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HERMITE-HADAMARD-TYPE INEQUALITIES FOR REAL α-STAR s-CONVEX MAPPINGS

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ABSTRACT. In this article some generalized refinements of some inequalities for real quasi-cinvex, convex, concave, s-convex, s-concave, and α-star s-convex mappings are obtained.

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1. Peliminaries

For a convex mapping $f : \mathbb{I} \subseteq \mathbb{R} \to \mathbb{R}$ defined on the interval $\mathbb{I}$ of real numbers and a, b in $\mathbb{I}$ with $a < b$, define the integral arithmetic mean $\frac{1}{b-a}\int_a^b f(x)dx$ of f and the arithmetic mean $\frac{a+b}{2}$ on an interior subinterval (a, b) of $\mathbb{I}$, respectively.

The classical Hermite-Hadamard's inequality [1,2,3,4,6,8,9,10] assert that:

$$f(\frac{a+b}{2}) \le \frac{1}{b-a}\int_a^b f(x)dx \le \frac{f(a)+f(b)}{2}. \tag{1}$$

A function $f : \mathbb{I} \subseteq \mathbb{R}^+ \to \mathbb{R}$ is said to be s-convex and α-star s-convex on $\mathbb{I}$ if

$$f(tx + (1-t)y) \le t^s f(x) + (1-t)^s f(y)$$

and

$$f(tx + (1-t)y) \le t^s f(x) + \{(1-t)\alpha\}^s f(y)$$

for any $x, y \in \mathbb{I}$ and $t, \alpha \in [0, 1]$, respectively. For the definitions of s-concave and α-star s-concave functions on $\mathbb{I}$ the inequalities in (2) are reversed.

For the simplicities of notations, define $R_f(a, b)$ and $L_f(a, b)$ by

$$R_f(a,b) = \frac{1}{b-a}[\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx]$$

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and

$$L_f(a,b) = \frac{1}{b-a}\left[\frac{1}{b-a}\int_a^b f(x)dx - f\left(\frac{a+b}{2}\right)\right].$$

In recent years, many authors established several inequalities connected to Hadamard's inequality. For recent results, refinements, counterparts, generalizations and new Hadamard's-type inequalities, see [1-13].

J. Pecaric [1,9,12], S.S. Dragomir [2,4,5], M. Alomari [2,3], M. Darus [3], R.P. Agarwal [4] and U.S. Kirmaci [3,8,9,10] obtained inequalities for differentiable convex mapping which are connected with Hermite-Hadamard's inequality, and they used the following lemma to prove them:

Lemma 1. *Let $f : \mathbb{I} \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping defined on the interior $\mathbb{I}^0$ of $\mathbb{I}$ and a, b in $\mathbb{I}$ with $a < b$. If $f' \in L([a,b])$, then the following equalities hold:*

$$(a) R_f(a,b) = \frac{1}{2}\int_0^1 (1-2t)f'(ta+(1-t)b)dt, \tag{2}$$

$$(b) L_f(a,b) = \left[\int_0^{\frac{1}{2}} tf'(ta+(1-t)b)dt + \int_{\frac{1}{2}}^1 (t-1)f'(ta+(1-t)b)dt\right]. \tag{3}$$

The main inequality in [4] pointed out as follows:

Theorem 1. *Let $f : \mathbb{I} \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping defined on the interior $\mathbb{I}^0$ of $\mathbb{I}$ and a, b in $\mathbb{I}$ with $a < b$. If $f' \in L([a,b])$ and $\mid f' \mid$ is convex on $\mathbb{I}$, then the following inequalities hold: $\mid R_f(a,b) \mid \leq \frac{1}{8}(\mid f'(a) \mid + \mid f'(b) \mid)$.*

In [12], C.E.M. Pearce and J. Pečarić proved the following theorem by using the equalities (a) and (c) in Lemma 1.

Theorem 2. *Let $f : \mathbb{I} \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping defined on the interior $\mathbb{I}^0$ of $\mathbb{I}$ and a, b in $\mathbb{I}$ with $a < b$ and $p \geq 1$. Then the following inequalities hold:*

(a) If $\mid f' \mid^p$ is convex on $\mathbb{I}$, then $\mid R_f(a,b) \mid \leq \frac{1}{4}\left[\frac{|f(a)|^p + |f(b)|^p)}{2}\right]^{\frac{1}{p}}$,

(b) If $\mid f' \mid^p$ is concave on $\mathbb{I}$, then $\mid R_f(a,b) \mid \leq \frac{1}{4} \mid f'(\frac{a+b}{2}) \mid$.

A function $f : \mathbb{I} \to \mathbb{R}$ is said to be quasi-convex on $\mathbb{I}$ if

$$f(tx + (1-t)y) \leq \sup\{f(x), f(x)\}$$

for any $x, y \in \mathbb{I}$ and $t \in [0,1]$.

Clearly any convex function is a quasi-convex function. Furthermore there exist quasi-convex functions which are not convex [7].

Recently, D.A. Ion [7] introduced two inequalities of the right hand side of Hadamard's type for quasi-convex functions, as follow:

Theorem 3. *Let $f : \mathbb{I} \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping defined on the interior $\mathbb{I}^0$ of $\mathbb{I}$ and a, b in $\mathbb{I}$ with $a < b$ and $p \geq 1$. Then the following inequalities hold:*

(a) $\mid R_f(a,b) \mid\leq \frac{1}{4}\sup\{\mid f'(a) \mid, \mid f'(b) \mid\}$ *if* $\mid f' \mid$ *is quasi-convex on* $\mathbb{I}$.

(b) $\mid R_f(a,b) \mid\leq \frac{1}{2}(\frac{p-1}{2p-2})^{\frac{p-1}{p}}\sup\{\mid f(a) \mid^p, \mid f(b) \mid^p\}^{\frac{1}{p}}$ *if* $\mid f' \mid^p$ *is quasi-convex* $\mathbb{I}$.

By the equality (b) in lemma 1, M. Alomari, M. Darus and U.S. Kirmaci [2,3,8,10] established refinements inequalities of the right hand side of Hadamard's type for quasi-convex functions:

Theorem 4 (8,9,10,11). *Let* $f : \mathbb{I} \subseteq \mathbb{R} \to \mathbb{R}$ *be a differentiable mapping defined on the interior* $\mathbb{I}^0$ *of* $\mathbb{I}$ *and* a, b *in* $\mathbb{I}$ *with* $a < b$. *Then the following inequalities hold:*

(a) *If* $\mid f' \mid$ *is quasi-convex on* $\mathbb{I}$, *then*

$$\mid R_f(a,b) \mid\leq \frac{1}{8}[\sup\{\mid f'(\frac{a+b}{2}) \mid, \mid f'(a) \mid\} + \sup\{\mid f'(\frac{a+b}{2}) \mid, \mid f'(b) \mid\}],$$

(b) *If* $\mid f' \mid^{\frac{p}{p-1}}$ *is quasi-convex on* $\mathbb{I}$ *for* $p > 1$, *then*

$$\mid R_f(a,b) \mid\leq \frac{1}{4}(\frac{p-1}{2p-1})^{\frac{p-1}{p}}[(\sup\{\mid f'(\frac{a+b}{2}) \mid^p, \mid f'(a) \mid^p\})^{\frac{1}{p}} + (\sup\{\mid f'(\frac{a+b}{2}) \mid^p, \mid f'(b) \mid^p\})^{\frac{1}{p}}].$$

Theorem 5 (2). *Let* $f : \mathbb{I} \subseteq \mathbb{R} \to \mathbb{R}$ *be a differentiable mapping defined on the interior* $\mathbb{I}^0$ *of* $\mathbb{I}$ *and* a, b *in* $\mathbb{I}$ *with* $a < b$. *If* $\mid f' \mid^p$ *is quasi-convex on* $\mathbb{I}$ *for* $p \geq 1$, *then*

$$\mid R_f(a,b) \mid \leq \frac{1}{8}[(\sup\{\mid f'(\frac{a+b}{2}) \mid^p, \mid f'(a) \mid^p\})^{\frac{1}{p}} + (\sup\{\mid f'(\frac{a+b}{2}) \mid^p, \mid f'(b) \mid^p\})^{\frac{1}{p}}].$$

Theorem 6 (6). *Suppose that* $f : \mathbb{I} \subseteq \mathbb{R}^+ \to \mathbb{R}^+$ *is a mapping defined on the interior* $\mathbb{I}^0$ *of* $\mathbb{I}$, a, b *in* $\mathbb{I}$ *with* $a < b$, $s \in (0,1)$, *and* $f \in L([a,b])$.

(a) If f *is a convex mapping on* $\mathbb{I}$, *then*

$$\mid 2^{s-1}f(\frac{a+b}{2}) \mid\leq\mid \frac{1}{b-a}\int_a^b f(x)dx \mid\leq \frac{2}{s+1} \mid \frac{f(a)+f(b)}{2} \mid, \qquad (4)$$

(b) If f *is a concave mapping on* $\mathbb{I}$, *then*

$$\frac{2}{s+1} \mid \frac{f(a)+f(b)}{2} \mid\leq\mid \frac{1}{b-a}\int_a^b f(x)dx \mid\leq 2^{s-1} \mid f(\frac{a+b}{2}) \mid. \qquad (5)$$

2. Hermite-Hadamard's inequalities for $\alpha-$star $s-$convex functions

In this article we will use the following new equalities not used in other articles:

Lemma 2. *Let $f : \mathbb{I} \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping defined on $\mathbb{I}^{\nvdash}$ of real numbers and a, b in $\mathbb{I}$ with $a < b$. If $f' \in L([a, b])$, then the following equalities hold:*

$$(a) R_f(a,b)$$

$$= \frac{1}{4}[\int_0^1 (-t) f'(\frac{1+t}{2}a + \frac{1-t}{2}b)dt + \int_0^1 t f'(\frac{1+t}{2}b + \frac{1-t}{2}a)dt], \tag{6}$$

$$= \frac{1}{2}[\int_0^1 (1-t) f'(ta + (1-t)b)dt + \int_0^1 (t-1) f'(tb + (1-t)a)dt], \tag{7}$$

$$(b) L_f(a,b)$$

$$= \frac{1}{4}[\int_0^1 (1-t) f'(\frac{1+t}{2}a + \frac{1-t}{2}b)dt + \int_0^1 (t-1) f'(\frac{1+t}{2}b + \frac{1-t}{2}a)dt], \tag{8}$$

$$= -[\int_0^{\frac{1}{2}} t f'(tb + (1-t)a)dt + \int_{\frac{1}{2}}^1 t f'(ta + (1-t)b)dt]. \tag{9}$$

In the following theorem, we shall propose some new upper and lower bound for the left-hand and right-hand sides of Hermite-Hadamard's inequality for quasi-convex, convex, and concave mapping, which is better than the inequality had done in other articles.

Theorem 7. *Let $f : \mathbb{I} \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on the interior $\mathbb{I}^0$ of $\mathbb{I}$ such that $f' \in L([a, b])$, where $a, b \in \mathbb{I}$ with $a < b$ and $s \in (0, 1]$.*
(a)If $\mid f' \mid$ is an s-convex mapping on $\mathbb{I}= [a, b]$, then

$$(i) \mid R_f(a,b) \mid \leq \frac{1}{4}(\frac{2s + 2^{1-s}}{(s+1)(s+2)})(\mid f'(a) \mid + \mid f'(b) \mid) \leq \frac{1}{2}(\mid f'(a) \mid + \mid f'(b) \mid),$$

or

$$\mid R_f(a,b) \mid \leq \frac{1}{2}(\frac{1}{s+1}[\mid f'(a) \mid + \mid f'(b) \mid] \leq \frac{1}{4}[\mid f'(a) \mid + \mid f'(b) \mid].$$

$$(ii) \mid L_f(a,b) \mid \leq \frac{1}{2}(\frac{2 - 2^{-s}}{(s+1)(s+2)})(\mid f'(a) \mid + \mid f'(b) \mid) \leq \frac{1}{4}(\mid f'(a) \mid + \mid f'(b) \mid).$$

(b)If $\mid f' \mid^p$ is an s-convex mapping on $\mathbb{I} = [\ni,]$ for $p > 1$, then

$$(i) \mid R_f(a,b) \mid \leq \frac{1}{4}(\frac{p-1}{2p-1})^{\frac{p-1}{p}}(\frac{1}{s+1})^{\frac{1}{p}}[\{\mid f'(\frac{a+b}{2}) \mid^p + \mid f'(a) \mid^p\}^{\frac{1}{p}} + \{\mid f'(b) \mid^p + \mid f'(\frac{a+b}{2}) \mid^p\}^{\frac{1}{p}}],$$

or

$$| R_f(a,b) | \leq \frac{1}{4}(\frac{p-1}{2p-1})^{\frac{p-1}{p}} 2^{\frac{s-1}{p}} [| f'(\frac{a+3b}{4}) |^{\frac{1}{p}} + | f'(\frac{3a+b}{4}) |^{\frac{1}{p}}].$$

$$(ii) \ | L_f(a,b) | \leq \frac{1}{4}[\{| f'(\frac{a+b}{2}) |^p + | f'(a) |^p\}^{\frac{1}{p}} + \{| f'(b) |^p + | f'(\frac{a+b}{2}) |^p\}^{\frac{1}{p}}],$$

Proof.(a) By (6), (7) and (8) in Lemma 2, we have:

$$\begin{aligned}
&(i) \ | R_f(a,b) | \\
&\leq \frac{1}{4}[\int_0^1 | -t || f'(\frac{1+t}{2}a + \frac{1-t}{2}b) | dt + \int_0^1 | t || f'(\frac{1+t}{2}b + \frac{1-t}{2}a) | dt] \\
&\leq \frac{1}{4}[| f'(a) | \int_0^1 t(\frac{1+t}{2})^s dt + | f'(b) | \int_0^1 t(\frac{1-t}{2})^s dt + \\
&\qquad + | f'(a) | \int_0^1 t(\frac{1-t}{2})^s dt + | f'(b) | \int_0^1 t(\frac{1+t}{2})^s dt] \\
&\leq \frac{1}{4}(\frac{2s+2^{1-s}}{(s+1)(s+2)})(| f'(a) | + | f'(b) |) \leq \frac{1}{4}(| f'(a) | + | f'(b) |),
\end{aligned}$$

or

$$\begin{aligned}
&| R_f(a,b) | \\
&\leq \frac{1}{2}[\int_0^1 | 1-t || f'(ta+(1-t)b) | dt + \int_0^1 | t-1 || f'(tb+(1-t)a) | dt] \\
&= \frac{1}{2}[| f'(a) | (\frac{1}{(s+1)(s+2)} + \frac{1}{s+2}) + | f'(b) | (\frac{1}{(s+1)(s+2)} + \frac{1}{s+2})] \\
&= \frac{1}{2}\frac{1}{s+1}[| f'(a) | + | f'(b) |] \quad \leq \frac{1}{4}(| f'(a) | + | f'(b) |).
\end{aligned}$$

$$\begin{aligned}
&(ii) \ | L_f(a,b) | \\
&\leq \frac{1}{4}[\int_0^1 (1-t)\{(\frac{1+t}{2})^s | f'(a) | + (\frac{1-t}{2})^s | f'(b) |\} dt \\
&\qquad + \int_0^1 (1-t)\{(\frac{1+t}{2})^s | f'(a) | + (\frac{1-t}{2})^s | f'(b) |\} dt] \\
&+ \{| f'(a) | \int_0^1 (1-t)(\frac{1-t}{2})^s dt + | f'(b) | \int_0^1 (1-t)(\frac{1+t}{2})^s dt\}] \\
&= \frac{1}{2}(\frac{2-2^{-s}}{(s+1)(s+2)})(| f'(a) | + | f'(b) |).
\end{aligned}$$

(b) By (6) and (8) in Lemma 2, we have:

$$
\begin{aligned}
(i)\ &| R_f(a,b) | \\
&\le \frac{1}{4}\Big(\frac{p-1}{2p-1}\Big)^{\frac{p-1}{p}}\Big[\Big\{\frac{| f'(\frac{a+b}{2}) |^p + | f'(a) |^p}{s+1}\Big\}^{\frac{1}{p}} + \Big\{\frac{| f'(\frac{a+b}{2}) |^p + | f'(b) |^p}{s+1}\Big\}^{\frac{1}{p}}\Big] \\
&\le \frac{1}{4}\Big(\frac{p-1}{2p-1}\Big)^{\frac{p-1}{p}}\Big(\frac{1}{s+1}\Big)^{\frac{1}{p}}\Big[\Big\{| f'(\frac{a+b}{2}) |^p + | f'(a) |^p\Big\}^{\frac{1}{p}} \\
&\qquad\qquad + \Big\{| f'(b) |^p + | f'(\frac{a+b}{2}) |^p\Big\}^{\frac{1}{p}}\Big]
\end{aligned}
$$

or

$$
\begin{aligned}
&| R_f(a,b) | \\
&\le \frac{1}{4}\Big(\frac{p-1}{2p-1}\Big)^{\frac{p-1}{p}}\Big[\Big(\int_0^1 | f'(\frac{1+t}{2}a + \frac{1-t}{2}b) |^p\, dt\Big)^{\frac{1}{p}} \\
&\qquad\qquad + \Big(\int_0^1 | t || f'(\frac{1+t}{2}b + \frac{1-t}{2}a) | \,dt\Big] \\
&\le \frac{1}{4}\Big(\frac{p-1}{2p-1}\Big)^{\frac{p-1}{p}}\Big[\Big\{\frac{2}{b-a}\int_a^{\frac{a+b}{2}} | (f')^p | \,dt\Big\}^{\frac{1}{p}} + \Big\{\frac{2}{b-a}\int_{\frac{a+b}{2}}^b | (f')^p | \,dt\Big\}^{\frac{1}{p}}\Big] \\
&\le \frac{1}{4}\Big(\frac{p-1}{2p-1}\Big)^{\frac{p-1}{p}}\Big[\Big(2^{s-1} | f'(\frac{a+3b}{4}) |\Big)^{\frac{1}{p}} + \Big(2^{s-1} | f'(\frac{3a+b}{4}) |\Big)^{\frac{1}{p}}\Big].
\end{aligned}
$$

$$
\begin{aligned}
(ii)\ &| L_f(a,b) | \\
&\le \frac{1}{4}\Big[\Big(\int_0^1 (1-t)^{\frac{p}{p-1}} dt\Big)^{\frac{p-1}{p}}\Big(\int_0^1 | f'(\frac{1+t}{2}a + \frac{1-t}{2}b) |^p\, dt\Big)^{\frac{1}{p}} \\
&\qquad + \Big(\int_0^1 (1-t)^{\frac{p}{p-1}} dt\Big)^{\frac{p-1}{p}}\Big(\int_0^1 | f'(\frac{1+t}{2}b + \frac{1-t}{2}a) |^p\, dt\Big)^{\frac{1}{p}}\Big] \\
&\le \frac{1}{4}\Big(\frac{p-1}{2p-1}\Big)^{\frac{p-1}{p}}\Big[\Big\{\int_a^{\frac{a+b}{2}} | f'(t) |^p\, dt\Big\}^{\frac{1}{p}} + \Big\{\int_{\frac{a+b}{2}}^b | f'(t) |^p\, dt\Big\}^{\frac{1}{p}}\Big] \\
&\le \frac{1}{4}\Big[\Big\{\frac{| f'(\frac{a+b}{2}) |^p + | f'(a) |^p}{s+1}\Big\}^{\frac{1}{p}} + \Big\{\frac{| f'(b) |^p + | f'(\frac{a+b}{2}) |^p}{s+1}\Big\}^{\frac{1}{p}}\Big] \\
&\le \frac{1}{4}\Big[\Big\{| f'(\frac{a+b}{2}) |^p + | f'(a) |^p\Big\}^{\frac{1}{p}} + \Big\{| f'(b) |^p + | f'(\frac{a+b}{2}) |^p\Big\}^{\frac{1}{p}}\Big].
\end{aligned}
$$

Theorem 8. *Let $f : \mathbb{I} \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on the interior $\mathbb{I}^0$ of $\mathbb{I}$ such that $f' \in L([a,b])$, where $a,b \in \mathbb{I}$ with $a < b$, $s \in (0,1]$, and $p \ge 1$. If*

$| f' |^p$ *is an s-concave mapping on* $\mathbb{I} = [\eth,]$, *then the following inequalities hold:*

$$(i)\ | R_f(a,b) | \leq \frac{1}{4}(\frac{p-1}{2p-1})^{\frac{p-1}{p}}(\frac{1}{2})^{\frac{1-s}{p}}[| f'(\frac{3a+b}{4}) | + | f'(\frac{a+3b}{4}) |]$$

or

$$| R_f(a,b) | \leq (\frac{p-1}{2p-1})^{\frac{p-1}{p}} 2^{\frac{s-1}{p}} \ | f'(\frac{a+b}{2}) |],$$

$$(ii)\ | L_f(a,b) | \leq \frac{1}{4}(\frac{p-1}{2p-1})^{\frac{p-1}{p}}(\frac{1}{2})^{\frac{1-s}{p}}[| f'(\frac{3a+b}{4}) | + | f'(\frac{a+3b}{4}) |]$$

Proof. By theorem 1.6(b), we have the following inequalities;

$$\int_0^1 | f'(\frac{1+t}{2}a + \frac{1-t}{2}b) |^p \ dt \leq 2^{s-1} \ | f'(\frac{3a+b}{4}) |^p,$$
$$\int_0^1 | f'(\frac{1+t}{2}b + \frac{1-t}{2}a) |^p \ dt \leq 2^{s-1} \ | f'(A(\frac{a+3b}{4}) |^p \ .$$

By using (6), (7) and (8) in Lemma 2 and the fact that $\frac{1}{2} \leq (\frac{p-1}{2p-1})^{\frac{p-1}{p}} \leq 1$ and $\frac{1}{2} \leq (\frac{1}{2})^{\frac{1-s}{p}} \leq 1$, we have:

$$\begin{aligned}
&(i)\ | R_f(a,b) | \\
&\leq \frac{1}{4}(\frac{p-1}{2p-1})^{\frac{p-1}{p}}[(\int_0^1 | f'(\frac{1+t}{2}a + \frac{1-t}{2}b) |^p \ dt)^{\frac{1}{p}} \\
&\qquad + (\int_0^1 | f'(\frac{1+t}{2}b + \frac{1-t}{2}a) |^p \ dt)^{\frac{1}{p}}] \\
&\leq \frac{1}{4}(\frac{p-1}{2p-1})^{\frac{p-1}{p}}(\frac{1}{2})^{\frac{1-s}{p}}[| f'(\frac{3a+b}{4}) | + | f'(\frac{a+3b}{4}) |]
\end{aligned}$$

or

$$\begin{aligned}
&| R_f(a,b) | \\
&\leq \frac{1}{2}[(\int_0^1 (1-t)^{\frac{p}{p-1}} dt)^{\frac{p-1}{p}}(\int_0^1 | f'(ta + (1-t)b) |^p \ dt)^{\frac{1}{p}} \\
&\qquad + (\int_0^1 (1-t)^{\frac{p}{p-1}} dt)^{\frac{p-1}{p}}(\int_0^1 | f'(tb + (1-t)a) |^p \ dt)^{\frac{1}{p}}] \\
&\leq \frac{1}{2}(\frac{p-1}{2p-1})^{\frac{p-1}{p}}[2(2^{s-1}) \ | f'(\frac{a+b}{2}) |^p)^{\frac{1}{p}}] \\
&\leq (\frac{p-1}{2p-1})^{\frac{p-1}{p}} 2^{\frac{s-1}{p}} \ | f'(\frac{a+b}{2}) |],
\end{aligned}$$

by the fact that $\int_0^1 \mid f'(ta+(1-t)b) \mid^p dt \leq 2^{s-1} \mid f'(\frac{a+b}{2}) \mid^p$, $\int_0^1 \mid f'(tb+(1-t)a) \mid^p dt \leq 2^{s-1} \mid f'(\frac{a+b}{2}) \mid^p$, by Theorem 1.2(b).

$$(ii) \mid L_f(a,b) \mid \leq \frac{1}{4}(\frac{p-1}{2p-1})^{\frac{p-1}{p}}[(\int_0^1 \mid f'(\frac{1+t}{2}a+\frac{1-t}{2}b) \mid^p dt)^{\frac{1}{p}} + (\int_0^1 \mid f'(\frac{1+t}{2}b+\frac{1-t}{2}a) \mid^p dt)^{\frac{1}{p}}]$$

$$\leq \frac{1}{4}(\frac{p-1}{2p-1})^{\frac{p-1}{p}}[(2^{s-1} \mid f'(\frac{3a+b}{4}) \mid^p)^{\frac{1}{p}} + (2^{s-1} \mid f'(\frac{a+3b}{4}) \mid^p)^{\frac{1}{p}}]$$
$$\leq \frac{1}{4}(\frac{p-1}{2p-1})^{\frac{p-1}{p}}(\frac{1}{2})^{\frac{1-s}{p}}[\mid f'(\frac{3a+b}{4}) \mid + \mid f'(\frac{a+3b}{4}) \mid].$$

Theorem 9. *Let $f : \mathbb{I} \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on the interior $\mathbb{I}^0$ of $\mathbb{I}$ such that $f' \in L([a,b])$, where $a, b \in \mathbb{I}$ with $a < b$ and $\alpha, s \in (0,1]$.*

(a)If $\mid f' \mid$ is an α-star s-convex mapping on $\mathbb{I}= [a,b]$, then

$$(i) \mid R_f(a,b) \mid \leq \frac{1}{4}\{\frac{1+2^{s+1}s+\alpha^s}{2^{-s}(s+1)(s+2)}\}(\mid f'(a) \mid + \mid f'(b) \mid),$$

or

$$\mid R_f(a,b) \mid \leq \frac{1}{2}\frac{\alpha^s(s+1)+1}{(s+1)(s+2)}[\mid f'(a) \mid + \mid f'(b) \mid],$$
$$(ii) \mid L_f(a,b) \mid \leq \frac{1}{8}\{\frac{2^{s+2}-s+3+(s+1)\alpha^s}{(s+1)(s+2)2^s}\}(\mid f'(a) \mid + \mid f'(b) \mid).$$

(b)If $\mid f' \mid^p$ is an α-star s-convex mapping on $\mathbb{I} = [\supset,]$ for $p > 1$, then

$$(i) \mid R_f(a,b) \mid \leq \frac{1}{4}(\frac{p-1}{2p-1})^{\frac{p-1}{p}}\{(\frac{2-2^{-s}}{s+1})^{\frac{1}{p}} + (\frac{2^{-s}}{s+1})^{\frac{1}{p}}\}(\mid f'(a) \mid + \mid f'(b) \mid),$$
$$(ii) \mid L_f(a,b) \mid \leq \frac{1}{4}(\frac{p-1}{2p-1})^{\frac{p-1}{p}}\{(\frac{2-2^{-s}}{s+1})^{\frac{1}{p}} + (\frac{2^{-s}}{s+1})^{\frac{1}{p}}\}(\mid f'(a) \mid + \mid f'(b) \mid).$$

Proof. By (6), (7) and (8) in Lemma 2, we have:

$$(a)(i) \mid R_f(a,b) \mid \leq \frac{1}{4}\{\int_0^1 t(\mid \frac{1+t}{2} \mid)^s dt + \int_0^1 t(\alpha \mid \frac{1-t}{2} \mid)^s dt\}(\mid f'(a) \mid + \mid f'(b) \mid)$$
$$\leq \frac{1}{4}(\frac{1+2^{s+1}s+\alpha^s}{2^{-s}(s+1)(s+2)})(\mid f'(a) \mid + \mid f'(b) \mid),$$

or

$$
\begin{aligned}
&| R_f(a,b) | \\
&\le \frac{1}{2}[\int_0^1 (1-t)\{(t^s \mid f'(a) \mid +((1-t)\alpha)^s \mid f'(b) \mid\}dt \\
&\qquad + \int_0^1 (1-t)\{(t^s \mid f'(b) \mid +((1-t)\alpha)^s \mid f'(a) \mid\}dt] \\
&\le \frac{1}{2}\frac{\alpha^s(s+1)+1}{(s+1)(s+2)}[\mid f'(a) \mid + \mid f'(b) \mid]
\end{aligned}
$$

$$
\begin{aligned}
&(ii) \mid L_f(a,b) \mid \\
&\le \frac{1}{4}[\int_0^1 (1-t)\{(\frac{1+t}{2})^s \mid f'(a) \mid +(\frac{1-t}{2})^s \mid f'(b) \mid\}dt \\
&\qquad + \int_0^1 (1-t)\{(\frac{1+t}{2})^s \mid f'(b) \mid +(\frac{1-t}{2})^s \mid f'(a) \mid\}dt] \\
&\le \frac{1}{4}[\mid f'(a) \mid \{\int_0^1 (1-t)\{(\frac{1+t}{2})^s dt + \int_0^1 (1-t)\{(\alpha\frac{1-t}{2})^s dt\} \\
&\qquad + \mid f'(b) \mid \{\int_0^1 (1-t)\{(\alpha\frac{1+t}{2})^s dt + \int_0^1 (1-t)\{(\frac{1+t}{2})^s dt\}] \\
&= \frac{1}{8}\{\frac{2^{s+2}-s+3+(s+1)\alpha^s}{(s+1)(s+2)2^s}\}(\mid f'(a) \mid + \mid f'(b) \mid).
\end{aligned}
$$

$$
\begin{aligned}
&(b)(i) \mid R_f(a,b) \mid \\
&\le \frac{1}{4}[(\int_0^1 t^{\frac{p}{p-1}}dt)^{\frac{p-1}{p}}(\int_0^1 \mid f'(\frac{1+t}{2}a+\frac{1-t}{2}b) \mid^p dt)^{\frac{1}{p}} \\
&\qquad + (\int_0^1 t^{\frac{p}{p-1}}dt)^{\frac{p-1}{p}}(\int_0^1 \mid f'(\frac{1+t}{2}b+\frac{1-t}{2}a) \mid^p dt)^{\frac{1}{p}} \\
&\le \frac{1}{4}(\frac{p-1}{2p-1})^{\frac{p-1}{p}}[\{\mid f'(a) \mid^p (\frac{2-2^{-s}}{s+1}) + \mid f'(b) \mid^p \alpha^s(\frac{2^{-s}}{s+1})\}^{\frac{1}{p}} \\
&\qquad + \{\mid f'(a) \mid^p \alpha^s(\frac{2^{-s}}{s+1}) + \mid f'(b) \mid^p (\frac{2-2^{-s}}{s+1})\}^{\frac{1}{p}}] \\
&\le \frac{1}{4}(\frac{p-1}{2p-1})^{\frac{p-1}{p}}\{(\frac{2-2^{-s}}{s+1})^{\frac{1}{p}} + (\frac{2^{-s}}{s+1})^{\frac{1}{p}}\}(\mid f'(a) \mid + \mid f'(b) \mid) \\
&\le \frac{1}{2}(\mid f'(a) \mid + \mid f'(b) \mid),
\end{aligned}
$$

or

$$
\begin{aligned}
&| R_f(a,b) | \\
&\le \frac{1}{2}(\frac{p-1}{2p-1})^{\frac{p-1}{p}}[(\int_0^1 \{(t^s \mid f'(a) \mid^p +((1-t)\alpha)^s \mid f'(b) \mid^p\}dt)^{\frac{1}{p}} \\
&\qquad\qquad + (\int_0^1 \{(t^s \mid f'(b) \mid^p +((1-t)\alpha)^s \mid f'(a) \mid^p\}dt)^{\frac{1}{p}} \\
&= \frac{1}{2}(\frac{p-1}{2p-1})^{\frac{p-1}{p}}(\frac{1}{s+1})^{\frac{1}{p}}[(\mid f'(a) \mid^p +\alpha^s \mid f'(b) \mid^p)^{\frac{1}{p}} \\
&\qquad\qquad + (\mid f'(b) \mid^p +\alpha^s \mid f'(a) \mid^p)^{\frac{1}{p}}] \\
&\le \frac{1}{2}(\frac{1}{s+1})^{\frac{1}{p}}[(\mid f'(a) \mid^p +\alpha^s \mid f'(b) \mid^p)^{\frac{1}{p}} \\
&\qquad + (\mid f'(b) \mid^p +\alpha^s \mid f'(a) \mid^p)^{\frac{1}{p}}].
\end{aligned}
$$

$$
\begin{aligned}
&(ii) \mid L_f(a,b) \mid \\
&\le \frac{1}{4}(\frac{p-1}{2p-1})^{\frac{p-1}{p}}[(\int_0^1 \mid f'(\mid \frac{1+t}{2}a + \frac{1-t}{2}b \mid) \mid^p dt)^{\frac{1}{p}} \\
&\qquad\qquad + \int_0^1 \mid f'(\mid \frac{1+t}{2}b + \frac{1-t}{2}a \mid) \mid^p dt)^{\frac{1}{p}}] \\
&\le \frac{1}{4}(\frac{p-1}{2p-1})^{\frac{p-1}{p}}[\{\mid f'(a) \mid^p \int_0^1 (\mid \frac{1+t}{2} \mid)^s dt + \mid f'(b) \mid^p \int_0^1 (\alpha \mid \frac{1-t}{2} \mid)^s dt\}^{\frac{1}{p}} \\
&\qquad + \{\mid f'(a) \mid^p \int_0^1 (\alpha \mid \frac{1-t}{2} \mid)^s dt + \mid f'(b) \mid^p \int_0^1 (\mid \frac{1+t}{2} \mid)^s dt\}^{\frac{1}{p}}] \\
&\le \frac{1}{2}(\frac{p-1}{2p-1})^{\frac{p-1}{p}}(\mid f'(a) \mid + \mid f'(b) \mid) \le \frac{1}{2}(\mid f'(a) \mid + \mid f'(b) \mid).
\end{aligned}
$$

By using the theorem 1.6, we have the following inequalities;

Theorem 10. *Let $f : \mathbb{I} \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on the interior $\mathbb{I}^0$ of $\mathbb{I}$ such that $f' \in L([a,b])$, where $a, b \in \mathbb{I}$ with $a < b$ and $\alpha, s \in (0,1]$. If $\mid f' \mid^p$ is an α-star s-convex mapping on $\mathbb{I}= [a,b]$, then*

$$
\begin{aligned}
&(i) \mid R_f(a,b) \mid \\
&\le \frac{1}{4}(\frac{p-1}{2p-1})^{\frac{p-1}{p}}(\frac{1}{s+1})^{\frac{1}{p}}[\{\mid f'(\frac{a+b}{2}) \mid^p + \mid f'(a) \mid^p\}^{\frac{1}{p}} \\
&\qquad\qquad + \{\mid f'(b) \mid^p + \mid f'(\frac{a+b}{2}) \mid^p\}^{\frac{1}{p}}],
\end{aligned}
$$

or

$$
\mid R_f(a,b) \mid \quad \le (\frac{p-1}{2p-1})^{\frac{p-1}{p}}(\frac{1}{s+1})^{\frac{1}{p}}\{\mid f'(a) \mid^p + \mid f'(b) \mid^p\}^{\frac{1}{p}}.
$$

(ii) $\mid L_f(a,b) \mid$

$$\leq \frac{1}{4}(\frac{p-1}{2p-1})^{\frac{p-1}{p}}(\frac{1}{s+1})^{\frac{1}{p}}[\{\mid f'(\frac{a+b}{2}) \mid^p + \mid f'(a) \mid^p\}^{\frac{1}{p}} + \{\mid f'(b) \mid^p + \mid f'(\frac{a+b}{2}) \mid^p\}^{\frac{1}{p}}].$$

Proof. By (6), (7) and (8) in Lemma 2, we have:

(i) $\mid R_f(a,b) \mid$

$$\begin{aligned}
&\leq \frac{1}{4}(\frac{p-1}{2p-1})^{\frac{p-1}{p}}(\frac{2}{b-a})^{\frac{1}{p}}[\{\int_a^{\frac{a+b}{2}} \mid f'(x) \mid^p dx\}^{\frac{1}{p}} + \{\int_{\frac{a+b}{2}}^b \mid f'(x) \mid^p dx\}^{\frac{1}{p}}] \\
&\leq \frac{1}{4}(\frac{p-1}{2p-1})^{\frac{p-1}{p}}[\{\frac{\mid f'(\frac{a+b}{2}) \mid^p + \mid f'(a) \mid^p}{s+1}\}^{\frac{1}{p}} + \{\frac{\mid f'(b) \mid^p + \mid f'(\frac{a+b}{2}) \mid^p}{s+1}\}^{\frac{1}{p}}] \\
&\leq \frac{1}{4}(\frac{p-1}{2p-1})^{\frac{p-1}{p}}(\frac{1}{s+1})^{\frac{1}{p}}[\{\mid f'(\frac{a+b}{2}) \mid^p \\
&\qquad + \mid f'(a) \mid^p\}^{\frac{1}{p}} + \{\mid f'(b) \mid^p + \mid f'(\frac{a+b}{2}) \mid^p\}^{\frac{1}{p}}] \\
&\leq \frac{1}{4}[\{\mid f'(\frac{a+b}{2}) \mid^p + \mid f'(a) \mid^p\}^{\frac{1}{p}} + \{\mid f'(b) \mid^p + \mid f'(\frac{a+b}{2}) \mid^p\}^{\frac{1}{p}}],
\end{aligned}$$

or

$\mid R_f(a,b) \mid$

$$\begin{aligned}
&\leq \frac{1}{2}(\frac{p-1}{2p-1})^{\frac{p-1}{p}}[(\int_0^1 \mid f'(ta+(1-t)b) \mid^p dt)^{\frac{1}{p}} \\
&\qquad + (\int_0^1 \mid f'(tb+(1-t)a) \mid^p dt)^{\frac{1}{p}}] \\
&\leq (\frac{p-1}{2p-1})^{\frac{p-1}{p}}(\frac{1}{s+1})^{\frac{1}{p}}[\mid f'(a) \mid^p + \mid f'(b) \mid^p]^{\frac{1}{p}}.
\end{aligned}$$

(ii) $\mid L_f(a,b) \mid$

$$\begin{aligned}
&\leq \frac{1}{4}(\frac{p-1}{2p-1})^{\frac{p-1}{p}}[\{\frac{2}{b-a}\int_a^{\frac{a+b}{2}} \mid (f')^p \mid dt\}^{\frac{1}{p}} + \{\frac{2}{b-a}\int_{\frac{a+b}{2}}^b \mid (f')^p \mid dt\}^{\frac{1}{p}}] \\
&\leq \frac{1}{4}(\frac{p-1}{2p-1})^{\frac{p-1}{p}}(\frac{1}{s+1})^{\frac{1}{p}}[\{\mid f'(\frac{a+b}{2}) \mid^p \\
&\qquad + \mid f'(a) \mid^p\}^{\frac{1}{p}} + \{\mid f'(b) \mid^p + \mid f'(\frac{a+b}{2}) \mid^p\}^{\frac{1}{p}}] \\
&\leq \frac{1}{4}[\{\mid f'(\frac{a+b}{2}) \mid^p + \mid f'(a) \mid^p\}^{\frac{1}{p}} + \{\mid f'(b) \mid^p + \mid f'(\frac{a+b}{2}) \mid^p\}^{\frac{1}{p}}].
\end{aligned}$$

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