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A Dynamic Lot-Sizing Problem with Backlogging for Minimum Replenishment Policy

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최소공급량 정책을 위한 추후조달 롯사이징 문제

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This paper considers a dynamic lot-sizing problem with backlogging under a minimum replenishment policy. For general concave production costs, we propose an $O(T^5)$ dynamic programming algorithm. If speculative motive is not allowed, in this case, a more efficient $O(T^4)$ algorithm is developed.

Keywords: Dynamic Lot-sizing, Inventory/Production, Minimum Replenishment Quantity

1. Introduction

A large-amount production in a single setup has not only the benefit of the economies of scale but also the effect of stabilizing the production system. As a result, each production system has a tendency to consolidate demands over multi-periods into a single period. The demand consolidation also has positive benefit to customers because a single order involves only a single transaction cost no matter how many item units they order. Recognizing these benefits, suppliers establish a minimum replenishment policy to prohibit production under a certain predefined quantity and thus encourage demand consolidation. For further managerial implications of minimum replenishment policy, we refer to Hwang (2009a).

In this paper we consider a dynamic lot-sizing problem with backlogging under a minimum replenishment policy. The dynamic lot-sizing model for minimum replenishment policy is first introduced by Lee (2004). His model deals with not only production cost but also transportation cost for shipping produced units via cargo. Hwang (2009a) provides polynomial time algorithms for all combinations of minimum replenishment quantity and cargo capacity. The problem in this paper is a generalization over that in Lee (2004) and Hwang (2009a) in the sense that it allows backlogging; however, it is a special case of them because it does not consider transportation costs.

Since the seminal research of Wagner and Whitin (1958), there have been significant efforts to explore various production systems, including the system with backlogging (Zangwill, 1966). One of the production systems as opposed to minimum replenishment policy is a facility with production capacities. In the capacitated lot-sizing problem, we cannot produce more units over the given production capacity in each period (see Florian and Klein, 1971; Chung and Lin 1988; Van Hoesel and Wagelmans 1996). That is, the capacitated problem has an upper bound to production. In this regard, our prob-

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lem can be thought of as having lower bound to production. However, we should be careful in stating the term of the lower bound, since a period can stay without production; only when we have production in a period, the minimum replenishment quantity has the role of lower bound.

For the dynamic lot-sizing problem with backlogging under a minimum replenishment policy, we propose an $O(T^5)$ algorithm under concave costs and an $O(T^4)$ algorithm under *nonspeculative* costs in which speculative motive does not occur to hold or backlog inventory. Section 2 formulates the problem and Section 3 presents solution algorithms. Section 4 concludes the paper

2. Problem Formulation and Optimality Properties

Let T denote the length of planning horizon. For each period $t \in \{1, 2, \cdots, T\}$ we define the following notation.

- d_t : demand in t.
- R: minimum replenishment quantity.
- x_t: replenishment level in t.
- I_t: inventory level in t.
- $p_t(x_t)$: replenishment cost in t for the amount x_t.
- $h_t(I_t)$: inventory holding and backlogging cost in t for the amount I_t. If $I_t > 0$, we let $h_t(I_t) = a_t(I_t)$ otherwise if $I_t < 0$, we let $h_t(I_t) = b_t(-I_t)$ where $a_t(\cdot)$ and $b_t(\cdot)$ denote the holding and backlogging cost in period t, respectively. We let $a_t(0) = b_t(0) = 0$ for convenience.

We assume all the cost functions $p_t(\cdot)$, $a_t(\cdot)$ and $b_t(\cdot)$ are concave. If production and inventory costs can be given in the form below, the model is said to have a *fixed-charge* cost structure:

$$p_t(x_t) = K_t + c_t \cdot x_t \text{ if } x_t > 0,$$
$$h_t(I_t) = a_t I_t \text{ for } I_t \geq 0 \text{ and } h_t(I_t) = -b_t I_t \text{ for } I_t < 0$$

where K_t is the setup cost, c_t, a_t, and b_t are the unit production, holding and backlogging costs in period t, respectively. Moreover, if it holds that $c_t - b_t \leq c_{t+1} \leq c_t + a_t$ for $t = 1, 2, \cdots, T\text{-}1$, the cost structure is then said to be nonspeculative (Chand and Morton, 1986, Federgruen and Tzur, 1991). We note that this nonspeculative cost structure is about per-unit costs of item units in which setup cost is not included. The lot-sizing model of minimizing the total replenishment and inventory cost is formulated as:

$$\text{Min} \sum_{t=1}^{T} (p_t(x_t) + h_t(I_t)) \quad (1)$$

Subject to

$$I_{t-1} + x_t = d_t + I_t,\ t = 1, \cdots, T \quad (2)$$
$$x_t = 0 \text{ or } x_t \geq R, \quad t = 1, \cdots, T \quad (3)$$
$$I_0 = I_T = 0 \quad (4)$$

In this problem (1)~(4), it should be noted that we have no nonnegativity constraint on the inventory level I_t since we allow for backlogging. Lee (2004) and Hwang (2009a) considers the non-backlogging case with each $I_t \geq 0$. If the minimum replenishment quantity is zero ($R = 0$) in the constraint (3), our problem reduces to the uncapacitated dynamic lot-sizing problems (Wagner and Whitin, 1958; Zangwill, 1966).

For notational convenience, we let $v_{s,t} = v_s + v_{s+1} + \cdots + v_t$ if $s \leq t$ and $v_{s,t} = 0$ if $s > t$, for any sequence of values $v_s, v_{s+1}, \cdots, v_t$. Then, $x_{s,t}$ and $d_{s,t}$ represent the cumulative sums of replenishments and demands from s through t, respectively. For a given solution $x = (x_1, x_2, \cdots, x_T)$ to the problem we classify periods according to production and inventory levels:

- A period t is called a *regeneration period* if $I_t = 0$;
- A period t is called a *replenishment* (*production*) period if $x_t > 0$;
- A replenishment period t is called *R-period* if $x_t = R$; We also say that period t has an R production. Otherwise if $x_t > R$, it is called *free* (replenishment) period

We present the most important property for designating replenishment quantities, which can be easily derived using the results in Hwang (2009a, 2009b). The proof of Property 1 is given in Appendix.

Property 1: There exists an optimal solution such that if $\lambda\text{-}1$ and γ are consecutive regeneration periods, i.e., $I_{\lambda-1} = I_\gamma = 0$ and $I_t \neq 0$, $\lambda \leq t < \gamma$, then there exists at most one free replenishment period during $\lambda, \lambda+1, \cdots, \gamma$.

We note that Property 1 is an extended version of that in Lee (2004) for non-backlogging case. If there is no backlogging, we can designate the free period explicitly as the last period γ. However, although Property 1 restricts the number of free periods to at most one, we have no idea about the position of the free period during $[\lambda, \gamma]$. This makes our problem more difficult than that in Lee (2004) and Hwang (2009a), which needs different solution approach.

We introduce a (λ, γ)*-problem* which finds a minimum schedule for demands $d_\lambda, d_{\lambda+1}, \cdots, d_\gamma$ under the condition that periods λ-1 and γ are regeneration periods and at most one free replenishment period occurs during $[\lambda, \gamma]$. In this definition of (λ, γ)-problem, we need to note that we do not impose the constraint of $I_t \neq 0$, $\lambda \leq t < \gamma$, for simplicity of computations. Let $G(\lambda, \gamma)$ be the minimum cost of a (λ, γ)-problem. We let $F(\gamma)$ be the minimum cost in satisfying demands $d_1, d_2, \cdots, d_\gamma$ for $1 \leq \gamma \leq T$. Then, the optimum cost is $F(T)$. Given each cost $G(\lambda, \gamma)$, an optimum solution can be obtained by the following recursion:

$$F(0) = 0 \qquad (5)$$
$$F(\gamma) = \min_{1 \leq \lambda \leq \gamma} \{F(\lambda - 1) + G(\lambda, \gamma)\}, \quad \gamma = 1, \cdots, T.$$

We note that all the values $F(\gamma)$ are obtained in $O(T^2)$ using the dynamic program (5) provided that $G(\lambda, \gamma)$ are preprocessed. From now on, we focus on solving the (λ, γ)-problems.

3. Solution Procedure for (λ, γ)-Problem

The computation of the cost $G(\lambda, \gamma)$ of the (λ, γ)-problem depends on the assumed cost structure. Section 3.1 deals with the concave cost structure and Section 3.2 the nonspeculative cost structure. As common procedures to be used for both cost structures, we introduce $u(n|\lambda, \gamma)$ and $v(n|\lambda, \gamma)$:

- $u(n|\lambda, \gamma)$: the minimum cost of serving demands $d_\lambda, d_{\lambda+1}, \cdots, d_\gamma$ using only R-productions during $[\lambda, \gamma]$ where λ-1 is a regeneration period and the number of R-productions is n. After satisfying demands $d_\lambda, d_{\lambda+1}, \cdots, d_\gamma$, we see, from the balance equation (2), that the inventory level at the end of period γ is $nR - d_{\lambda,\gamma}$.
- $v(n|\lambda, \gamma)$: the minimum cost of serving demands $d_\lambda, d_{\lambda+1}, \cdots, d_\gamma$ using only R-productions during $[\lambda, \gamma]$ where γ is a regeneration period and the number of R-productions is n. The balance equation (2) in this case suggests that the inventory level at the end of period λ-1 be $d_{\lambda,\gamma}$-nR.

3.1 Concave Costs

We let τ be the free replenishment period if it exists during $[\lambda, \gamma]$; otherwise, we let τ be any replenishment period. Suppose that the number of R-productions during $[\lambda, \tau\text{-}1]$ is n_1, $0 \leq n_1 \leq \tau\text{-}\lambda$, and the number of R-productions during $[\tau+1, \gamma]$ is n_2, $0 \leq n_2 \leq \gamma - \tau$. It is then clear that the costs during $[\lambda, \tau\text{-}1]$ and during $[\tau+1, \gamma]$ are given by $u(n_1|\lambda, \tau\text{-}1)$ and $v(n_2|\tau+1, \gamma)$, respectively. It remains to obtain the cost in period τ. Note that period τ produces $d_{\lambda,\gamma} - (n_1 + n_2)R$ units, which should be at least the minimum replenishment quantity R, and its final inventory level is $d_{\tau+1,\gamma} - n_2R$. Hence, the cost $G(\lambda, \gamma)$ is computed by the following:

$$\begin{aligned} G(\lambda, \gamma) = \min\{&u(n_1|\lambda, \tau\text{-}1) + p_\tau(d_{\lambda,\gamma} - (n_1 + n_2)R) \\ &+ h_\tau(d_{\tau+1,\gamma} - n_2R) + v(n_2|\tau+1, \gamma): \\ &d_{\lambda,\gamma} - (n_1 + n_2)R \geq R, 0 \leq n_1 \leq \tau - \lambda, 0 \leq n_2 \\ &\leq \gamma - \tau, \lambda \leq \tau \leq \gamma\}. \end{aligned} \qquad (6)$$

Computing $u(n|\lambda, \gamma)$. Note that the inventory level at the end of period γ is $I_\gamma = nR\text{-}d_{\lambda,\gamma}$, which incurs inventory cost of $h_\gamma(nR - d_{\lambda,\gamma})$. If period γ has no production, then we need to have all the R-productions during $[\lambda, \gamma\text{-}1]$. Note that the inventory level of period γ-1 is $nR - d_{\lambda,\gamma-1}$. Thus the cost during $[\lambda, \gamma\text{-}1]$ is $u(n|\lambda, \gamma\text{-}1)$. On the other hand, if period γ has a production which must be R-production, then the number of R-productions during $[\lambda, \gamma\text{-}1]$ decreases to n-1. Since $(n\text{-}1)R\text{-}d_{\lambda,\gamma-1}$ units are stored or backlogged at the end of period γ-1, it follows that the cost during $[\lambda, \gamma\text{-}1]$ is $u(n\text{-}1|\lambda, \gamma\text{-}1)$. Hence we have, for $0 \leq n \leq \gamma\text{-}\lambda+1$, $1 \leq \lambda \leq \gamma \leq T$,

$$u(n|\lambda,\gamma) = \min\begin{cases} u(n|\lambda,\gamma-1) + h_\gamma(nR - d_{\lambda,\gamma}), \\ u(n-1|\lambda,\gamma-1) + p_\gamma(R) + h_\gamma(nR - d_{\lambda,\gamma}). \end{cases}$$

Computing $v(n|\lambda, \gamma)$: The computation of the cost $v(n|\lambda, \gamma)$ can be explained using symmetric arguments for the cost $u(n|\lambda, \gamma)$. We can therefore see that the cost $v(n|\lambda, \gamma)$ is given as, for $0 \leq n \leq \gamma\text{-}\lambda+1$, $1 \leq \lambda \leq \gamma \leq T$,

$$v(n|\lambda,\gamma) = \min\begin{cases} v(n|\lambda+1, \gamma) + h_\lambda(d_{\lambda,\gamma} - nR), \\ v(n-1|\lambda+1, \gamma) + p_\lambda(R) + h_\lambda(d_{\lambda,\gamma} - nR). \end{cases}$$

From the dynamic procedures for $u(n|\lambda, \gamma)$ and $v(n|\lambda, \gamma)$ given above, we can obtain every value $u(n|\lambda, \gamma)$ and $v(n|\lambda, \gamma)$ in $O(T^3)$. Given these values, we see that the complexity of procedure (6) for $G(\lambda, \gamma)$ is $O(T^5)$. Since the main procedure (5) has complexity of $O(T^2)$, an optimal solution can be obtained in $O(T^5)$.

3.2 Nonspeculative Costs

In this subsection, we focus on the nonspeculative cost structure, a special case of the fixed-charge cost structure with no speculative motive in holding or backlogging inventory. Under a nonspeculative cost structure, it is more profitable for a production period t to serve the demand d_t by its own production rather than by carried over units from the previous period or backlogged units from the following period. This feature of the nonspeculative cost structure leads to the following useful result, which is proved in Appendix.

Property 2 : For a (λ, γ)-problem with free production at period τ, we have

(a) if $I_{\tau-1} > 0$, then $I_{\tau-1} = nR - d_{\lambda,\tau-1}$ where $n = \lceil d_{\lambda,\tau-1}/R \rceil$, and

(b) if $I_\tau < 0$, then $I_\tau = d_{\tau+1,\gamma} - nR$ where $n = \lceil d_{\tau+1,\gamma}/R \rceil$.

By the fact $I_{\tau-1} > 0$ ($I_\tau < 0$), we mean that period τ has *inflow* supply from earlier (later) periods than τ, respectively. Property 2 enables us to improve procedure (6) for $G(\lambda, \gamma)$. For the purpose of efficient computation of $G(\lambda, \gamma)$, we introduce additional notations, $u'(n|\lambda, \gamma)$ and $v'(n|\lambda, \gamma)$:

- $u'(n|\lambda, \gamma)$: the minimum cost of serving demands d_λ, $d_{\lambda+1}$, $\cdots$, d_γ using only R-productions during $[\lambda, \gamma]$ where n is the maximum number of R-productions and λ-1 is a regeneration period. We assume that a production has been setup in period γ+1 and product units are supplied from period γ+1 ($I_\gamma < 0$) so that $n \le \lfloor d_{\lambda,\gamma}/R \rfloor$. Note that any supply from period γ+1 (i.e., I_γ units) does not incur setup cost $K_{\gamma+1}$, implying that $u'(n|\lambda, \gamma)$ does not include $K_{\gamma+1}$.
- $v'(n|\lambda, \gamma)$: the minimum cost of serving demands d_λ, $d_{\lambda+1}$, $\cdots$, d_γ using only R-productions during $[\lambda, \gamma]$ where n is the maximum number of R-productions and γ is a regeneration period. We assume that a production has been setup in period λ-1 and product units are supplied from period λ-1 ($I_{\lambda-1} > 0$) so that $n \le \lfloor d_{\lambda,\gamma}/R \rfloor$. Note that any supply from period λ-1 (i.e., $I_{\lambda-1}$ units) does not incur setup cost $K_{\lambda-1}$, implying that $v'(n|\lambda, \gamma)$ does not include $K_{\lambda-1}$.

The constraint of the maximum number of R-productions in $u'(n|\lambda, \gamma)$ is imposed to ensure the minimum replenishment quantity in period γ+1. Similarly, the constraint of the maximum number of R-productions in $v'(n|\lambda, \gamma)$ is used for the minimum replenishment quantity in period λ-1. Notice that the number n in $u(n|\lambda, \gamma)$ and $v(n|\lambda, \gamma)$ of the previous subsection means the exact number of R-productions during $[\lambda, \gamma]$ whereas the number n in $u'(n|\lambda, \gamma)$ and $v'(n|\lambda, \gamma)$ is the maximum restriction of R-productions during $[\lambda, \gamma]$.

We first show how to compute $u'(n|\lambda, \gamma)$. Suppose that the total number of R-production during $[\lambda, \gamma]$ is n_1. We note that the quantity of backlogged units supplied from period γ is $d_{\lambda,\gamma} - n_1R$ or $I_\gamma = n_1R - d_{\lambda,\gamma} < 0$. This suggests that the cost during $[\lambda, \gamma]$ be $u(n_1|\lambda, \gamma)$, resulting in the following formula:

$$u'(n|\lambda, \gamma) = \min\{u(n_1|\lambda, \gamma) + c_{\gamma+1}(d_{\lambda,\gamma} - n_1R): 0 \le n_1 \le n\}.$$

It should be observed that the backlogging cost $h_\gamma(n_1R - d_{\lambda,\gamma})$ is not counted in this formula, since it will be considered in the cost $u(n_1|\lambda, \gamma)$. Using analogous arguments for $u'(n|\lambda, \gamma)$, we obtain the formula for $v'(n|\lambda, \gamma)$ as follows:

$$v'(n|\lambda, \gamma) = \min\{c_{\lambda-1}(d_{\lambda,\gamma} - n_2R) + h_{\lambda-1}(d_{\lambda,\gamma} - n_2R) + v(n_2|\lambda, \gamma) : 0 \le n_2 \le n\}.$$

We notice, in this formula of $v'(n|\lambda, \gamma)$, that the holding cost $h_{\lambda-1}(d_{\lambda,\gamma} - n_2R)$ is taken into account. We note that it takes $O(T)$ to compute $u'(n|\lambda, \gamma)$ and $v'(n|\lambda, \gamma)$ by the procedures above given values $u(n|\lambda, \gamma)$ and $v(n|\lambda, \gamma)$. Hence, we can obtain every $u'(n|\lambda, \gamma)$ and $v'(n|\lambda, \gamma)$ in $O(T^4)$.

Now, we consider a (λ, γ)-problem in which the free production occurs at period τ. If we have no free production, we let τ be any production period during $[\lambda, \gamma]$. We consider four cases in regard to the inventory levels $I_{\tau-1}$ and I_τ.

Case 1 : $I_{\tau-1} > 0$ and $I_\tau < 0$: In this case, the production period τ has inflows from its earlier and later periods. By Property 2 we can determine the number of R-productions, n_1 and n_2 during $[\lambda, \tau\text{-}1]$ and $[\tau+1, \gamma]$, respectively; that is, $n_1 = \lceil d_{\lambda,\tau-1}/R \rceil$ and $n_2 = \lceil d_{\tau+1,\gamma}/R \rceil$. Thus procedure (6) is simplified as follows:

$$\begin{aligned} G(\lambda, \gamma) = \min\{&u(n_1|\lambda, \tau\text{-}1) + K_\tau + c_\tau(d_{\lambda,\gamma} - (n_1 + n_2)R) \\ &+ h_\tau(d_{\tau+1,\gamma} - n_2R) + v(n_2|\tau+1, \gamma) : \\ &d_{\lambda,\gamma} - (n_1 + n_2)R \ge R,\ n_1 = \lceil d_{\lambda,\tau-1}/R \rceil, \\ &n_2 = \lceil d_{\tau+1,\gamma}/R \rceil,\ \lambda \le \tau \le \gamma\}. \end{aligned}$$

Case 2 : $I_{\tau-1} \le 0$ and $I_\tau \ge 0$: This is the opposite of Case 1, which has no inflows to the period τ, implying that the production in period τ fulfills its demand d_τ.

Suppose that n_1 and n_2 productions occur during $[\lambda, \tau-1]$ and $[\tau+1, \gamma]$, respectively. Because of the minimum replenishment restriction on period τ, we have $d_{\lambda,\gamma}-(n_1+n_2)R \geq R$, which implies that $n_1 \leq (d_{\lambda,\gamma}-(n_2+1)R)/R$. That is, if we are given the number n_2 of productions during $[\tau+1, \gamma]$, the maximum of number of R-productions during $[\lambda, \tau-1]$ is $\lfloor (d_{\lambda,\gamma}-(n_2+1)R)/R \rfloor$. From the definitions of $u'(n_1|\lambda, \tau-1)$ and $v(n_2|\tau+1, \gamma)$, we can see that

$$G(\lambda, \gamma) = \min\{u'(n_1|\lambda, \tau-1) + K_\tau + c_\tau(d_\tau) + h_\tau(d_{\tau+1,\gamma} - n_2R) + v(n_2|\tau+1, \gamma) : d_{\lambda,\gamma} - (n_1+n_2)R \geq R, n_1 = \lfloor (d_{\lambda,\gamma} - (n_2+1)R)/R \rfloor, 0 \leq n_2 \leq \gamma - \tau, \lambda \leq \tau \leq \gamma\}.$$

Note that it takes $O(T^2)$ to compute $G(\lambda, \gamma)$ in this case.

Case 3 : $I_{\tau-1} > 0$ and $I_\tau \geq 0$: This case can be treated by the combination of Cases 1 and 2. Therefore, we have

$$G(\lambda, \gamma) = \min\{u(n_1|\lambda, \tau-1) + K_\tau + c_\tau(d_\tau) + v'(n_2|\tau+1, \gamma): d_{\lambda,\gamma} - (n_1+n_2)R \geq R, n_1 = \lceil d_{\lambda,\tau-1}/R \rceil, n_2 = \lfloor (d_{\lambda,\gamma} - (n_1+1)R)/R \rfloor, \lambda \leq \tau \leq \gamma\}.$$

In this formula, we need to note that the inventory cost of period τ is counted in the cost $v'(n_2|\tau+1, \gamma)$.

Case 4 : $I_{\tau-1} \leq 0$ and $I_\tau < 0$: Similar to Case 3, we have

$$G(\lambda, \gamma) = \min\{u'(n_1|\lambda, \tau-1) + K_\tau + c_\tau(d_\tau) + h_\tau(d_{\tau+1,\gamma} - n_2R) + v(n_2|\tau+1, \gamma) : d_{\lambda,\gamma} - (n_1+n_2)R \geq R, n_1 = \lfloor (d_{\lambda,\gamma} - (n_2+1)R)/R \rfloor, n_2 = \lceil d_{\tau+1,\gamma}/R \rceil, \lambda \leq \tau \leq \gamma\}.$$

From this procedure, it takes $O(T^4)$ to compute every $G(\lambda, \gamma)$. Hence, an optimal solution can be found in $O(T^4)$.

4. Concluding Remark

In this paper, we present solution algorithms for a dynamic lot-sizing problem with backlogging under a minimum replenishment policy. An $O(T^5)$ algorithm is proposed for concave costs and an efficient $O(T^4)$ algorithm is proposed for nonspeculative costs. It is an open question whether or not more efficient algorithms are possible than the $O(T^5)$ and $O(T^4)$ algorithms. Moreover, it would be interesting to see if the approach of the $O(T^4)$ algorithm can be applied to fixed-charge cost structures. Finally, it will be fruitful to extend our problem to a more generalized problem with transportation costs.

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Incorporating the four cases, we have a complete formula:

$$G(\lambda, \gamma) = \min \begin{cases} u(n_1 \mid \lambda, \tau-1) + K_\tau + c_\tau(d_{\lambda,\gamma} - (n_1+n_2)R) + h_\tau(d_{\tau+1,\gamma} - n_2R) + v(n_2 \mid \tau+1, \gamma): \\ \quad d_{\lambda,\gamma} - (n_1+n_2)R \geq R,\ n_1 = \lceil d_{\lambda,\tau-1}/R \rceil,\ n_2 = \lceil d_{\tau+1,\gamma}/R \rceil,\ \lambda \leq \tau \leq \gamma, \\ u'(n_1 \mid \lambda, \tau-1) + K_\tau + c_\tau(d_\tau) + h_\tau(d_{\tau+1,\gamma} - n_2R) + v(n_2 \mid \tau+1, \gamma): \\ \quad d_{\lambda,\gamma} - (n_1+n_2)R \geq R,\ n_1 = \lfloor (d_{\lambda,\gamma} - (n_2+1)R/R \rfloor, 0 \leq n_2 \leq \gamma - \tau,\ \lambda \leq \tau \leq \gamma, \\ u(n_1 \mid \lambda, \tau-1) + K_\tau + c_\tau(d_\tau) + v'(n_2 \mid \tau+1, \gamma): \\ \quad d_{\lambda,\gamma} - (n_1+n_2)R \geq R,\ n_1 = \lceil d_{\lambda,\tau-1}/R \rceil,\ n_2 = \lfloor (d_{\lambda,\gamma} - (n_1+1)R/R \rfloor,\ \lambda \leq \tau \leq \gamma, \\ u'(n_1 \mid \lambda, \tau-1) + K_\tau + c_\tau(d_\tau) + h_\tau(d_{\tau+1,\gamma} - n_2R) + v(n_2 \mid \tau+1, \gamma): \\ \quad d_{\lambda,\gamma} - (n_1+n_2)R \geq R,\ n_1 = \lfloor (d_{\lambda,\gamma} - (n_2+1)R/R \rfloor,\ n_2 = \lceil d_{\tau+1,\gamma}/R \rceil,\ \lambda \leq \tau \leq \gamma. \end{cases}$$

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<Appendix> Proofs for Properties

Note that $I_t = x_{1,t}$-$d_{1,t}$. Given two feasible solutions x' and x'', we say that they share the same *inventory carrying and backlogging status* when $x'_{1,t}$ - $d_{1,i} \geq 0$ if and only if $x''_{1,t}$ - $d_{1,i} \geq 0$ for all $i = 1, 2, \cdots, T$. In other words, the solution x' (x'') is referred to as an *inventory-status conserving* solution of x'' (x'), respectively. Thus in each period t, solutions x' and x'' choose the same inventory function of either $a_t(\cdot)$ or $b_t(\cdot)$. Given a feasible solution x, we can represent its total production with inventory cost $Z(x)$ in terms of x_t and d_t such that $Z(x) = \sum_t p_t(x_t) + \sum_{x_{1,t} d_{1,t}} a_t(x_{1,t} - d_{1,t}) + + \sum_{x_{1,t} < d_{1,t}} b_t(d_{1,t} - x_{1,t})$. With respect to a feasible solution x, we let $x'(\delta \mid s, t)$ be a perturbed solution around two periods s and t constructed as follows:

$$x'_s = x_s - \delta, \;\; x'_t = x_t + \delta,$$
$$\text{and } \; x'_i = x_i, \text{ for all } i \neq s, t.$$

If there is no ambiguity, we will use x' instead of $x'(\delta|s, t)$. From Hwang (2009b), we then have the following result from the concavity of cost functions.

Lemma 1 : With respect to a feasible solution x, assume that the perturbed solutions $x'(\delta' \mid s, t)$ and $x''(-\delta'' \mid s, t)$ conserve the inventory status of x where $\delta', \delta'' > 0$. Then, either x' or x'' has total production and inventory holding costs at most that of x; that is, $Z(x') \leq Z(x)$ or $Z(x'') \leq Z(x)$.

Proof of Property 1 : Suppose that we have at least two (consecutive) free periods s and t, $\lambda \leq s < t \leq \gamma$ in an optimal solution x, where λ-1 and γ are consecutive regeneration periods. With respect to the solution x, consider perturbed solutions $x'(\delta' \mid s, t)$ and $x''(\delta'' \mid s, t)$ where

$\delta' = \min\{x_s - R, I_i: I_i > 0, i = s, s+1, \cdots, t\text{-}1\}$ and
$\delta'' = \min\{x_t - R, -I_i: I_i < 0, i = s, s+1, \cdots, t\text{-}1\}$.

Then it is clear that both x' and x'' are feasible, and both conserve the inventory-status of x. By Lemma 1, we see that either x' or x'' has total production and holding cost at most that of x. Consequently, either x' or x'' is also an optimum solution. Now, we would like to show that x' and x'' each have number of free periods decreased from that of x. Consider the case that x' is an optimum solution. If $\delta' = I_i > 0$ for some $i = s, s+1, \cdots, t$-1, then period i in the new solution x' is a regeneration period and the periods $\{\lambda, \lambda+1, \cdots, \gamma\}$ are divided into $\{\lambda, \cdots, i\}$ and $\{i+1, \cdots, \gamma\}$. Note that the numbers of free periods in $\{\lambda, \cdots, i\}$ and $\{i+1, \cdots, \gamma\}$ are smaller than that of $\{\lambda, \lambda+1, \cdots, \gamma\}$. In the case that $\delta' = x_s$-R, period s is R-period and x' has the number of free periods fewer than that of x. Furthermore, we can see that x'' also has free periods less than those of x. Continuing this process on all other free periods, we finally come to a solution with at most one free period between consecutive regeneration periods. □

Proof of Property 2 : We first prove part (a). Let n be the number of R-productions during $[\lambda, \tau\text{-}1]$, from which the inventory level at the end of period τ-1 is given as $I_{\tau\text{-}1} = nR - d_{\lambda,\tau\text{-}1}$. Since $I_{\tau\text{-}1} > 0$, we have $nR > d_{\lambda,\tau\text{-}1}$, or $n \geq \lceil d_{\lambda,\tau\text{-}1}/R \rceil$. If $n = \lceil d_{\lambda,\tau\text{-}1}/R \rceil$, the part (a) is proven. Otherwise, if $n > \lceil d_{\lambda,\tau\text{-}1}/R \rceil$, then we can move the $(n - \lceil d_{\lambda,\tau\text{-}1}/R \rceil)R$ units from the productions during $[\lambda, \tau\text{-}1]$ to the production in period τ while keeping the inventory level at the end of period τ-1 no less than zero. This new solution is no worse than the original solution if we have no speculative motive. That is, the per- unit production and inventory cost of each carried over unit to period τ is no smaller than the per-unit cost in period τ. The new solution, obtained after reassigning the $(n\text{-}\lceil d_{\lambda,\tau\text{-}1}/R \rceil)R$ units to period τ, satisfies part (a). We can prove part (b) in a similar way for part (a). □