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SOME NEW ČEBYŠEV TYPE INEQUALITIES

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Abstract. Some new Čebyšev type inequalities have been developed by working on functions whose first derivatives are absolutely continuous and the second derivatives belong to the usual Lebesgue space $L_\infty [a,b]$. A unified treatment of the special cases is also given.

1. Introduction

For two measurable functions $f,\ g : [a,b] \to \mathbb{R}$, we define the functional,

$$T(f,g;a,b) := \frac{1}{b-a}\int_a^b f(x)\, g(x)\, dx - \left(\frac{1}{b-a}\int_a^b f(x)\, dx\right)\left(\frac{1}{b-a}\int_a^b g(x)\, dx\right), \tag{1.1}$$

which in literature is called the Čebyšev functional, provided the integrals in (1.1) exist.

Moreover, in 1882, P. L. Čebyšev (see [4], p. 297) proved that, if $f',\ g' \in L_\infty [a,b]$, then

$$|T(f,g;a,b)| \le \frac{1}{12}(b-a)^2 \|f'\|_\infty \|g'\|_\infty . \tag{1.2}$$

In the recent past, Čebyšev functional has remained an area of special interest for many researchers and has yielded many variants and generalizations in the field of inequalities. It has also played a key role in obtaining some new inequalities of Ostrowski type, for example, Ostrowski-Grüss type, Ostrowski-Čebyšev type, etc. The research papers [1, 2, 5] cover a comprehensive literature on the generalizations of Čebyšev functional and its associated bounds.

In [6], B. G. Pachpatte presented the following Čebyšev type inequality by using trapezoid like rules:

Theorem 1. *Let $f, g : [a,b] \to \mathbb{R}$ be differentiable functions so that $f',\ g'$ are absolutely continuous on $[a,b]$. Then*

$$\left|P\left(\bar{F},\bar{G},f,g\right)\right| \le \frac{(b-a)^4}{144}\|f'' - [f';a,b]\|_\infty \|g'' - [g';a,b]\|_\infty , \tag{1.3}$$

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where

$$\bar{F} = \frac{f(a)+f(b)}{2} - \frac{(b-a)^2}{12}[f';a,b],$$
$$\bar{G} = \frac{g(a)+g(b)}{2} - \frac{(b-a)^2}{12}[g';a,b]$$

and

$$P(\alpha,\beta,f,g) = \alpha\beta - \frac{1}{b-a}\left(\alpha\int_a^b g(t)\,dt + \beta\int_a^b f(t)\,dt\right) + \left(\frac{1}{b-a}\int_a^b f(t)\,dt\right)\left(\frac{1}{b-a}\int_a^b g(t)\,dt\right),$$
$$[f';a,b] = \frac{f'(b)-f'(a)}{b-a}.$$

Recently, in [3], Z. Liu presented the following generalization of (1.3):

Theorem 2. *Let the assumptions of Theorem 1 hold. Then for any* $\theta \in [0,1]$,

$$\left|P\left(\bar{\Gamma}_\theta, \bar{\Delta}_\theta, f, g\right)\right| \le (b-a)^4 I^2(\theta)\,\|f''-[f';a,b]\|_\infty \|g''-[g';a,b]\|_\infty, \tag{1.4}$$

where

$$I(\theta) = \begin{cases} \frac{\theta^3}{3} - \frac{\theta}{8} + \frac{1}{24}, & 0 \le \theta \le \frac{1}{2}, \\ \frac{1}{8}\left(\theta - \frac{1}{3}\right), & \frac{1}{2} < \theta \le 1, \end{cases}$$

and

$$\Gamma_\theta = \frac{\theta}{2}[f(a)+f(b)] + (1-\theta) f\left(\frac{a+b}{2}\right),$$
$$\Delta_\theta = \frac{\theta}{2}[g(a)+g(b)] + (1-\theta) g\left(\frac{a+b}{2}\right),$$
$$\bar{\Gamma}_\theta = \Gamma_\theta + \frac{1}{24}(1-3\theta)(b-a)^2[f';a,b],$$
$$\bar{\Delta}_\theta = \Delta_\theta + \frac{1}{24}(1-3\theta)(b-a)^2[g';a,b].$$

In this paper, we, by following an approach similar to that of [3] and [6], present some new Čebyšev type inequalities.

2. Main results

For suitable functions $f, g : [a,b] \to \mathbb{R}$ and $h \in [0,1]$, we use the following notations:

$$\bar{T}_{h,x} = \frac{1}{2}(2-h) f(x) - (1-h)\left(x - \frac{a+b}{2}\right) f'(x) + \frac{h}{2}\left(\frac{(x-a) f(a) + (b-x) f(b)}{b-a}\right),$$

$$\bar{S}_{h,x} = \frac{1}{2}(2-h)\,g(x) - (1-h)\left(x - \frac{a+b}{2}\right)g'(x) + \frac{h}{2}\left(\frac{(x-a)\,g(a) + (b-x)\,g(b)}{b-a}\right),$$

$$\begin{aligned}\bar{H}_{h,x} = {} & (1-h)\,f(x) + h\left(\frac{(x-a)\,f(a) + (b-x)\,f(b)}{b-a}\right) \\ & - (1-h)^2\left(x - \frac{a+b}{2}\right)f'(x) \\ & - \frac{h^2}{2}\left(\frac{(b-x)^2 f'(b) - (x-a)^2 f'(a)}{b-a}\right),\end{aligned}$$

$$\begin{aligned}\bar{L}_{h,x} = {} & (1-h)\,g(x) + h\left(\frac{(x-a)\,g(a) + (b-x)\,g(b)}{b-a}\right) \\ & - (1-h)^2\left(x - \frac{a+b}{2}\right)g'(x) \\ & - \frac{h^2}{2}\left(\frac{(b-x)^2 g'(b) - (x-a)^2 g'(a)}{b-a}\right),\end{aligned}$$

$$T_{h,x} = \bar{T}_{h,x} + \frac{1}{4}(2-3h)\,[f';a,b]\,(b-a)^2\,\Delta(x),$$

$$S_{h,x} = \bar{S}_{h,x} + \frac{1}{4}(2-3h)\,[g';a,b]\,(b-a)^2\,\Delta(x),$$

$$H_{h,x} = \bar{H}_{h,x} + \frac{1}{2}\left(3h^2 - 3h + 1\right)[f';a,b]\,(b-a)^2\,\Delta(x),$$

and

$$L_{h,x} = \bar{L}_{h,x} + \frac{1}{2}\left(3h^2 - 3h + 1\right)[g';a,b]\,(b-a)^2\,\Delta(x),$$

where

$$\Delta(x) = \frac{1}{12} + \left(\frac{x - \frac{a+b}{2}}{b-a}\right)^2, \tag{2.1}$$

and $[f';a,b]$ is defined as above.

Theorem 3. *Let the assumptions of Theorem 1 hold. Then for any* $h \in [0,1]$,

$$\begin{aligned}& |P(T_{h,x}, S_{h,x}, f, g)| \\ & \le \frac{1}{16}\omega^2(h)\,(b-a)^4\,\Delta^2(x)\,\|f'' - [f';a,b]\|_\infty \|g'' - [g';a,b]\|_\infty,\end{aligned} \tag{2.2}$$

where $\Delta(x)$, $T_{h,x}$ *and* $S_{h,x}$ *are defined as above and*

$$\omega(h) = 2h^3 - 3h + 2. \tag{2.3}$$

Proof. We define the following kernel:

$$K(x,t;h)=\begin{cases}\frac{1}{2}(t-a)(t-(1-h)a-hx), & t\in[a,x],\\ \frac{1}{2}(t-b)(t-hx-(1-h)b), & t\in(x,b].\end{cases}$$

Through simple calculations, it can be shown that

$$\frac{1}{b-a}\int_a^b f(t)\,dt-T_{h,x} = I(f',f'';a,b), \tag{2.4}$$

$$\frac{1}{b-a}\int_a^b g(t)\,dt-S_{h,x} = I(g',g'';a,b), \tag{2.5}$$

where

$$I(f',f'';a,b)=\frac{1}{b-a}\int_a^b K(x,t;h)\{f''(t)-[f';a,b]\}\,dt.$$

Multiplying the left and right hand side of (2.4) and (2.5), we get,

$$P(T_{h,x},S_{h,x},f,g)=I(f',f'';a,b)\,I(g',g'';a,b),$$

implies

$$|P(T_{h,x},S_{h,x},f,g)|=|I(f',f'';a,b)|\,|I(g',g'';a,b)|. \tag{2.6}$$

Following an approach similar to [6], we have

$$\begin{aligned}|I(f',f'';a,b)| &\le \frac{1}{b-a}\int_a^b |K(x,t;h)|\,|f''(t)-[f';a,b]|\,dt\\ &\le \frac{1}{b-a}\|f''(t)-[f';a,b]\|_\infty\int_a^b |K(x,t;h)|\,dt.\end{aligned} \tag{2.7}$$

Similarly, we have

$$|I(g',g'';a,b)|\le\frac{1}{b-a}\|g''(t)-[g';a,b]\|_\infty\int_a^b |K(x,t;h)|\,dt. \tag{2.8}$$

From the definition of $K(x,t;h)$, it follows that

$$\frac{1}{b-a}\int_a^b |K(x,t;h)|\,dt=\frac{1}{4}\omega(h)(b-a)^2\Delta(x), \tag{2.9}$$

where $\Delta(x)$ and $\omega(h)$ are defined by (2.1) and (2.3).

By using (2.6)-(2.9), (2.2) follows. □

The following corollary of Theorem 3 holds:

Corollary 4. *Let the assumptions of Theorem 1 hold. Then for any $h\in[0,1]$,*

$$\begin{aligned}&\left|P\left(T_{h,\frac{a+b}{2}},S_{h,\frac{a+b}{2}},f,g\right)\right|\\ &\le\frac{1}{2304}\omega^2(h)(b-a)^4\|f''-[f';a,b]\|_\infty\|g''-[g';a,b]\|_\infty,\end{aligned} \tag{2.10}$$

where

$$T_{h,\frac{a+b}{2}} = \frac{1}{2}(2-h)f\left(\frac{a+b}{2}\right)+\frac{h}{4}(f(a)+f(b)) +\frac{1}{48}(2-3h)[f';a,b](b-a)^2,$$

$$S_{h,\frac{a+b}{2}} = \frac{1}{2}(2-h)g\left(\frac{a+b}{2}\right)+\frac{h}{4}(g(a)+g(b)) +\frac{1}{48}(2-3h)[g';a,b](b-a)^2$$

and $\omega(h)$ *is defined by* (2.3).

Remark 1. It may observed that for $x=\frac{a+b}{2}$, the kernel defined in Theorem 3 takes the following form:

$$K\left(\frac{a+b}{2},t;h\right)=\begin{cases}\frac{1}{2}(t-a)\left(t-\left(a+h\frac{b-a}{2}\right)\right), & t\in\left[a,\frac{a+b}{2}\right],\\ \frac{1}{2}(t-b)\left(t-\left(b-h\frac{b-a}{2}\right)\right), & t\in\left(\frac{a+b}{2},b\right].\end{cases}$$

The following special cases of Corollary 4 hold:

Remark 2. 1. For $h=0$, (2.10) takes the form,

$$\left|P\left(T_{0,\frac{a+b}{2}},S_{0,\frac{a+b}{2}},f,g\right)\right| \le \frac{1}{576}(b-a)^4\left\|f''-[f';a,b]\right\|_\infty\left\|g''-[g';a,b]\right\|_\infty, \tag{2.11}$$

with

$$T_{0,\frac{a+b}{2}}=f\left(\frac{a+b}{2}\right)+\frac{1}{24}[f';a,b](b-a)^2$$

and

$$S_{0,\frac{a+b}{2}}=g\left(\frac{a+b}{2}\right)+\frac{1}{24}[g';a,b](b-a)^2.$$

2. For $h=1$, (2.10) takes the form,

$$\left|P\left(T_{1,\frac{a+b}{2}},S_{1,\frac{a+b}{2}},f,g\right)\right| \le \frac{1}{2304}(b-a)^4\left\|f''-[f';a,b]\right\|_\infty\left\|g''-[g';a,b]\right\|_\infty, \tag{2.12}$$

where

$$T_{1,\frac{a+b}{2}}=\frac{1}{4}\left(f(a)+2f\left(\frac{a+b}{2}\right)+f(b)\right)-\frac{1}{48}[f';a,b](b-a)^2$$

and

$$S_{1,\frac{a+b}{2}}=\frac{1}{4}\left(g(a)+2g\left(\frac{a+b}{2}\right)+g(b)\right)-\frac{1}{48}[g';a,b](b-a)^2.$$

3. For $h = \frac{2}{3}$, (2.10) takes the form,

$$\begin{aligned} & \left|P\left(T_{\frac{2}{3},\frac{a+b}{2}}, S_{\frac{2}{3},\frac{a+b}{2}}, f, g\right)\right| \\ & \leq \frac{1}{6561}(b-a)^4 \|f'' - [f';a,b]\|_\infty \|g'' - [g';a,b]\|_\infty , \end{aligned} \tag{2.13}$$

where

$$T_{\frac{2}{3},\frac{a+b}{2}} = \frac{1}{6}\left(f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right)$$

and

$$S_{\frac{2}{3},\frac{a+b}{2}} = \frac{1}{6}\left(g(a) + 4g\left(\frac{a+b}{2}\right) + g(b)\right).$$

It may also be noted that $\omega(h)$ is minimum for $h = \frac{1}{\sqrt{2}}$.

Theorem 5. *Let the assumptions of Theorem 1 hold. Then for any* $h \in [0,1]$,

$$\begin{aligned} & |P(H_{h,x}, L_{h,x}, f, g)| \\ & \leq \frac{1}{4}\eta^2(h)(b-a)^4 \Delta^2(x) \|f'' - [f';a,b]\|_\infty \|g'' - [g';a,b]\|_\infty , \end{aligned} \tag{2.14}$$

where $\Delta(x)$, $H_{h,x}$ *and* $L_{h,x}$ *are defined as above and*

$$\eta(h) = 3h^2 - 3h + 1. \tag{2.15}$$

Proof. We define the following kernel

$$K_1(x,t;h) = \begin{cases} \frac{1}{2}(t-(1-h)a-hx)^2, & t \in [a,x], \\ \frac{1}{2}(t-hx-(1-h)b)^2, & t \in (x,b]. \end{cases}$$

Through simple calculations, it can be shown that

$$\frac{1}{b-a}\int_a^b f(t)\,dt - H_{h,x} = J(f', f''; a,b), \tag{2.16}$$

$$\frac{1}{b-a}\int_a^b g(t)\,dt - L_{h,x} = J(g', g''; a,b), \tag{2.17}$$

where

$$J(f', f''; a,b) = \frac{1}{b-a}\int_a^b K_1(x,t;h)\{f''(t) - [f';a,b]\}\,dt.$$

Multiplying the left and right hand side of (2.16) and (2.17), we get,

$$P(H_{h,x}, L_{h,x}, f, g) = J(f', f''; a,b)\,J(g', g''; a,b),$$

implies

$$|P(H_{h,x}, L_{h,x}, f, g)| = |J(f', f''; a,b)|\,|J(g', g''; a,b)|. \tag{2.18}$$

Following an approach similar to [6], we calculate

$$|J(f', f''; a,b)| \leq \frac{1}{b-a}\int_a^b |K_1(x,t;h)|\,|f''(t) - [f';a,b]|\,dt$$

$$\leq \frac{1}{b-a}\left\|f''(t)-[f';a,b]\right\|_{\infty}\int_a^b |K_1(x,t;h)|\,dt. \tag{2.19}$$

Similarly, we have

$$|J(g',g'';a,b)| \leq \frac{1}{b-a}\left\|g''(t)-[g';a,b]\right\|_{\infty}\int_a^b |K_1(x,t;h)|\,dt. \tag{2.20}$$

From the definition of $K_1(x,t;h)$, it follows that

$$\frac{1}{b-a}\int_a^b |K_1(x,t;h)|\,dt = \frac{1}{2}\eta(h)(b-a)^2\Delta(x), \tag{2.21}$$

where $\Delta(x)$ and $\eta(h)$ are defined by (2.1) and (2.15).

Therefore (2.14) follows directly from (2.18)-(2.21). □

The following corollary of Theorem 5 holds:

Corollary 6. *Let the assumptions of Theorem 1 hold. Then for any* $h \in [0,1]$,

$$\begin{aligned} &\left|P\left(H_{h,\frac{a+b}{2}}, L_{h,\frac{a+b}{2}}, f, g\right)\right| \\ &\leq \frac{1}{576}\eta^2(h)(b-a)^4\left\|f''-[f';a,b]\right\|_{\infty}\left\|g''-[g';a,b]\right\|_{\infty}, \end{aligned} \tag{2.22}$$

where

$$H_{h,\frac{a+b}{2}} = (1-h)f\left(\frac{a+b}{2}\right)+\frac{h}{2}(f(a)+f(b))+\frac{1}{24}(1-3h)(b-a)^2[f';a,b]$$

and

$$L_{h,\frac{a+b}{2}} = (1-h)g\left(\frac{a+b}{2}\right)+\frac{h}{2}(g(a)+g(b))+\frac{1}{24}(1-3h)(b-a)^2[g';a,b],$$

$\eta(h)$ *is defined by* (2.15).

Remark 3. It may observed that for $x = \frac{a+b}{2}$, the kernel defined in Theorem 5 takes the following form:

$$K_1\left(\frac{a+b}{2},t;h\right) = \begin{cases} \frac{1}{2}\left(t-\left(a+h\frac{b-a}{2}\right)\right)^2, & t\in\left[a,\frac{a+b}{2}\right], \\ \frac{1}{2}\left(t-\left(b-h\frac{b-a}{2}\right)\right)^2, & t\in\left(\frac{a+b}{2},b\right]. \end{cases}$$

The following special cases of Corollary 6 hold:

Remark 4. 1. For $h=0$, (2.22) takes the form,

$$\begin{aligned} &\left|P\left(H_{0,\frac{a+b}{2}}, L_{0,\frac{a+b}{2}}, f, g\right)\right| \\ &\leq \frac{1}{576}(b-a)^4\left\|f''-[f';a,b]\right\|_{\infty}\left\|g''-[g';a,b]\right\|_{\infty}, \end{aligned} \tag{2.23}$$

with

$$H_{0,\frac{a+b}{2}} = f\left(\frac{a+b}{2}\right)+\frac{1}{24}[f';a,b](b-a)^2$$

and

$$L_{0,\frac{a+b}{2}}=g\left(\frac{a+b}{2}\right)+\frac{1}{24}\left[g';a,b\right](b-a)^2.$$

2. For $h=1$, (2.22) takes the form,

$$\begin{aligned}&\left|P\left(H_{1,\frac{a+b}{2}},L_{1,\frac{a+b}{2}},f,g\right)\right|\\ &\leq\frac{1}{576}(b-a)^4\left\|f''-[f';a,b]\right\|_\infty\left\|g''-[g';a,b]\right\|_\infty,\end{aligned}\tag{2.24}$$

where

$$H_{1,\frac{a+b}{2}}=\frac{1}{2}\left(f(a)+f(b)\right)-\frac{1}{12}\left[f';a,b\right](b-a)^2$$

and

$$L_{1,\frac{a+b}{2}}=\frac{1}{2}\left(g(a)+g(b)\right)-\frac{1}{12}\left[g';a,b\right](b-a)^2.$$

3. For $h=\frac{1}{2}$, (2.22) takes the form,

$$\begin{aligned}&\left|P\left(H_{\frac{1}{2},\frac{a+b}{2}},L_{\frac{1}{2},\frac{a+b}{2}},f,g\right)\right|\\ &\leq\frac{1}{9216}(b-a)^4\left\|f''-[f';a,b]\right\|_\infty\left\|g''-[g';a,b]\right\|_\infty,\end{aligned}\tag{2.25}$$

where

$$H_{\frac{1}{2},\frac{a+b}{2}}=\frac{1}{4}\left(f(a)+2f\left(\frac{a+b}{2}\right)+f(b)\right)-\frac{1}{48}\left[f';a,b\right](b-a)^2$$

and

$$L_{\frac{1}{2},\frac{a+b}{2}}=\frac{1}{4}\left(g(a)+2g\left(\frac{a+b}{2}\right)+g(b)\right)-\frac{1}{48}\left[g';a,b\right](b-a)^2.$$

It may also be noted that $\eta(h)$ is minimum for $h=\frac{1}{2}$.

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