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A Two-Plan Sampling System for Life Testing Under Weibull Distribution

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Abstract. A two-plan sampling system is proposed for a failure-censored life testing when the lifetime follows a Weibull distribution with known shape parameter. The proposed sampling system is based on a switching rule, for switching between the tightened and the normal inspection levels when lots are submitted for inspection in the order of production or in some other systematic way. The design parameters of the proposed sampling system are determined by the two-point approach considering the producer's risks and the consumer's at the specified acceptable reliability level and the lot tolerance reliability level, respectively. It has been observed that the proposed system requires only a single failure for the observation.

Keywords: Acceptable Reliability Level, Consumer's Risk, Lot Tolerance Reliability Level, OC Curve, Producer's Risk, Sampling by Variables

Glossary of Symbols

p_1 = Acceptable reliability level
p_2 = Lot tolerance reliability level
α = Producer's risk
β = Consumer's risk
L = Lower specification limit regarding the lifetime of a part
λ = Scale parameter of the Weibull distribution
m = Shape parameter of the Weibull distribution
n = Sample size
r = Failure number
k_T = Acceptance criterion under tightened inspection
k_N = Acceptance criterion under normal inspection
P_T = Proportion of lots expected to be accepted under tightened inspection
P_N = Proportion of lots expected to be accepted under normal inspection
$G_\varphi(.)$ = Cumulative distribution function of Chi-square distribution with degrees of freedom φ
s = Criterion for switching to tightened inspection
t = Criterion for switching to normal inspection
$P_a(p)$ = Lot acceptance probability when the lot quality is p

1. INTRODUCTION

A manufacturer of products performs a life testing in order to check whether the quality of their products meets the customer's requirements such as the minimum lifetime or reliability. In most life testing, a common restriction is the duration of the total time spent on test-

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ing. In order to reduce the testing time, many types of censoring schemes such as type-I (or time-censored), type-II (or failure-censored), mixed of type-I and type-II, and progressive censoring are usually adopted.

Although various sampling plans including single, double and sequential plans are available for normally distributed quality characteristics (see Schilling, 1982), most of the plans for a life testing are based on the single sampling plan. A single sampling plan under a failure censored (type-II) or a time censored (type-I) scheme is usually adopted for a life testing. Weibull distribution is popularly adopted as a lifetime distribution since the real failure data are known to fit this distribution quite well. Fertig and Mann (1980) and Schneider (1989) developed failure-censored sampling plans under a Weibull distribution having unknown shape parameter. Their approach is to utilize the extreme value distribution and the maximum likelihood estimation. As a result, designing a sampling plan is quite complicated for being used in practice. Particularly when the number of failure data is small, the estimators of unknown parameters may not be valid. Jun *et al.* (2006) proposed the single and double sampling plans for a Weibull distribution with known shape parameter under a sudden death testing. It is demonstrated that a more efficient sampling plan can be constructed if the shape parameter of a Weibull distribution is known.

In this study, we propose a two-plan sampling system for a failure censoring scheme under a Weibull distribution having known shape parameter. The assumption of known shape parameter in a Weibull distribution sometimes makes theoretical statisticians uncomfortable but it enables us to design various more efficient sampling plans. This assumption is not unrealistic because an estimate of the shape parameter can be readily available in practice from the past failure data and engineering experience. It is also assumed that lots are submitted for inspection serially in the order of production as in a continuous process.

A sampling system is comprised of two or more sampling plans, which has a rule for switching between the sampling plans to achieve advantageous features of each sampling plan. A sampling system involving the normal and the tightened inspection will be referred to as a tightened-normal-tightened (TNT) system or simply a two-plan system. The tightened inspection will be used when the quality deteriorates and the normal inspection will be used when the quality is found to be good. Dodge (1965), Hald and Thyregod (1965) and Stephens and Larson (1967) have studied the attributes two-plan systems using different switching rules to achieve the desired operating characteristics. Calvin (1977) considered the zero acceptance number TNT scheme for the application of attributes quality characteristics. Soundararajan and Vijayaraghavan (1990) and Vijayaraghavan and Soundararajan (1996) demonstrated that the TNT systems give the desired protection with smaller sample size. Muthuraj and Senthilkumar (2006) proposed the variables TNT scheme when the quality characteristic follows a normal distribution and more recently Balamurali and Jun (2009) proposed an optimization method for determining the design parameters.

In this paper, a new variables two-plan sampling system will be proposed for life testing and described in Section 2. The design parameters will be obtained and the related tables are given in Section 3.

2. A VARIABLES TWO-PLAN SYSTEM

Suppose that the quality characteristic of interest is the time to a failure and that it follows a Weibull distribution with known shape parameter m and unknown scale parameter λ such that the cumulative distribution function is given by

$$F(x) = 1 - \exp(-(\lambda x)^m), \quad x \geq 0 \tag{1}$$

Also, it is assumed that there is a lower specification L so that an item having the lifetime less than L is regarded as non-conforming. The fraction non-confor-ming (or unreliability at time L) is obtained by

$$p = 1 - \exp(-(\lambda L)^m) \tag{2}$$

The usual single sampling plan based on a failure censoring can be proceeded as follows:

1) Draw a random sample of size n from a lot and put them on test.
2) Perform testing until $r (\leq n)$ failures are observed and record $X_{(i)}$, the i-th failure time ($i = 1, \cdots, r$).
3) Calculate the quantity

$$v = \sum_{i=1}^{r} \left(X_{(i)}\right)^m + (n-r)\left(X_{(r)}\right)^m \tag{3}$$

4) Accept the lot if $v \geq kL^m$ and reject the lot, otherwise.

Therefore, the lot acceptance probability based on the variables single sampling plan, P_{single}, is given by

$$P_{\text{single}} = P\{v \geq kL^m\} \tag{4}$$

Note that v follows a Gamma distribution with parameters (r, λ^m) and that the quantity $2\lambda^m v$ follows a chi-square distribution with degree of freedom 2r (Jun *et al.*, 2006). So,

$$P_{\sin gle} = 1 - G_{2r}(2kw) \tag{5}$$

where G_φ is the distribution function of a Chi-square random variable with degree of freedom ϕ and w is given by

$$w = (\lambda L)^m = -\ln(1-p) \tag{6}$$

Now, we propose a new sampling system called a two-plan system, which has a switching rule for switching between the tightened inspection and the normal inspection levels. The proposed two-plan system has the following two-step procedure:

Step 1: Start with the tightened inspection level using the single sampling plan with a failure number r and the acceptance criterion k_T. Accept the lot if $\upsilon \geq k_T$ and reject the lot if $\upsilon < k_T$, where υ is calculated as in (3). If $t (\geq 2)$ consecutive lots are accepted under the tightened level, then switch to the normal inspection level as in Step 2.

Step 2: During the normal inspection level, inspect lots using the single sampling plan with the same failure number r and the acceptance criterion k_N (< k_T). Accept the lot if $\upsilon \geq k_N$ and reject the lot if $\upsilon < k_N$. Switch to the tightened level if an additional lot is rejected in the next s (≥ 1) lots after the rejection of a lot.

The above system involves three design parameters, namely r, k_T, k_N and the resulting system is designated as TNT-(r, k_T, k_N) when the switching criteria t and s are specified. Obviously, the two-plan system reduces to a single sampling plan as t goes to infinity. If $k_T = k_N$, then it reduces to a single sampling plan.Let P_T and P_N be the proportion of lots expected to be accepted under the tightened and the normal levels, respectively. Then, they will be obtained by

$$P_T = 1 - G_{2r}(2k_T w) \tag{7}$$

$$P_N = 1 - G_{2r}(2k_N w) \tag{8}$$

According to Balamurali and Jun (2009), the OC function of the proposed two-plan sampling system is given by

$$L(p) = \frac{\xi P_T + \delta P_N}{\xi + \delta} \tag{9}$$

where

$$\xi = (1 - P_T^t)(1 - P_N^s)(1 - P_N)$$

$$\delta = P_T^t(1 - P_T)(2 - P_N^s)$$

It should be noted that the proportions of lots to be inspected under the tightened level and the normal level will be

$$\frac{\xi}{\xi + \delta} \text{ and } \frac{\delta}{\xi + \delta}, \text{ respectively.}$$

3. DETERMINATION OF PARAMETERS

The design parameters (r, k_T, k_N) of the proposed sampling system can be determined by the two-point approach, in which its operating characteristics (OC) curve passes through two points $(p_1, 1-\alpha)$ and (p_2, β) where p_1 is the acceptable reliability level (ARL), p_2 is the lot tolerance reliability level (LTRL), α is the producer's risk and β is the consumer's risk (Fertig and Mann, 1980). A producer wants that the probability of acceptance of a lot should be at least (1-α) when the fraction non-conforming is at ARL and a consumer demands that this probability should not be greater than β when the fraction non-conforming is at LTRL. So, the following two inequalities should be considered:

$$L(p_1) \geq 1 - \alpha \tag{10a}$$

$$L(p_2) \leq \beta \tag{10b}$$

In general, s and t are selected in such a way that t > s. It is possible to construct tables for any number of combinations of s and t. For each set of combinations of s and t, the TNT-(r, k_T, k_N) scheme will result in a composite OC function. Hence, for several sets of combinations of s and t, a desirable OC curve (curve of desired discrimination) must be chosen. An ideal condition for selecting a desirable OC curve is that the OC curve should immediately discriminate against any lot submitted which contains a fraction nonconforming above the good quality level.

Figure 1 shows five composite OC curves for (s = 1, t = 1), (s = 1, t = 2), (s = 2, t = 3), (s = 3, t = 4) and (s = 4, t = 5) combinations, together with normal and tightened plan OC curves. It can be observed that the scheme OC curves at good quality, i.e. for lesser values of p, are very nearly the same as when operated under normal inspection, which is a desirable feature. It can also be seen that the scheme OC curve for the combination s = 4 and t = 5 is more discriminating than are the other sets of combinations. The scheme OC curve drops immediately towards the OC curve of tightened plan when quality deteriorates. Hence, from several sets of combinations of s and t, one set (namely s = 4, t = 5) can be singled out, in view of a better discriminating OC. Moreover, the scheme OC that results from the combination of s = 4 and t = 5 is the same as that of MIL-STD-105D system which involves only tightened and normal switching. However, we restrict our discussions to construct tables only for three combinations of s and t, namely (s = 1, t = 1), (s = 1, t = 2), (s = 3, t = 4). Table 1 shows k_T, the acceptance criterion in the tightened inspection and k_N, the acceptance criterion in the normal inspection for each of three arbitrarily selected combinations of (t, s), that is, (1, 1), (2, 1) and (4, 3). Various levels of ARL and LTRL were considered by assuming the producer's risk is 5 percent and the consumer's risk is 10 percent. It turned out for all cases considered here that the value of r was determined as 1, so it was omitted in this table.

It is quite surprising that the number of failures to be observed for the proposed sampling system is just 1. For the comparison purpose we prepared Table 2, which shows the number of failures required for the observation and acceptance number under the single sampling plan. When comparing with the single sampling plan, the proposed system reduces the number of failures to be observed significantly as the ratio of LTRL/ARL is relatively small.

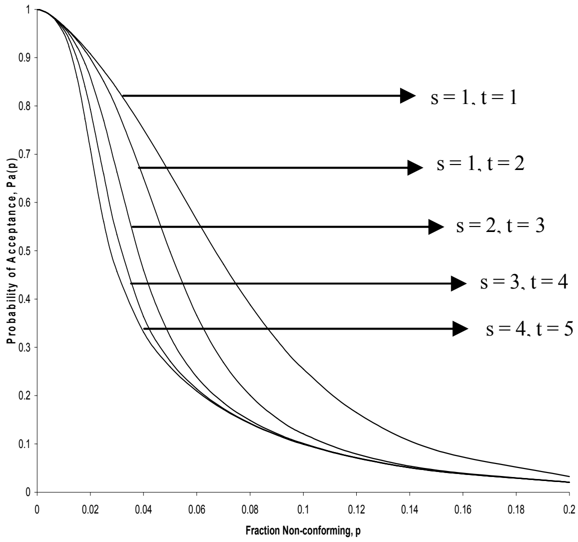

Figure 1. OC curves of TNT-(1; 37.33, 5.01) system for different combinations of s and t.

4. COMPARISON OF OC CURVES

Figure 2 shows the OC curves of tightened and normal inspection plans, and the composite OC curve of the TNT-(1, 21.88, 4.5) scheme with s = 3 and t = 4. From this figure, it can be easily observed that, for good quality, i.e. for smaller values of fraction nonconforming, the composite OC curve (OC curve of the Two plan system) coincides with the OC curve of the one stage plan or single sampling plan (1, 4.5). As quality deteriorates the OC curve of the composite OC curve moves toward that for the one stage sampling plan (1, 21.88) and comes close to it beyond the indifference quality level.

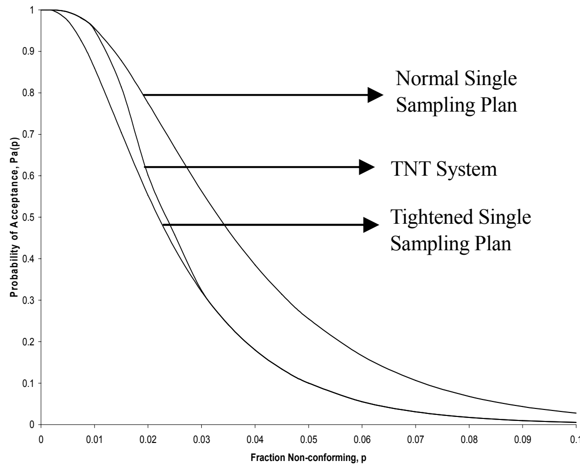

Figure 2. OC curves of two plan system and one stage sampling Plans.

In order to analyze the behavior of the proposed system further, we fixed the number of failures (r) to 2 and obtained the acceptance criteria at the tightened and the normal levels as in Table 3. It shows that the acceptance criteria at two levels become closer than the case of r = 1. As the ratio of LTRL/ARL increases the proposed system tends to behave like a single sampling plan.

Table 1. Variables two-plan systems (r = 1).

p_1	p_2	t = 1, s = 1		t = 2, s = 1		t = 4, s = 3	
		k_T	k_N	k_T	k_N	k_T	k_N
0.001	0.002	4274.90	21.15	2200.01	19.76	1302.70	9.11
	0.005	1052.70	47.50	591.01	46.90	462.01	35.63
	0.010	398.10	50.30	256.60	50.01	229.40	45.30
	0.015	223.83	50.80	162.01	50.60	152.50	48.01
	0.020	148.80	51.01	118.10	50.90	114.00	49.01
	0.030	84.60	51.20	76.50	51.10	75.60	50.60
0.005	0.010	850.30	4.24	435.2	4.07	259.60	1.81
	0.025	207.70	9.50	116.80	9.38	91.46	7.15
	0.050	77.40	10.05	50.14	9.99	44.94	9.07
	0.100	28.08	10.18	22.58	10.16	21.86	9.88
	0.150	15.54	10.21	14.31	10.21	14.17	9.90
0.010	0.020	423.90	2.09	216.80	2.01	128.90	0.92
	0.050	102.51	4.69	57.60	4.65	45.14	3.53
	0.100	37.33	5.01	24.33	4.98	21.88	4.50
	0.200	12.97	5.08	10.62	5.07	10.32	4.90
	0.300	6.88	5.10	6.50	5.10	6.46	5.06
0.050	0.100	79.55	0.44	40.83	0.42	24.51	0.19
	0.250	17.56	0.94	10.10	0.90	8.04	0.73
	0.500	5.13	0.98	3.58	0.98	3.33	0.92
0.100	0.200	36.67	0.23	18.85	0.22	11.44	0.10
	0.500	6.88	0.46	4.02	0.46	3.34	0.38

Example 1. Suppose that a certain type of bearing is regarded as conforming if its lifetime is greater than 10 (thousand cycles). For the decision of lot acceptance the manufacturer wants to use the two-plan sampling system with t = 4 and s = 3. The lifetime of a bearing is known to follow a Weibull distribution with m = 2. The ARL is selected as 0.01 at which the producer's risk is 5 percent and the LTRL is selected as 0.05 at which the consumer's risk is 10 percent. The design parameters for this example are obtained from Table 1 as r = 1, $k_T =$ 45.14 and $k_N = 3.53$. The sampling system operates as follows. Suppose now that the tightened inspection level is being used and that the first failure occurred at 12.7 when six bearings were put on test initially. Then, v is obtained by

$$v = (12.7)^2 + 5(12.7)^2 = 967.74 < k_T L^m = 45.14(10)^2 = 4514 .$$

So, the current lot will be rejected and the next lot should be inspected under the tightened level again.

5. CONCLUSION

A two-plan sampling system was proposed for a failure-censored life testing when the lifetime follows a Weibull distribution with known shape parameter, which

Table 2. Variables single sampling plans.

p_1	p_2	parameters	p_1	p_2	parameters
0.001	0.002	19 (12437)	0.010	0.020	19 (1226)
	0.005	4 (1333)		0.050	4 (131)
	0.010	3 (530)		0.100	2 (51)
	0.015	2 (258)		0.200	2 (18)
	0.020	2 (193)		0.300	2 (11)
	0.030	2 (128)	0.050	0.100	18 (225)
0.005	0.010	19 (2464)		0.250	4 (24)
	0.025	4 (264)		0.500	2 (6)
	0.050	3 (104)	0.100	0.200	17 (101)
	0.100	2 (37)		0.500	4 (10)
	0.150	2 (24)			

Table 3. Variables two-plan systems (r = 2).

p_1	p_2	t = 1, s = 1		t = 2, s = 1		t = 4, s = 3	
		k_T	k_N	k_T	k_N	k_T	k_N
0.001	0.002	4400.15	287.81	2623.51	291.02	1980.41	208.31
	0.005	1048.02	352.01	813.81	351.31	776.31	336.01
	0.010	395.71	354.91	387.84	355.02	386.97	354.17
	0.015	257.40	257.20	257.41	257.11	257.41	256.15
	0.020	192.63	192.50	192.61	190.03	192.61	192.15
	0.030	128.01	126.51	127.73	126.05	127.72	126.11
0.005	0.010	878.02	57.08	521.91	58.21	394.28	41.71
	0.025	206.34	70.05	161.01	70.11	153.73	67.11
	0.050	77.11	70.90	75.94	70.90	75.83	70.71
	0.100	37.01	36.60	37.01	33.61	36.92	36.19
	0.150	24.01	23.64	24.01	21.31	23.94	23.11
0.010	0.020	434.70	28.73	259.51	29.01	196.11	20.86
	0.050	101.57	35.01	79.35	35.01	75.88	33.53
	0.100	37.31	35.34	36.95	35.33	36.92	35.30
	0.200	17.54	17.00	17.44	17.11	17.43	17.15
	0.300	11.01	10.52	11.01	8.29	10.91	10.11
0.050	0.100	82.60	5.63	49.20	5.77	37.52	4.20
	0.250	17.31	6.88	14.01	6.86	13.53	6.65
	0.500	5.64	5.63	5.62	5.61	5.99	5.90
0.100	0.200	37.93	2.84	22.85	2.88	17.67	2.11
	0.500	6.69	3.36	5.74	3.35	5.62	3.28

is based on switching rules, switching between the tightened and the normal inspection levels. The design parameters were determined by the two-point approach considering the consumer's and producer's risk simultaneously, which are not dependent on the shape parameter as long as the ARL and the LTRL are specified. It was found that the number of failures required for the observation under the proposed sampling system is just one.

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REFERENCES

Balamurali, S. and Jun, C-H. (2009), Designing of a variables two plan system by minimizing the average sample number, *Journal of Applied Statistics*, **36**, 1159-1172.

Calvin, T. W. (1977), TNT zero acceptance number sampling, *American Society for Quality Control Annual Technical Conference Transactions*, Philadelphia, PA, 35-39.

Dodge, H. F. (1965), Evaluation of a sampling system having rules for switching between normal and tightened inspection, Technical Report No.14, Statistics Center, Rutgers University, Piscataway, NJ.

Fertig, K. W. and Mann, N. R. (1980), Life-test sampling plans for two-parameter Weibull populations, *Technometrics*, **22**, 165-177.

Hald, A. and Thyregod, P. (1965), The composite operating characteristic under normal and tightened sampling inspection by attributes, *Bulletin of the International Statistical Institute*, **41**, 517-529.

Jun, C-H., Balamurali, S., and Lee, S-H. (2006), Variables sampling plans for Weibull distributed lifetimes under sudden death testing, *IEEE Transactions on Reliability*, **55**(1), 53-58.

Muthuraj, D. and Senthilkumar, D. (2006), Designing and construction of tightened-normal-tightened variables sampling scheme, *Journal of Applied Statistics*, **33**, 101-111.

Schneider, H. (1989), Failure-censored variables-sampling plans for lognormal and Weibull distributions, *Technometrics*, **31**, 199-206.

Soundararajan, V. and Vijayaraghavan, R. (1990), Construction and Selection of Tightened-Normal-Tightened (TNT) Plans, *Journal of Quality Technology*, **22**, 146-153.

Stephens, K. S. and Larson, K. E. (1967), An evaluation of the MIL-STD 105D system of sampling plans, *Industrial Quality Control*, **23**, 310-319.

Vijayaraghavan, R. and Soundararajan, V. (1996), Procedures and tables for the selection of tightened-normal-tightened (TNT(n; c_1, c_2)) sampling schemes, *Journal of Applied Statistics*, **23**, 69-79.