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PROPERTY (D_k) IN BANACH SPACES†

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Abstract. In this paper, we define property (D_k) and get the following strict implications.

$$\text{(UC)} \Rightarrow (D_2) \Rightarrow (D_3) \Rightarrow \cdots \Rightarrow (D_\infty) \Rightarrow \text{(BS)}.$$

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1. Introduction

Let $(X, \|\cdot\|)$ be a real Banach space and X^* the dual space of X. By B_X, we denote the closed unit ball of X. For a Banach space X with a usual unit basis (e_n), if $x = \sum_{n=1}^{\infty} a_n e_n$, we define the support of x, $\text{supp}(x) = \{n : a_n \neq 0\}$. For $x, y \in X$, we write $x < y$ for $\max \text{supp}(x) < \min \text{supp}(y)$.

$(X, \|\cdot\|)$ is said to be uniformly convex (UC) if for all $\epsilon > 0$, there exists a $\delta > 0$ such that for $x,\ y \in B_X$ with $\|x - y\| \geq \epsilon$,

$$\left\| \frac{1}{2}(x+y) \right\| \leq 1 - \delta.$$

A Banach space is said to have Banach-Saks property (BS) if any bounded sequence in the space admits a subsequence whose arithmetic means converges in norm. S. Kakutani [4] showed that Uniform convexity implies Banach-Saks property. And T. Nishiura and D. Waterman [5] proved that Banach-Saks property implies reflexivity in Banach spaces. A Banach space X is said to have weak Banach-Saks property if every weakly null sequence (x_n) in X admits a subsequence whose arithmetic means converges in norm. It is easy to see that Banach-Saks property implies weak Banach-Saks property. Since every bounded sequence in reflexive Banach spaces has weakly convergent subsequence, weak

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Banach-Saks property is equivalent to Banach-Saks property in reflexive Banach spaces.

2. Main Results

We start with the following definition.

Definition 1. A Banach space X is said to have *property* (D_k), where $k \geq 2$ if it is reflexive and there exists a number α, $0 < \alpha < 1$, such that for a weakly null sequence (x_n) in B_X, there exist $n_1 < n_2 < \cdots < n_k$ with

$$\left\| \frac{1}{k} \sum_{i=1}^{k} (-1)^{i+1} x_{n_i} \right\| < \alpha.$$

It is easy to see that property (D_k) implies property (D_{k+1}).

Proposition 2. *If X has property (D_k), then it has property (D_{k+1}).*

Proof. Suppose that X has property (D_k). Then X is reflexive and there exists a number α, $0 < \alpha < 1$, such that for a weakly null sequence (x_n) in B_X, there exist $n_1 < n_2 < \cdots < n_k$ with

$$\left\| \frac{1}{k} \sum_{i=1}^{k} (-1)^{i+1} x_{n_i} \right\| < \alpha.$$

Let $n_{k+1} = n_k + 1$. Then

$$\begin{aligned} \left\| \frac{1}{k+1} \sum_{i=1}^{k+1} (-1)^{i+1} x_{n_i} \right\| &\leq \frac{k}{k+1} \left\| \frac{1}{k} \sum_{i=1}^{k} (-1)^{i+1} x_{n_i} \right\| + \frac{1}{k+1} \|x_{n_{k+1}}\| \\ &\leq \frac{k}{k+1}\alpha + \frac{1}{k+1} = \frac{1}{k+1}(k\alpha + 1) < 1. \end{aligned}$$

Letting $\beta = \frac{1}{k+1}(k\alpha + 1)$, we get the result. □

The following Proposition 3 can be found in [2].

Proposition 3. *If X is uniformly convex, then it has property (D_2). The converse dose not hold.*

The following Definition 4 and Lemma 5 can be found in [1].

Definition 4. A Banach space X is said to have *alternate signs weak Banach−Saks property* if every weakly null sequence (x_n) in X there exists a subsequence (x'_n) of (x_n) and a sequence (ϵ_n) of $\{\pm 1\}$ such that $(1/n)\sum_{i=1}^{n} \epsilon_i x'_i$ converges in norm.

Lemma 5. *A Banach space has weak Banach-Saks property if and only if it has alternate signs weak Banach-Saks property.*

Banach spaces with property (D_k) have alternate Banach-Saks property.

Proposition 6. *If X has property (D_k), it has alternate signs weak Banach-Saks property.*

Proof. Suppose that X has property (D_k). Then there exists $0 < \alpha < 1$ such that for all weakly null sequence (x_n) in B_X, there exist $n_1 < n_2 < \cdots < n_k$ with

$$\left\| \frac{1}{k} \sum_{i=1}^{k} (-1)^{i+1} x_{n_i} \right\| < \alpha.$$

Suppose (x_n) is a weakly null sequence in X. Without loss of generality, we may assume that $\|x_n\| \leq 1$. Then there exist $n_1 < n_2 < \cdots < n_k$ such that

$$\left\| \frac{1}{k} \sum_{i=1}^{k} (-1)^{i+1} x_{n_i} \right\| < \alpha.$$

Since $(x_n)_{n > n_k}$ is weakly null and $\|x_n\| \leq 1$ for $n > n_k$, there exist $(n_k <) n_{k+1} < n_{k+2} < \cdots < n_{2k}$ such that

$$\left\| \frac{1}{k} \sum_{i=k+1}^{2k} (-1)^{i+1} x_{n_i} \right\| < \alpha.$$

Continue this process, we obtain a subsequence (x_{n_m}) for which given any $k \in \mathbb{N}$

$$\left\| \frac{1}{k} \sum_{i=jk+1}^{(j+1)k} (-1)^{i+1} x_{n_i} \right\| < \alpha,$$

for all $j \in \mathbb{N} \cup \{\nvdash\}$. Now, using Kakutani's result [4], we conclude that there exists a subsequence (x'_n) of (x_n) such that

$$\left\| \frac{1}{n} \sum_{i=1}^{n} (-1)^{i+1} x'_i \right\| \to 0 \qquad \text{as } n \to \infty.$$

This means that X has alternate weak Banach-Saks property. □

By Lemma 5 and Proposition 6, if X has property (D_k) then X has weak Banach-Saks property. Since weak Banach-Saks property is equivalent to Banach-Saks property in reflexive Banach spaces, we get the following.

Corollary 7. *If X has property (D_k), then it has Banach-Saks property.*

By Proposition 2, Proposition 3 and Corollary 7, we get the following implications.

$$\text{(UC)} \Rightarrow (D_2) \Rightarrow (D_3) \Rightarrow \cdots \Rightarrow (D_\infty) \Rightarrow \text{(BS)}.$$

We will now show that the implications are not reversible. The following Example 8 can be found in [6].

Example 8. For $x = (a_n) \in l_2$, we define a norm $\|x\|_{(s)}$ by

$$\|x\|_{(s)} = \left(\sup_{n_1 < n_2 < \cdots < n_s} \left(\sum_{i=1}^{s} |a_{n_i}| \right)^2 + \sum_{n \neq n_1, n_2 \cdots, n_s} |a_n|^2 \right)^{\frac{1}{2}}.$$

Then $\|x\|_2 \leq \|x\|_{(s)} \leq \sqrt{s}\|x\|_2$. Let $X_s = (l_2, \|\cdot\|_{(s)})$.

The following Lemma 9 can be found in [3].

Lemma 9. *If X is a Banach space with basis (e_n) and (x_n) is a weakly null sequence in X, then for all $\epsilon > 0$ there exists a subsequence $(x_{n_i}$ of (x_n) and block sequence (u_i) of (e_n) such that $\|x_{n_i} - u_i\| < \frac{\epsilon}{2^{i+1}}$.*

We need the following lemma.

Lemma 10. *If $x_1, x_2, \cdots, x_k, x_{k+1} \in B_{X_k}$ and $x_1 < x_2 < \cdots < x_k < x_{k+1}$ then*

$$\left\| \sum_{i=1}^{k+1} (-1)^{i+1} x_i \right\|_{(k)} \leq \sqrt{k^2+1}.$$

Proof. This is proved by straightforward computation using the following inequality

$$(n-1)\sum_{i=1}^{n} a_i^2 \geq 2 \sum_{1\leq i<j\leq n} a_i a_j,$$

where (a_i) is a real sequence. For simplicity, we give the proof in case $k = 2$. Suppose that $x = (a_n),\ y = (b_n),\ z = (c_n) \in B_{X_2}$ and $x < y < z$. Without loss of generality, it suffices to consider the following two cases.
Case 1: $\|x-y+z\|_{(2)}^2 = \sup_{n_1,n_2} \left(|a_{n_1}| + |a_{n_2}|\right)^2 + \sum_{n\neq n_1,n_2} |a_n|^2 + \sum_n |b_n|^2 + \sum_n |c_n|^2$.

$$\begin{aligned}\|x-y+z\|_{(2)}^2 &= \sup_{n_1,n_2} \left(|a_{n_1}| + |a_{n_2}|\right)^2 + \sum_{n\neq n_1,n_2} |a_n|^2 + \sum_n |b_n|^2 + \sum_n |c_n|^2 \\ &\leq \|x\|_{(2)}^2 + \|y\|_2^2 + \|z\|_2^2 \leq \|x\|_{(2)}^2 + \|y\|_{(2)}^2 + \|z\|_{(2)}^2 = 3.\end{aligned}$$

Case 2: $\|x-y+z\|_{(2)}^2 = \sup_{n_1,n_2} \left(|a_{n_1}| + |b_{n_2}|\right)^2 + \sum_{n\neq n_1} |a_n|^2 + \sum_{n\neq n_2} |b_n|^2 + \sum_n |c_n|^2$.

$$\begin{aligned}\|x-y+z\|_{(2)}^2 &= \sup_{n_1,n_2} \left(|a_{n_1}| + |b_{n_2}|\right)^2 + \sum_{n\neq n_1} |a_n|^2 + \sum_{n\neq n_2} |b_n|^2 + \sum_n |c_n|^2 \\ &\leq 2\sup_{n_1,n_2} \left(|a_{n_1}|^2 + |b_{n_2}|^2\right) + \sum_{n\neq n_1} |a_n|^2 + \sum_{n\neq n_2} |b_n|^2 + \sum_n |c_n|^2 \\ &\leq \sup_{n_1,n_2} \left(|a_{n_1}|^2 + |b_{n_2}|^2\right) + \sum_n |a_n|^2 + \sum_n |b_n|^2 + \sum_n |c_n|^2 \\ &\leq \|x\|_2^2 + \|y\|_2^2 + \|x\|_2^2 + \|y\|_2^2 + \|z\|_2^2 = 5\end{aligned}$$

This implies that $\|x-y+z\|_{(2)} \leq \sqrt{5}$. □

By the above lemmas, we get the following.

Proposition 11. *Property (D_{k+1}) does not imply Property (D_k)*

Proof. Since the space X_k is isomorphic to l_2, unit vector basis (e_n) is weakly null in X_k. But

$$\left\|\sum_{i=1}^{k}(-1)^{i+1} e_{n_i}\right\|_{(k)} = k$$

for all choice of n_i. This means that X_k does not have property (D_k).

Let (x_n) be a weak null sequence in B_{X_k}. By Lemma 9, for all $\epsilon > 0$ there exists a subsequence (x_{n_i}) of (x_n) and block sequence (u_i) of (e_n) such that $\|x_{n_i} - u_i\| < \frac{\epsilon}{2^{i+1}}$. We note that

$$\|\sum_{i=1}^{k+1}(-1)^{i+1}u_i\|_{(k)} \leq \sqrt{k^2+1},$$

by Lemma 10. For some large $i_1 < i_2 < \cdots < i_k < i_{k+1}$,

$$\|x_{n_{i_j}} - u_{i_j}\| < \frac{1}{k+1}\left(\sqrt{k^2+2} - \sqrt{k^2+1}\right),$$

where $j = 1, 2, \ldots, k+1$. Then we have

$$\begin{aligned}\left\|\sum_{j=1}^{k+1}(-1)^{i+1}x_{n_{i_j}}\right\| &\leq \sum_{j=1}^{k+1}\left\|x_{n_{i_j}} - u_{i_j}\right\| + \left\|\sum_{j=1}^{k+1}(-1)^{i+1}u_{n_j}\right\| \\ &\leq \sqrt{k^2+2}.\end{aligned}$$

Let $\alpha = \frac{\sqrt{k^2+2}}{k+1}$. Then $\alpha < 1$ and this leads that the space X_k has property (D_{k+1}). □

To get that the following implications hold and strict, the remaining proof is that Banach-Saks property does not imply Property (D_∞).

$$\text{(UC)} \Rightarrow (D_2) \Rightarrow (D_3) \Rightarrow \cdots \Rightarrow (D_\infty) \Rightarrow \text{(BS)}.$$

Proposition 12. *Banach-Saks property does not imply Property* (D_∞).

Proof. Consider $\left(\prod_{s\geq 2} C_s\right)_{l_2}$. Then $\left(\prod_{s\geq 2} C_s\right)_{l_2}$ has Banach-Saks property [6].

Let $k \in \mathbb{N}$. If $x^{(n)} = (0, 0, \cdots, 0, e_n, 0, \cdots)$ where usual unit vector e_n in k-th coordinate is only nonzero element of $x^{(n)}$, then $x^{(n)} \in \left(\prod_{s\geq 2} C_s\right)_{l_2}$ and $\|x^{(n)}\|_{\left(\prod_{s\geq 2} C_s\right)_{l_2}} = 1$. We note that $x^{(n)}$ is weakly null in $\left(\prod_{s\geq 2} C_s\right)_{l_2}$. But

$$\left\|\sum_{j=1}^{k}(-1)^{i+1}x^{(n_i)}\right\|_{\left(\prod_{s\geq 2} C_s\right)_{l_2}} = \left\|\sum_{j=1}^{k}(-1)^{i+1}e_{n_i}\right\|_{(k)} = k.$$

This means that $\left(\prod_{s\geq 2} C_s\right)_{l_2}$ has no property (D_∞). □

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