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MODIFIED MANN'S ALGORITHM BASED ON THE CQ METHOD FOR PSEUDO-CONTRACTIVE MAPPINGS

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Abstract. IIn this paper, we suggest and analyze a modified Mann's algorithm based on the CQ method for pseudo-contractive mappings in Hilbert spaces. Further, we prove a strong convergence theorem according to the proposed algorithm for pseudo-contractive mappings.

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1. Introduction

Let X be a real Banach space and C be a nonempty closed convex subset of X. Let $T : C \to C$ be a mapping. Recall that T is said to be pseudo-contraction if

$$\langle Tx - Ty, j(x-y)\rangle \leq \|x-y\|^2$$

for all $x, y \in C$. We use $F(T)$ to denote the set of fixed points of T.

Note that the class of pseudo-contractions includes the class of nonexpansive mappings which are mappings T on C such that

$$\|Tx - Ty\| \leq \|x-y\|$$

for all $x, y \in C$.

Construction of fixed points of nonexpansive mappings via Mann's algorithm has extensively been investigated in literature, please see, e.g., [1-2,6] and the references therein. Related works can also be found in [3-5,7,22-26]. Mann's algorithm generates, initializing with an arbitrary $x_0 \in C$, a sequence according to the recursive manner

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n Tx_n, \quad n \geq 0, \tag{1}$$

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where $\{\alpha_n\}$ is a real sequence in the interval $(0, 1)$. If T is a nonexpansive mapping with a fixed point and if the control sequence $\{\alpha_n\}$ is chosen so that $\sum_{n=0}^{\infty} \alpha_n(1-\alpha_n) = \infty$, then the sequence $\{x_n\}$ generated by Mann's algorithm (1) converges weakly to a fixed point of T. We note that this result is true in a uniformly convex Banach space with a Frechet differentiable norm [6].

Apart from being an important generalization of nonexpansive mappings, interest in pseudo-contractive mappings stems mainly from their firm connection with the important class of nonlinear accretive operators, where a mapping S with domain $D(S)$ and range $R(S)$ in X is called accretive if the inequality

$$\|x - y\| \leq \|x - y + r(Sx - Sy)\|,$$

holds for every $x, y \in D(S)$ and for all $r > 0$. It is well known (see, e.g.,[8]) that if S is accretive, then the solutions of the equation $Sx = 0$ correspond to the equilibrium points of some evolution systems. Consequently, considerable research efforts, especially within the past 30 years or so, have been devoted to iterative methods for approximating fixed points of T when T is pseudo-contractive (see for example [9-18] and the references therein).

Our concern now is the following: Is it possible to approximate the fixed points of a pseudo-contractive mapping T by Mann's algorithm (1)? In this connection, in 2001, Chidume and Mutangadura [19] have given an example of a Lipschitz pseudo-contractive self-mapping of a compact convex subset of a Hilbert space with a unique fixed point for which no Mann sequence converges and they stated there "This resolves a long standing open problem".

In order to get strong convergence for k-strict pseudo-contractive mapping T (Recall that $T : C \to C$ is said to be a strict pseudo-contraction if there exists a constant $0 \leq k < 1$ such that $\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|(I - T)x - (I - T)y\|^2$ for all $x, y \in C$), Marino and Xu [20] proposed the following modified Mann's algorithm: Let C be a closed convex subset of a real Hilbert space H and $T : C \to C$ a k-strict pseudo-contractive mapping such that $F(T) \neq \emptyset$, let $\{x_n\}$ be the sequence generated by

$$\begin{cases} x_0 \in C \ \text{ chosen arbitrarily,} \\ y_n = \alpha_n x_n + (1 - \alpha_n)Tx_n, \\ C_n = \{z \in C : \|y_n - z\|^2 \leq \|x_n - z\|^2 + (1 - \alpha_n)(k - \alpha_n)\|x_n - Tx_n\|^2\}, \\ Q_n = \{z \in C : \langle x_n - z, x_0 - x_n\rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_0, \end{cases} \tag{2}$$

which is referred to as the CQ method which has been studied by many authors. Marino and Xu [20] proved that the sequence $\{x_n\}$ generated by (2) converges strongly to $P_{F(T)}x_0$ under mild assumptions on parameter. At this point, we give the following remarks.

Remark 1.1. (i) Without the demicompactness condition imposed on $I - T$, Marino and Xu still obtained strong convergence of the sequence $\{x_n\}$ generated by the modified Mann's algorithm (2) by projecting the initial guess x_0 onto the intersection of two appropriately constructed closed convex subsets C_n and Q_n.

(ii) A natural problem rises: Is it possible to construct a modified Mann's algorithm to approximate the fixed points of pseudo-contractive mappings?

It is the purpose of this paper to suggest and analyze a modified Mann's algorithm based on the CQ method for pseudo-contractive mappings in Hilbert spaces. Further, we prove a strong convergence theorem according to the proposed algorithm for pseudo-contractive mappings.

2. Preliminaries

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$ and let C be a closed convex subset of H. For every point $x \in H$ there exists a unique nearest point in C, denoted by $P_C x$, such that

$$\|x - P_C x\| \leq \|x - y\| \text{ for all } y \in C,$$

where P_C is called the metric projection of H onto C. We know that P_C is a nonexpansive mapping. It is also known that H satisfies Opial's condition, i.e., for any sequence $\{x_n\}$ with $x_n \to x$ weakly the inequality

$$\liminf_{n \to \infty} \|x_n - x\| < \liminf_{n \to \infty} \|x_n - y\|$$

holds for every $y \in H$ with $y \neq x$.

For given sequence $\{x_n\} \subset C$, let $\omega_w(x_n) = \{x : \exists x_{n_j} \to x \text{ weakly}\}$ denote the weak ω-limit set of $\{x_n\}$.

Now we collects some lemmas which will be used in the proof of the main result in the next section. We note that Lemmas 2.1 and 2.2 are well-known.

Lemma 2.1. *Let H ba a real Hilbert space. There holds the following identity*

$$\|x - y\|^2 = \|x\|^2 - \|y\|^2 - 2\langle x - y, y \rangle \quad \forall x, y \in H.$$

Lemma 2.2. *Let C be a closed convex subset of real Hilbert space H. Given $x \in H$ and $z \in C$. Then $z = P_C x$ if and only if there holds the relation*

$$\langle x - z, y - z \rangle \leq 0 \quad \textit{for all } y \in C.$$

Lemma 2.3 ([4])**.** *Let C be a closed convex subset of H. Let $\{x_n\}$ be a sequence in H and $u \in H$. Let $q = P_C u$. If $\{x_n\}$ is such that $\omega_w(x_n) \subset C$ and satisfies the condition*

$$\|x_n - u\| \leq \|u - q\| \textit{ for all } n.$$

Then $x_n \to q$.

Lemma 2.4 ([21])**.** *Let X be a real reflexive Banach space which satisfies Opial's condition. Let C be a nonempty closed convex subset of X, and $T : C \to C$ be a continuous pseudo-contractive mapping. Then, $I - T$ is demiclosed at zero.*

3. Main results

Let C be a closed convex subset of a real Hilbert space H. Let $T : C \to C$ be a pseudo-contractive mapping. Now we suggest the following form of the modified Mann's algorithm based on the CQ method for pseudo-contractive mappings:

$$\begin{cases} x_0 \in C \text{ chosen arbitrarily}, \\ y_n = (1-\alpha_n)x_n + \alpha_n T x_n, \\ C_n = \{z \in C : \|\alpha_n(y_n - Ty_n)\|^2 \le 2\alpha_n \langle x_n - z, y_n - Ty_n\rangle\}, \\ Q_n = \{z \in C : \langle x_n - z, x_0 - x_n\rangle \ge 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_0, \end{cases} \tag{3}$$

where $\{\alpha_n\}$ is a real sequence in $(0, 1)$.

Now we state and prove our main result as follows.

Theorem 3.1. *Let C be a closed convex subset of a real Hilbert space H. Let $T : C \to C$ be a L-Lipschitz pseudo-contractive mapping such that $F(T) \neq \emptyset$. Assume the sequence $\alpha_n \in [a, b]$ for some $a, b \in (0, \frac{1}{L+1})$. Then the sequences $\{x_n\}$ and $\{y_n\}$ generated by (3) converge strongly to $P_{F(T)}x_0$.*

Proof. First, from [21] and [20], we have that $F(T)$ is closed and convex. Indeed, by [21], we can define a mapping $g : C \to C$ by $g(x) = (2I - T)^{-1}$ for every $x \in C$. It is clear that g is a nonexpansive self-mapping such that $F(T) = F(g)$. Hence, by [20, Proposition 2.1 (iii)], we conclude that $F(g) = F(T)$ is a closed convex set. This implies that the projection $P_{F(T)}$ is well defined. It is obvious that C_n is convex. Next, we show that $F(T) \subset C_n$ for all n.

Setting $A = I - T$. Taking $p \in F(T)$, we note that $Ap = 0$ and

$$\langle Ax - Ay, x - y\rangle \ge 0$$

for all $x, y \in C$.

Using Lemma 2.1, we obtain

$$\begin{aligned} \|x_n - p - \alpha_n Ay_n\|^2 &= \|x_n - p\|^2 - \|\alpha_n Ay_n\|^2 - 2\alpha_n\langle Ay_n, x_n - p - \alpha_n Ay_n\rangle \\ &= \|x_n - p\|^2 - \|\alpha_n Ay_n\|^2 - 2\alpha_n\langle Ay_n - Ap, y_n - p\rangle \\ &\quad - 2\alpha_n\langle Ay_n, x_n - y_n - \alpha_n Ay_n\rangle \\ &\le \|x_n - p\|^2 - \|\alpha_n Ay_n\|^2 - 2\alpha_n\langle Ay_n, x_n - y_n - \alpha_n Ay_n\rangle \\ &= \|x_n - p\|^2 - \|x_n - y_n + y_n - x_n + \alpha_n Ay_n\|^2 \\ &\quad - 2\alpha_n\langle Ay_n, x_n - y_n - \alpha_n Ay_n\rangle \\ &= \|x_n - p\|^2 - \|x_n - y_n\|^2 - \|y_n - x_n + \alpha_n Ay_n\|^2 \\ &\quad - 2\langle x_n - y_n, y_n - x_n + \alpha_n Ay_n\rangle \\ &\quad + 2\alpha_n\langle Ay_n, y_n - x_n + \alpha_n Ay_n\rangle \\ &= \|x_n - p\|^2 - \|x_n - y_n\|^2 - \|y_n - x_n + \alpha_n Ay_n\|^2 \\ &\quad - 2\langle x_n - y_n - \alpha_n Ay_n, y_n - x_n + \alpha_n Ay_n\rangle. \end{aligned} \tag{4}$$

Since T is L-Lipschitz continuous mapping, we have

$$\begin{aligned} \|Ax - Ay\| &\le \|x - y\| + \|Tx - Ty\| \\ &\le (L+1)\|x - y\|. \end{aligned} \tag{5}$$

From (3), we observe that $x_n - y_n = \alpha_n A x_n$. Hence, from (5), we obtain

$$\begin{aligned} \langle x_n - y_n - \alpha_n A y_n, &y_n - x_n + \alpha_n A y_n\rangle \\ &= \alpha_n \langle A x_n - A y_n, y_n - x_n + \alpha_n A y_n\rangle \\ &\le \alpha_n \|A x_n - A y_n\| \|y_n - x_n + \alpha_n A y_n\| \\ &\le \alpha_n (L+1)\|x_n - y_n\| \|y_n - x_n + \alpha_n A y_n\| \\ &\le \frac{\alpha_n(L+1)}{2}(\|x_n - y_n\|^2 + \|y_n - x_n + \alpha_n A y_n\|^2). \end{aligned} \tag{6}$$

Combining (4) and (6), we get

$$\begin{aligned} \|x_n - p - \alpha_n A y_n\|^2 &\le \|x_n - p\|^2 - \|x_n - y_n\|^2 - \|y_n - x_n + \alpha_n A y_n\|^2 \\ &\quad + \alpha_n(L+1)(\|x_n - y_n\|^2 + \|y_n - x_n + \alpha_n A y_n\|^2) \\ &= \|x_n - p\|^2 + [\alpha_n(L+1) - 1](\|x_n - y_n\|^2 \\ &\quad + \|y_n - x_n + \alpha_n A y_n\|^2) \\ &\le \|x_n - p\|^2. \end{aligned} \tag{7}$$

At the same time,

$$\|x_n - p - \alpha_n A y_n\|^2 = \|x_n - p\|^2 - 2\alpha_n \langle x_n - p, A y_n\rangle + \|\alpha_n A y_n\|^2. \tag{8}$$

Therefore, from (7) and (8), we have

$$\|\alpha_n A y_n\|^2 \le 2\alpha_n \langle x_n - p, A y_n\rangle,$$

which implies that

$$p \in C_n,$$

that is to say,

$$F(T) \subset C_n \quad \text{for all} \quad n.$$

Next, we show that

$$F(T) \subset Q_n \quad \text{for all} \quad n. \tag{9}$$

We prove this by induction. For $n = 0$, we have $F(T) \subset C = Q_0$. Assume that $F(T) \subset Q_n$. Since x_{n+1} is the projection of x_0 onto $C_n \cap Q_n$, by Lemma 2.2 we have

$$\langle x_{n+1} - z, x_0 - x_{n+1}\rangle \ge 0 \quad \forall z \in C_n \cap Q_n.$$

As $F(T) \subset C_n \cap Q_n$ by the induction assumption, the last inequality holds, in particular, for all $z \in F(T)$. This together with the definition of Q_{n+1} implies that $F(T) \subset Q_{n+1}$. Hence (9) holds for all n.

From the definition of Q_n, we note that $x_n = P_{Q_n} x_0$. Since $F(T) \subset Q_n$, we have $\|x_n - x_0\| \le \|z - x_0\|$ for all $z \in F(T)$. In particular, x_n is bounded and

$$\|x_n - x_0\| \le \|q - x_0\|, \quad \text{where} \quad q = P_{F(T)} x_0. \tag{10}$$

The fact that $x_{n+1} \in Q_{n+1}$ implies that $\langle x_{n+1} - x_n, x_n - x_0 \rangle \geq 0$ and at the same time we note that $\|x_n - x_0\| \leq \|x_{n+1} - x_0\|$ which implies that $\lim_{n\to\infty} \|x_n - x_0\|$ exists. Hence, applying Lemma 2.1, we obtain

$$\begin{aligned}\|x_{n+1} - x_n\|^2 &= \|(x_{n+1} - x_0) - (x_n - x_0)\|^2 \\ &= \|x_{n+1} - x_0\|^2 - \|x_n - x_0\|^2 - 2\langle x_{n+1} - x_n, x_n - x_0 \rangle \\ &\leq \|x_{n+1} - x_0\|^2 - \|x_n - x_0\|^2 \to 0.\end{aligned}$$

Since $x_{n+1} \in C_n$, we have

$$\begin{aligned}\|\alpha_n(y_n - Ty_n)\|^2 &\leq 2\alpha_n \langle x_n - x_{n+1}, y_n - Ty_n \rangle \\ &\leq 2\alpha_n \|x_n - x_{n+1}\| \|y_n - Ty_n\| \to 0,\end{aligned}$$

this implies that

$$\|y_n - Ty_n\| \to 0.$$

We note that

$$\begin{aligned}\|x_n - Tx_n\| &\leq \|x_n - y_n\| + \|y_n - Ty_n\| + \|Ty_n - Tx_n\| \\ &\leq (L+1)\|x_n - y_n\| + \|y_n - Ty_n\| \\ &\leq \alpha_n(L+1)\|x_n - Tx_n\| + \|y_n - Ty_n\|,\end{aligned}$$

i.e.,

$$\|x_n - Tx_n\| \leq \frac{1}{1 - \alpha_n(L+1)} \|y_n - Ty_n\| \to 0. \tag{11}$$

Now (11) and Lemma 2.4 guarantee that every weak limit point of $\{x_n\}$ is a fixed point of T. That is, $\omega_w(x_n) \subset F(T)$. This fact, the inequality (10) and Lemma 2.3 ensure the strong convergence of $\{x_n\}$ to $P_{F(T)}x_0$. Note that

$$\|y_n - x_n\| \leq \alpha_n \|x_n - Tx_n\| \to 0.$$

Consequently, $y_n \to P_{F(T)}x_0$. This completes the proof. □

As direct consequence of Theorem 3.1, we obtain the following.

Corollary 3.2. *Let C be a closed convex subset of a real Hilbert space H. Let $T : C \to C$ be a nonexpansive mapping such that $F(T) \neq \emptyset$. Assume the sequence $\alpha_n \in [a,b]$ for some $a, b \in (0, \frac{1}{2})$. Then the sequences $\{x_n\}$ generated by (3) converges strongly to $P_{F(T)}x_0$.*

Remark 3.3. Without the demicompactness condition imposed on $I - T$, we also obtain strong convergence of the sequence $\{x_n\}$ generated by the modified Mann's algorithm (3) by projecting the initial guess x_0 onto the intersection of two appropriately constructed closed convex subsets C_n and Q_n.

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