compute the **index** of $F(\mathbf{u})$ at $\tilde{\mathbf{u}}$. A direct computation shows that

$$D_{\mathbf{u}} F(\tilde{\mathbf{u}}) = I - (I - \Delta)^{-1}\{\Phi_{\mathbf{u}}^{-1}(\tilde{\mathbf{u}}) G_{\mathbf{u}}(\tilde{\mathbf{u}}) + I\}.$$

If $D_{\mathbf{u}} F(\tilde{\mathbf{u}})$ is invertible, then

$$\mathrm{index}(F(\cdot), \tilde{\mathbf{u}}) = (-1)^\gamma,$$

where γ is the number of negative eigenvalues of $D_{\mathbf{u}} F(\tilde{\mathbf{u}})$, see [9] in details. Then we will consider the eigenvalues of $D_{\mathbf{u}} F(\tilde{\mathbf{u}})$. Denote $E(\mu_i)$ be the eigenspace corresponding to μ_i in $C^1(\bar{\Omega})$, $\{\phi_{ij} : j = 1, \ldots, \dim E(\mu_i)\}$ be an orthonormal basis of $E(\mu_i)$ and $X_{ij} = \{\mathbf{c}\phi_{ij} \mid \mathbf{c} \in \mathcal{R}^2\}$. Then, $X = \bigoplus_{i=0}^{\infty} X_i$ and $X_i = \bigoplus_{j=1}^{\dim E(\mu_i)} X_{ij}$. For each integer $i \geqslant 0$ and each integer $1 \leqslant j \leqslant$

$\dim E(\mu_i)$, X_{ij} is invariant under $D_{\mathbf{u}}F(\tilde{\mathbf{u}})$, and λ is an eigenvalue of $D_{\mathbf{u}}F(\tilde{\mathbf{u}})$ on X_{ij} if and only if it is an eigenvalue of the matrix

$$I - \frac{\Phi_{\mathbf{u}}^{-1}(\tilde{\mathbf{u}})G_{\mathbf{u}}(\tilde{\mathbf{u}}) + I}{1+\mu_i} = \frac{1}{1+\mu_i}[\mu_i I - \Phi_{\mathbf{u}}^{-1}(\tilde{\mathbf{u}})G_{\mathbf{u}}(\tilde{\mathbf{u}})].$$

Thus $D_{\mathbf{u}}F(\tilde{\mathbf{u}})$ is invertible if and only if the matrix $I - \frac{\Phi_{\mathbf{u}}^{-1}(\tilde{\mathbf{u}})G_{\mathbf{u}}(\tilde{\mathbf{u}})+I}{1+\mu_i}$ is non-singular for all $i \geq 0$. Writing

$$H(\mu') = H(\tilde{\mathbf{u}};\mu') \triangleq \det\{\mu' I - \Phi_{\mathbf{u}}^{-1}(\tilde{\mathbf{u}})G_{\mathbf{u}}(\tilde{\mathbf{u}})\}.$$

We note that if $H(\mu_i) \neq 0$, then for each $1 \leqslant j \leqslant \dim E(\mu_i)$, the number of negative eigenvalues of $D_{\mathbf{u}}F(\tilde{\mathbf{u}})$ on X_{ij} is odd if and only if $H(\mu_i) < 0$. In conclusion, we have the following:

$$\operatorname{index}(F(\cdot),\tilde{\mathbf{u}}) = (-1)^{\gamma}, \text{ where } \gamma = \sum_{i\geq 0, H(\mu_i)<0} \dim E(\mu_i).$$

Therefore, to compute $\operatorname{index}(F(\cdot),\tilde{\mathbf{u}})$, we only consider the sign of $H(\mu_i)$. We note that

$$H(\mu') = \det\{\Phi_{\mathbf{u}}^{-1}(\tilde{\mathbf{u}})\}\det\{\mu'\Phi_{\mathbf{u}}(\tilde{\mathbf{u}}) - G_{\mathbf{u}}(\tilde{\mathbf{u}})\}. \tag{4.2}$$

Since $\det\{\Phi_{\mathbf{u}}^{-1}(\tilde{\mathbf{u}})\}$ is positive, we will only need to consider $\det\{\mu'\Phi_{\mathbf{u}}(\tilde{\mathbf{u}}) - G_{\mathbf{u}}(\tilde{\mathbf{u}})\}$.

Direct calculations show that

$$\Phi_{\mathbf{u}}(\tilde{\mathbf{u}}) = \begin{pmatrix} d_1 & 0 \\ d_2 d_3 \tilde{v} & d_2(1+d_3\tilde{u}) \end{pmatrix},$$

$$G_{\mathbf{u}}(\tilde{\mathbf{u}}) = \begin{pmatrix} p(\tilde{u})h'(\tilde{u}) & -p(\tilde{u}) \\ \mu & -\mu \end{pmatrix},$$

then we have

$$\begin{aligned} & \det\{\mu'\Phi_{\mathbf{u}}(\tilde{\mathbf{u}}) - G_{\mathbf{u}}(\tilde{\mathbf{u}})\} \\ = \; & A_2(d_1,d_2,d_3)(\mu')^2 + A_1(d_1,d_2,d_3)\mu' + \det\{G_{\mathbf{u}}(\tilde{\mathbf{u}})\} \\ \triangleq \; & \Psi(d_1,d_2,d_3;\mu'), \end{aligned} \tag{4.3}$$

where

$$\begin{aligned} A_2(d_1,d_2,d_3) &= d_1 d_2(1+d_3\tilde{u}), \\ A_1(d_1,d_2,d_3) &= d_1\mu - d_2(1+d_3\tilde{u})p(\tilde{u})h'(\tilde{u}) - d_2 d_3 p(\tilde{u})\tilde{v}, \\ \det\{G_{\mathbf{u}}(\tilde{\mathbf{u}})\} &= \mu p(\tilde{u})(1-h'(\tilde{u})). \end{aligned}$$

We consider the dependence of Ψ on d_2.

Set $\tilde{\mu}_1(d_2)$, $\tilde{\mu}_2(d_2)$ be two roots of $\Psi(d_1,d_2,d_3;\mu') = 0$, which satisfy $\mathrm{Re}\{\tilde{\mu}_1(d_2)\} \leq \mathrm{Re}\{\tilde{\mu}_2(d_2)\}$. Then we can obtain $\tilde{\mu}_1\tilde{\mu}_2 = \frac{\det\{G_{\mathbf{u}}(\tilde{\mathbf{u}})\}}{A_2(d_1,d_2,d_3)}$. Let $h'(\tilde{u}) \leq 1$, we note that $\det\{G_{\mathbf{u}}(\tilde{\mathbf{u}})\} > 0$ and $A_2(d_1,d_2,d_3) > 0$. Thus, the product of two roots is positive.

Next consider the following limits:

$$\begin{aligned}
&\lim_{d_2\to\propto} \tfrac{A_2(d_1,d_2,d_3)}{d_2} = d_1(1+d_3\tilde{u}) \triangleq a_2(d_1,d_3),\\
&\lim_{d_2\to\propto} \tfrac{A_1(d_1,d_2,d_3)}{d_2} = -(1+d_3\tilde{u})p(\tilde{u})h'(\tilde{u}) - d_3\tilde{v}p(\tilde{u}) \triangleq a_1(d_1,d_3),\\
&\lim_{d_2\to\propto} \tfrac{\Psi(d_1,d_2,d_3;\mu')}{d_2} = \mu'(a_2(d_1,d_3)\mu' + a_1(d_1,d_3)).
\end{aligned}$$

Let $-1 < h'(\tilde{u}) \le 0$ and fix $d_3 > \frac{h'(\tilde{u})}{-\tilde{u}(h'(\tilde{u})+1)}$. Then $a_1 < 0$. So we have that

Proposition 4.1. *Let $-1 < h'(\tilde{u}) \le 0$ and fix $d_3 > \frac{h'(\tilde{u})}{-\tilde{u}(h'(\tilde{u})+1)}$. Then there exists a positive constant D_2 such that when $d_2 \ge D_2$, $\tilde{\mu}_1(d_2)$ and $\tilde{\mu}_2(d_2)$ satisfy*

$$\begin{aligned}
&\lim_{d_2\to\propto} \tilde{\mu}_1(d_2) = 0,\\
&\lim_{d_2\to\propto} \tilde{\mu}_2(d_2) = -\tfrac{a_1}{a_2} \triangleq \mu^* > 0.
\end{aligned} \tag{4.4}$$

Moreover, for all $d_2 \ge D_2$, we have that

$$\begin{cases}
0 < \tilde{\mu}_1(d_2) < \tilde{\mu}_2(d_2),\\
\Psi(d_1,d_2,d_3;\mu') < 0, \quad \mu' \in (\tilde{\mu}_1(d_2), \tilde{\mu}_2(d_2)),\\
\Psi(d_1,d_2,d_3;\mu') > 0, \quad \mu' \in (-\infty, \tilde{\mu}_1(d_2)) \cup (\tilde{\mu}_2(d_2), +\infty).
\end{cases} \tag{4.5}$$

From above arguments, we can obtain the results of the existence of positive non-constant solutions of (1.5) as follows:

Theorem 4.1. *Fix $d_1 > \lambda/\mu_1$, $d_3 > \frac{h'(\tilde{u})}{-\tilde{u}(h'(\tilde{u})+1)}$, let μ^* be given by the limit (4.4), we have that if $-1 < h'(\tilde{u}) \le 0$, $\sigma_n = \sum_{i=1}^{n} \dim E(\mu_i)(n \ge 1)$ is odd. Then there exists a positive constant D_2^1 such that (1.5) admits at least one non-constant positive solution provided that $d_2 \ge D_2^1$.*

Proof. By Proposition 4.1, there exists a positive constant D_2 such that if $d_2 \ge D_2$,

$$\mu^* \in (\mu_n, \mu_{n+1}). \tag{4.6}$$

It follows from simple computations that

$$\begin{cases}
A_2(d_1,d_2,0) = d_1 d_2 > 0,\\
A_1(d_1,d_2,0) = d_1\mu - d_2 p(\tilde{u})h'(\tilde{u}),
\end{cases} \tag{4.7}$$

since $-1 < h'(\tilde{u}) \le 0$, then there exist $\hat{d}_1$, $\hat{d}_2$ and $\hat{d}_3 = 0$ such that for any $i \ge 0$,

$$\Psi(\mu_i, \hat{d}_1, \hat{d}_2, 0) > 0. \tag{4.8}$$

Fix $\hat{d}_1 > \lambda/\mu_1$, it follows from Remark 3.1 that there exists $\bar{D}_2 > 0$ such that (1.5) has no positive non-constant solution if $\hat{d}_2 > \bar{D}_2$.

Next we will verify that (1.5) has at least one non-constant positive solution when $d_2 \ge D_2^1 = \max\{\bar{D}_2, D_2\}$ and $d_3 > \frac{h'(\tilde{u})}{-\tilde{u}(h'(\tilde{u})+1)}$. The proof which is by contradiction is based on the homotopy invariance of the topological degree. Assume that the result is not true for some $d_2 \ge D_2^1$, $d_3 > \frac{h'(\tilde{u})}{-\tilde{u}(h'(\tilde{u})+1)}$.

For $t \in [0,1]$, define

$$\begin{aligned}\alpha_i(t) &= \hat{d}_i + t(d_i - \hat{d}_i), \quad i = 1,2,3,\\ \Phi(t;\mathbf{u}) &= (\alpha_1(t)u,\ \alpha_2(t)v + \alpha_3(t)uv)^T,\end{aligned}$$

and consider the problem

$$\begin{cases} -\Delta\Phi(t;\mathbf{u}) = G(\mathbf{u}), & x \in \Omega, \quad 0 \le t \le 1, \\ \partial_\eta \mathbf{u} = 0, & x \in \partial\Omega. \end{cases} \tag{4.9}$$

Then $\mathbf{u}$ is a positive non-constant solution of (1.5) if and only if it is such a solution of (4.9) for $t = 1$. Clearly $\tilde{\mathbf{u}}$ is the unique constant positive solution of (4.9) for any $0 \le t \le 1$. Moreover $\mathbf{u}$ is a positive solution of (4.9) if and only if it is such a solution of the following operator equation

$$F(t;\mathbf{u}) \triangleq \mathbf{u} - (I-\Delta)^{-1}\{\Phi_{\mathbf{u}}^{-1}(t;\mathbf{u})[G(\mathbf{u}) + \bigtriangledown_{\mathbf{u}}\Phi_{\mathbf{uu}}(t;\mathbf{u})\bigtriangledown\mathbf{u}] + \mathbf{u}\} = 0, \quad \mathbf{u} \in Z^+.$$

It is obvious that $F(1;\mathbf{u}) = F(\mathbf{u})$. By a direct computation,

$$D_{\mathbf{u}}F(t;\tilde{\mathbf{u}}) = I - (I-\Delta)^{-1}\{\Phi_{\mathbf{u}}^{-1}(t;\tilde{\mathbf{u}})G_{\mathbf{u}}(\tilde{\mathbf{u}}) + I\}.$$

In particular,

$$\begin{aligned}D_{\mathbf{u}}F(0;\tilde{\mathbf{u}}) &= I - (I-\Delta)^{-1}\{D^{-1}G_{\mathbf{u}}(\tilde{\mathbf{u}}) + I\},\\ D_{\mathbf{u}}F(1;\tilde{\mathbf{u}}) &= I - (I-\Delta)^{-1}\{\Phi_{\mathbf{u}}^{-1}(1;\tilde{\mathbf{u}})G_{\mathbf{u}}(\tilde{\mathbf{u}}) + I\} = D_{\mathbf{u}}F(\tilde{\mathbf{u}}),\end{aligned}$$

where $D = \operatorname{diag}(\hat{d}_1, \hat{d}_2)$. From (4.2) and (4.3), we have that

$$H(\mu') = \det\{\Phi_{\mathbf{u}}^{-1}(\tilde{\mathbf{u}})\}\Psi(d_1,d_2,d_3;\mu'). \tag{4.10}$$

In view of (4.4) and (4.5), it follows from (4.10) that

$$\begin{cases} H(\mu_0) = H(0) > 0, & \\ H(\mu_i) < 0, & 1 \le i \le n, \\ H(\mu_i) > 0, & i \ge n+1. \end{cases}$$

Thus, zero is not an eigenvalue of the matrix $\mu_i I - \Phi_{\mathbf{u}}^{-1}G(\tilde{\mathbf{u}})$ for all $i \ge 0$, and

$$\sum_{i \ge 0, H(\mu_i) < 0} \dim E(\mu_i) = \sum_{i=1}^{n} \dim E(\mu_i) = \sigma_n$$

which is odd. Therefore

$$\operatorname{index}(F(1;\cdot),\tilde{\mathbf{u}}) = (-1)^\gamma = (-1)^{\sigma_n} = -1. \tag{4.11}$$

Similarly, using (4.8) yields that

$$\operatorname{index}(F(0;\cdot),\tilde{\mathbf{u}}) = (-1)^0 = 1. \tag{4.12}$$

It follows from Theorem 2.1 and the definitions of $\alpha_i(t)$ that there exists a positive constant C for all $0 \le t \le 1$, such that the positive solutions of (4.9) satisfy $\frac{1}{C} < u, v < C$, where the positive C is independent of t. Therefore $F(t;\mathbf{u}) \ne 0$ on $\partial B(C)$ for all $0 \le t \le 1$. By the homotopy invariance of the topological degree,

$$\deg(F(1;\cdot),0,B(C)) = \deg(F(0;\cdot),0,B(C)). \tag{4.13}$$

On the other hand, both equations $F(1;\mathbf{u}) = 0$ and $F(0;\mathbf{u}) = 0$ have only the positive solution $\tilde{\mathbf{u}}$ in $B(C)$. It follows from (4.11) and (4.12) that

$$\deg(F(1;\cdot), 0, B(C)) = \text{index}(F(1;\cdot), \tilde{\mathbf{u}}) = -1,$$
$$\deg(F(0;\cdot), 0, B(C)) = \text{index}(F(0;\cdot), \tilde{\mathbf{u}}) = 1.$$

This contradicts (4.13). The proof is completed. □

5. Concluding remarks

In this paper, we have considered a predator-prey model with Holling type III response function and cross-diffusion. The biological implication of cross-diffusion means that the prey species exercise as self-defense mechanism to protect themselves from the attack of the predator. From Theorems 3.1, 4.1 and Remark 3.1, we see that only if fix $d_3 > \frac{h'(\tilde{u})}{-\tilde{u}(h'(\tilde{u})+1)}$, the problem (1.5) has at least one non-constant positive solution for large d_2, otherwise if fix $0 \le d_3 < \frac{h'(\tilde{u})}{-\tilde{u}(h'(\tilde{u})+1)}$, the problem (1.5) has no non-constant positive solution for large d_1. Our results show that non-constant positive steady states can exist due to the emergence of cross-diffusion and cross-diffusion can induce the spatial pattern.

It is known that many chemical, physical or biological processes with space dispersion can be described by the reaction-diffusion systems. The diffusion in these systems usually represents the natural dispersive force of movement of an individual, recently cross-diffusion has been introduced to describe the mutual interferences between individuals and more work has been done to consider the corresponding strongly coupled systems and study the effect of cross-diffusion.

One of the main problems for the strongly coupled systems is the existence of non-constant positive solution for the strongly coupled systems. The fixed point index method gives an effective technique, which has been used extensively in literature. Compared to existing results such as the existence of non-constant positive solution for the strongly coupled elliptic systems, to the best of our knowledge, there have been very few results for the long time behaviors of the corresponding strongly coupled parabolic systems, and therefore there still remain many challenging tasks that deserve much more attention.

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