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FUZZY ALMOST r-M CONTINUOUS FUNCTIONS ON FUZZY r-MINIMAL STRUCTURES

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ABSTRACT. We introduce the concept of fuzzy almost r-M continuous function on fuzzy r-minimal structures, and investigate characterizations and properties for such a function.

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1. Introduction

The concept of fuzzy set was introduced by Zadeh [11]. Chang [2] defined fuzzy topological spaces using fuzzy sets. In [3], Chattopadhyay, Hazra and Samanta introduced a smooth topological space which is a generalization of fuzzy topological space. In [10], Yoo et al. introduced the concept of fuzzy r-minimal space which is an extension of the smooth topological space. The concepts of fuzzy r-open sets, fuzzy r-semiopen sets, fuzzy r-preopen sets, r-fuzzy β-open sets and fuzzy r-regular open sets were introduced in [1, 4, 5, 6], which are kinds of fuzzy r-minimal structures. The concept of fuzzy r-M continuity was also introduced and investigated in [10]. The author [7] introduced and studied the concept of fuzzy weak r-M continuity which is a generalization of fuzzy r-M continuity. In this paper, we introduce and study the concept of fuzzy almost r-M continuity which is also a generalization of fuzzy r-M continuity. Finally, we investigate the relationships among fuzzy r-M continuity, fuzzy weak r-M continuity and fuzzy almost r-M continuity.

2. Preliminaries

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Let I be the unit interval $[0,1]$ of the real line. A member A of I^X is called a fuzzy set of X. By $\tilde{\mathbf{0}}$ and $\tilde{\mathbf{1}}$ we denote constant maps on X with value 0 and 1, respectively. For any $A \in I^X$, A^c denotes the complement $\tilde{\mathbf{1}} - A$. All other notations are standard notations of fuzzy set theory.

An *fuzzy point* x_α in X is a fuzzy set x_α defined as follows

$$x_\alpha(y) = \begin{cases} \alpha \text{ if } y = x \\ 0 \text{ if } y \neq x. \end{cases}$$

A fuzzy point x_α is said to belong to a fuzzy set A in X, denoted by $x_\alpha \in A$, if $\alpha \leq A(x)$ for $x \in X$.

A fuzzy set A in X is the union of all fuzzy points which belong to A.

Let $f : X \to Y$ be a function and $A \in I^X$ and $B \in I^Y$. Then $f(A)$ is a fuzzy set in Y, defined by

$$f(A)(y) = \begin{cases} \sup\limits_{z \in f^{-1}(y)} A(z), & \text{if } f^{-1}(y) \neq \emptyset, \\ 0, & \text{otherwise,} \end{cases}$$

for $y \in Y$ and $f^{-1}(B)$ is a fuzzy set in X, defined by $f^{-1}(B)(x) = B(f(x)), x \in X$.

A *smooth topology* [3,9] on X is a map $\mathcal{T} : \mathcal{I}^{\mathcal{X}} \to \mathcal{I}$ which satisfies the following properties:

(1) $\mathcal{T}(\tilde{\mathbf{0}}) = \mathcal{T}(\tilde{\mathbf{1}}) = 1$.
(2) $\mathcal{T}(A_1 \wedge A_2) \geq \mathcal{T}(A_1) \wedge \mathcal{T}(A_2)$.
(3) $\mathcal{T}(\vee A_i) \geq \wedge \mathcal{T}(A_i)$.

The pair $(X, \mathcal{T})$ is called a *smooth topological space.*

Definition 1 ([10]). *Let X be a nonempty set and $r \in (0,1] = I_0$. A fuzzy family $\mathcal{M} : I^X \to I$ on X is said to have a fuzzy r-minimal structure if the family*

$$\mathcal{M}_r = \{A \in I^X \mid \mathcal{M}(A) \geq r\}$$

contains $\tilde{\mathbf{0}}$ and $\tilde{\mathbf{1}}$.

Then the $(X, \mathcal{M})$ is called a fuzzy r-minimal space (simply r-FMS) if $\mathcal{M}$ has a fuzzy r-minimal structure. Every member of $\mathcal{M}_\nabla$ is called a fuzzy r-minimal open set. A fuzzy set A is called a fuzzy r-minimal closed set if the complement of A (simply, A^c) is a fuzzy r-minimal open set.

Let $(X, \mathcal{M})$ be an r-FMS and $r \in I_0$. The fuzzy r-minimal closure and the fuzzy r-minimal interior of A [10], denoted by $mC(A, r)$ and $mI(A, r)$, respectively, are defined as

$$mC(A, r) = \cap\{B \in I^X : B^c \in \mathcal{M}_r \text{ and } A \subseteq B\},$$

$$mI(A, r) = \cup\{B \in I^X : B \in \mathcal{M}_r \text{ and } B \subseteq A\}.$$

Theorem 1 ([10]). *Let $(X, \mathcal{M})$ be an r-FMS and A, B in I^X.*

(1) $mI(A, r) \subseteq A$ and if A is a fuzzy r-minimal open set, then $mI(A, r) = A$.

(2) $A \subseteq mC(A, r)$ and if A is a fuzzy r-minimal closed set, then $mC(A, r) = A$.

(3) If $A \subseteq B$, then $mI(A, r) \subseteq mI(B, r)$ and $mC(A, r) \subseteq mC(B, r)$.

(4) $mI(A, r) \cap mI(B, r) \supseteq mI(A \cap B, r)$ and $mC(A, r) \cup mC(B, r) \subseteq mC(A \cup B, r)$.

(5) $mI(mI(A, r), r) = mI(A, r)$ and $mC(mC(A, r), r) = mC(A, r)$.

(6) $\tilde{\mathbf{1}} - mC(A, r) = mI(\tilde{\mathbf{1}} - A, r)$ and $\tilde{\mathbf{1}} - mI(A, r) = mC(\tilde{\mathbf{1}} - A, r)$.

Let $(X, \mathcal{M}_X)$ and $(Y, \mathcal{M}_Y)$ be two r-FMS's. Then a function $f : X \to Y$ is said to be

(1) *fuzzy r-M continuous* [10] if for every fuzzy r-minimal open set A in Y, $f^{-1}(A)$ is fuzzy r-minimal open in X,

(2) *fuzzy weakly r-M continuous* [7] if for fuzzy point x_α in X and each fuzzy r-minimal open set V containing $f(x_\alpha)$, there is a fuzzy r-minimal open set U containing x_α such that $f(U) \subseteq mC(V, r)$.

3. Fuzzy Almost r-M Continuous Functions

Definition 2. *Let $f : X \to Y$ be a function between r-FMS's $(X, \mathcal{M}_X)$ and $(Y, \mathcal{M}_Y)$. Then f is said to be fuzzy almost r-M continuous if for fuzzy point x_α in X and each fuzzy r-minimal open set V containing $f(x_\alpha)$, there is a fuzzy r-minimal open set U containing x_α such that $f(U) \subseteq mI(mC(V, r), r)$.*

fuzzy r-M continuity $\Rightarrow$ fuzzy almost r-M continuity $\Rightarrow$ fuzzy weakly r-M continuity

Example 1. *Let $X = I$, let A, B and C be fuzzy sets defined as follows*

$$A(x) = \frac{1}{2}x, \quad x \in I;$$

$$B(x) = -\frac{1}{2}(x - 1), \quad x \in I;$$

$$C(x) = \begin{cases} \frac{1}{2}(x + 1), & \text{if } 0 \le x \le \frac{1}{2}, \\ -\frac{1}{2}(x - 2), & \text{if } \frac{1}{2} < x \le 1; \end{cases}$$

and

$$D(x) = \begin{cases} -\frac{1}{2}(2x - 1), & \text{if } 0 \le x \le \frac{1}{2}, \\ \frac{1}{2}(2x - 1), & \text{if } \frac{1}{2} < x \le 1. \end{cases}$$

Consider two fuzzy families $\mathcal{M}_1$, $\mathcal{M}_2$, $\mathcal{N}$ defined as the following:

$$\mathcal{M}_1(\mu) = \begin{cases} \frac{2}{3}, & \text{if } \mu = \tilde{\mathbf{1}}, \\ \frac{1}{2}, & \text{if } \mu = \tilde{\mathbf{0}}, C, \\ 0, & \text{otherwise}; \end{cases}$$

$$\mathcal{M}_2(\mu) = \begin{cases} \frac{2}{3}, & \text{if } \mu = \tilde{\mathbf{1}}, \\ \frac{1}{2}, & \text{if } \mu = \tilde{\mathbf{0}}, D, \\ 0, & \text{otherwise}; \end{cases}$$

$$\mathcal{N}(\mu) = \begin{cases} \frac{1}{2}, & \text{if } \mu = \tilde{\mathbf{0}}, \tilde{\mathbf{1}}, \\ \frac{2}{3}, & \text{if } \mu = A, B, \\ 0, & \text{otherwise.} \end{cases}$$

(1) The identity function $f : (X, \mathcal{N}) \to (X, \mathcal{M}_1)$ is fuzzy almost $\frac{1}{2}$-M continuous but not fuzzy $\frac{1}{2}$-M continuous.

(2) The identity function $g : (X, \mathcal{N}) \to (X, \mathcal{M}_2)$ is fuzzy weakly $\frac{1}{2}$-M continuous but not fuzzy almost $\frac{1}{2}$-M continuous.

Theorem 2. *Let $f : X \to Y$ be a function between r-FMS's $(X, \mathcal{M}_X)$ and $(Y, \mathcal{M}_Y)$. Then the following statements are equivalent:*

(1) f is fuzzy almost r-M continuous.

(2) $f^{-1}(B) \subseteq mI(f^{-1}(mI(mC(B, r), r)), r)$ for each fuzzy r-minimal open set B of Y.

(3) $mC(f^{-1}(mC(mI(F, r), r)), r) \subseteq f^{-1}(F)$ for each fuzzy r-minimal closed set F in Y.

Proof. (1) $\Rightarrow$ (2) Let B be a fuzzy r-minimal open set in Y. Then for each $x_\alpha \in f^{-1}(V)$, there exists a fuzzy r-minimal open set U of x_α such that $f(U) \subseteq mI(mC(B, r), r)$. Since $x_\alpha \in U \subseteq f^{-1}(mI(mC(B, r), r))$,
$x_\alpha \in mI(f^{-1}(mI(mC(B, r), r)), r)$. Thus the statement (2) is obtained.

(2) $\Rightarrow$ (1) Let x_α be a fuzzy point in X and V a fuzzy r-minimal open set containing $f(x_\alpha)$. Then by (2), $x_\alpha \in mI(f^{-1}(mI(mC(V, r), r)), r)$,
and so there exists a fuzzy r-minimal open set U containing x_α such that $U \subseteq f^{-1}(mI(mC(V, r), r))$. From the fact, we have the following:

$$f(U) \subseteq f(f^{-1}(mI(mC(V, r), r))) \subseteq mI(mC(V, r), r).$$

Hence f is fuzzy almost r-M-continuous.

(2) $\Rightarrow$ (3) Let F be any fuzzy r-minimal closed set of Y. Then from (2), it follows

$$\begin{aligned} f^{-1}(\tilde{\mathbf{1}} - F) &\subseteq mI(f^{-1}(mI(mC(\tilde{\mathbf{1}} - F, r), r)), r) \\ &= mI(f^{-1}(\tilde{\mathbf{1}} - mC(mI(F, r), r)), r) \\ &= mI(\tilde{\mathbf{1}} - f^{-1}(mC(mI(F, r), r)), r) \\ &= \tilde{\mathbf{1}} - mC(f^{-1}(mC(mI(F, r), r)), r). \end{aligned}$$

Hence $mC(f^{-1}(mC(mI(F, r), r)), r) \subseteq f^{-1}(F)$.

(3) $\Rightarrow$ (2) Obvious.

□

Let X be a nonempty set and $\mathcal{M} : I^X \to I$ a fuzzy family on X. The fuzzy family $\mathcal{M}$ is said to have the property $(\mathcal{U})$ [10] if for $A_i \in \mathcal{M}$ $(i \in J)$,

$$\mathcal{M}(\cup A_i) \geq \wedge \mathcal{M}(A_i).$$

Theorem 3 ([10])**.** *Let $(X, \mathcal{M})$ be an r-FMS with the property $(\mathcal{U})$. Then*
(1) For $A \in I^X$, $mI(A, r) = A$ if and only if A is fuzzy r-minimal open.
(2) For $F \in I^X$, $mC(F, r) = F$ if and only if F is fuzzy r-minimal closed.

Corollary 1. *Let $f : X \to Y$ be a function between r-FMS's $(X, \mathcal{M}_X)$ and $(Y, \mathcal{M}_Y)$. If $\mathcal{M}_Y$ have property $(\mathcal{U})$, then the following statements are equivalent:*
(1) f is fuzzy almost r-M continuous.
(2) $f^{-1}(B) \subseteq mI(f^{-1}(mI(mC(B, r), r)), r)$ for each fuzzy r-minimal open set B of Y.
(3) $f^{-1}(mI(B, r)) \subseteq mI(f^{-1}(mI(mC(mI(B, r), r), r)), r)$ for each $B \subseteq Y$.
(4) $mCl(f^{-1}(mC(mI(mC(B, r), r), r)), r) \subseteq f^{-1}(mC(B, r))$ for each $B \subseteq Y$.

Definition 3. *Let $(X, \mathcal{M})$ be an r-FMS and $A \in I^X$. Then a fuzzy set A is said to be fuzzy r-minimal regular open (resp., fuzzy r-minimal regular closed if $A = mI(mC(A, r), r)$ (resp., $A = mC(mI(A, r), r)$).*

Theorem 4. *Let $f : X \to Y$ be a function between r-FMS's $(X, \mathcal{M}_X)$ and $(Y, \mathcal{M}_Y)$. If $\mathcal{M}_Y$ has property $(\mathcal{U})$, then the following statements are equivalent:*
(1) f is fuzzy almost r-M continuous.
(2) $f^{-1}(F) = mC(f^{-1}(F), r)$ for an fuzzy r-minimal regular closed set F in Y.
(3) $f^{-1}(V) = mI(f^{-1}(V), r)$ for an fuzzy r-minimal regular open set V in Y.

Proof. (1) $\Rightarrow$ (2) Let F be any fuzzy r-minimal regular closed set of Y. Then since Y has the property $(\mathcal{U})$, F is $F = mC(mI(F, r), r)$ and fuzzy r-minimal closed, so by Theorem 2(3), $mC(f^{-1}(F), r) = mC(f^{-1}(mC(mI(F, r), r)), r) \subseteq f^{-1}(F)$. Hence $f^{-1}(F) = mC(f^{-1}(F), r)$.

(2) $\Rightarrow$ (3) Obvious.

(3) $\Rightarrow$ (1) Let V be a fuzzy r-minimal open set containing $f(x_\alpha)$. Since $mI(mC(V, r), r)$ is fuzzy r-minimal regular open, by (3),

$f^{-1}(mI(mC(V, r), r)) = mI(f^{-1}(mI(mC(V, r), r)), r)$

and so there is a fuzzy r-minimal open set U containing x_α such that $U \subseteq f^{-1}(mI(mC(V, r), r))$. Then this implies $f(U) \subseteq mI(mC(V, r), r)$ so that f is fuzzy almost r-M-continuous. □

Corollary 2. *Let $f : X \to Y$ be a function between r-FMS's $(X, \mathcal{M}_X)$ and $(Y, \mathcal{M}_Y)$. If $\mathcal{M}_X$ and $\mathcal{M}_Y$ have property $(\mathcal{U})$, then the following statements are equivalent:*
(1) f is fuzzy almost r-M continuous.
(2) $f^{-1}(B)$ is fuzzy r-minimal open for each fuzzy r-minimal regular open set B of Y.

(3) $f^{-1}(B)$ is fuzzy r-minimal closed for each fuzzy r-minimal regular closed set B of Y.

Definition 4. *Let $(X, \mathcal{M})$ be an r-FMS and $A \in I^X$. Then a fuzzy set A is said to be*

(1) fuzzy r-minimal semiopen [8] if $A \subseteq mC(mI(A, r), r)$;
(2) fuzzy r-minimal preopen if $A \subseteq mI(mC(A, r), r)$;
(3) fuzzy r-minimal β-open if $A \subseteq mC(mI(mC(A, r), r), r)$.

A fuzzy set A is called a *fuzzy r-minimal semiclosed* (resp., *fuzzy r-minimal preclosed* , *fuzzy r-minimal β-closed*) set if the complement of A is a fuzzy r-minimal semiopen (resp., fuzzy r-minimal preopen , fuzzy r-minimal β-open) set.

Theorem 5. *Let $f : X \to Y$ be a function on r-FMS's $(X, \mathcal{M}_X)$ and $(Y, \mathcal{M}_Y)$. If $\mathcal{M}_Y$ has the property $(\mathcal{U})$, then the following statements are equivalent:*

(1) *f is fuzzy almost r-M continuous.*
(2) *$mC(f^{-1}(G), r) \subseteq f^{-1}(mC(G, r))$ for each fuzzy r-minimal β-open set G in Y.*
(3) *$mC(f^{-1}(G), r) \subseteq f^{-1}(mC(G, r))$ for each fuzzy r-minimal semiopen set G in Y.*

Proof. (1) $\Rightarrow$ (2) For a fuzzy r-minimal β-open set G, $G \subseteq mC(mI(mC(G, r), r), r)$ and $mC(G, r)$ is fuzzy r-minimal regular closed. Hence from Theorem 4, it follows

$$mC(f^{-1}(G, r)) \subseteq mC(f^{-1}(mC(G, r)), r) = f^{-1}(mC(G, r)).$$

(2) $\Rightarrow$ (3) Since every fuzzy r-minimal semiopen set is fuzzy r-minimal β-open, it is obvious.

(3) $\Rightarrow$ (1) Let F be a fuzzy r-minimal regular closed set. Then F is fuzzy r-minimal semiopen, and so from (3), we have

$$mC(f^{-1}(F), r) \subseteq f^{-1}(mC(F, r)) = f^{-1}(F).$$

Hence, from Theorem 4, f is a fuzzy almost r-M continuous mapping. □

Theorem 6. *Let $f : X \to Y$ be a function on r-FMS's $(X, \mathcal{M}_X)$ and $(Y, \mathcal{M}_Y)$. If $\mathcal{M}_Y$ has the property $(\mathcal{U})$ then f is fuzzy almost r-M continuous if and only if $mC(f^{-1}(mC(mI(mC(G, r), r), r)), r) \subseteq f^{-1}(mC(G, r))$ for each fuzzy r-minimal preopen set G in Y.*

Proof. Suppose f is fuzzy almost r-M continuous. Let G be a fuzzy r-minimal preopen set in Y. Then we have

$$mC(G, r) = mC(mI(mC(G, r), r), r),$$

so $mC(G, r)$ is fuzzy r-minimal regular open. From Theorem 4, we have

$$\begin{aligned} f^{-1}(mC(G, r)) &= mC(f^{-1}(mC(G, r)), r) \\ &= mC(f^{-1}(mC(mI(mC(G, r), r), r)), r). \end{aligned}$$

Thus it implies
$mC(f^{-1}(mC(mI(mC(G,r),r),r)),r) \subseteq f^{-1}(mC(G,r))$.

For the converse, let A be a fuzzy r-minimal regular closed set in Y. Then $mI(A,r)$ is fuzzy r-minimal preopen. From hypothesis and $A = mC(mI(A,r),r)$, it follows

$$\begin{aligned} f^{-1}(A) &= f^{-1}(mC(mI(A,r),r)) \\ &\supseteq mC(f^{-1}(mC(mI(mC(mI(A,r),r),r),r)),r) \\ &= mC(f^{-1}(mC(mI(A,r),r)),r) \\ &= mC(f^{-1}(A),r). \end{aligned}$$

This implies $f^{-1}(A) = mCl(f^{-1}(A),r)$, and hence by Theorem 4, f is fuzzy almost r-M continuous. □

Theorem 7. *Let $f : X \to Y$ be a function on r-FMS's $(X, \mathcal{M}_X)$ and $(Y, \mathcal{M}_Y)$. If $\mathcal{M}_Y$ has the property $(\mathcal{U})$, then f is fuzzy almost r-M continuous if and only if $f^{-1}(G) \subseteq mI(f^{-1}(mI(mC(G,r),r)),r)$ for each fuzzy r-minimal preopen set G in Y.*

Proof. Suppose f is fuzzy almost r-M continuous and let G be a fuzzy r-minimal preopen set in Y. Then $mI(mC(G,r),r)$ is fuzzy r-minimal regular open. From Theorem 4, it follows $f^{-1}(G) \subseteq f^{-1}(mI(mC(G,r),r)) = mI(f^{-1}(mI(mC(G,r),r)),r)$.

For the converse, let U be fuzzy r-minimal regular open. Then U is obviously fuzzy r-minimal preopen. By hypothesis and $A = mI(mC(A,r),r)$, $f^{-1}(U) \subseteq mI(f^{-1}(mI(mC(U,r),r)),r) = mI(f^{-1}(U),r)$. So $f^{-1}(U) = mI(f^{-1}(U),r)$ and by Theorem 4, f is fuzzy almost r-M continuous. □

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