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CONVERGENCE OF TWO-STAGE MULTISPLITTING AND ILU-MULTISPLITTING METHODS WITH PREWEIGHTING

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Abstract. In this paper, we study convergence of both two-stage multisplitting method with preweighting and ILU-multisplitting method with preweighting for solving a linear system.

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1. Introduction

In this paper, we consider two-stage multisplitting and ILU-multisplitting methods with preweighting for solving a linear system of the form

$$Ax = b, \quad x, b \in \mathbb{R}^n, \tag{1}$$

where $A \in \mathbb{R}^{n\times n}$ is a monotone matrix or an H-matrix.

For a vector $x \in \mathbb{R}^n$, $x \geq 0$ ($x > 0$) denotes that all components of x are nonnegative (positive), and $|x|$ denotes the vector whose components are the absolute values of the corresponding components of x. For two vectors x, $y \in \mathbb{R}^n$, $x \geq y$ ($x > y$) means that $x - y \geq 0$ ($x - y > 0$). These definitions carry immediately over to matrices. For a square matrix A, $\text{diag}(A)$ denotes a diagonal matrix whose diagonal part coincides with the diagonal part of A. Let $\rho(A)$ denote the *spectral radius* of a square matrix A.

A matrix $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ is called a *Z-matrix* if $a_{ij} \leq 0$ for $i \neq j$. A matrix $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ is called *monotone* if A is nonsingular and $A^{-1} \geq 0$. A matrix $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ is called an *M-matrix* if it is a monotone Z-matrix.

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The *comparison matrix* $\langle A\rangle = (\alpha_{ij})$ of a matrix $A = (a_{ij})$ is defined by

$$\alpha_{ij} = \begin{cases} |a_{ij}| & \text{if } i = j \\ -|a_{ij}| & \text{if } i \neq j \end{cases}.$$

A matrix A is called an *H-matrix* if $\langle A\rangle$ is an M-matrix.

A representation $A = M - N$ is called a *splitting* of A if M is nonsingular. A splitting $A = M - N$ is called *regular* if $M^{-1} \geq 0$ and $N \geq 0$, *the first type weak regular* if $M^{-1} \geq 0$ and $M^{-1}N \geq 0$, and *the second type weak regular* if $M^{-1} \geq 0$ and $NM^{-1} \geq 0$. A splitting $A = M - N$ is called *convergent* if $\rho(M^{-1}N) < 1$. It is well known that if $A = M - N$ is the first type weak regular splitting of A, then $\rho(M^{-1}N) < 1$ if and only if $A^{-1} \geq 0$ [10]. A splitting $A = M - N$ is called an *H-compatible splitting* of A if $\langle A\rangle = \langle M\rangle - |N|$. It was shown in [5] that if A is an H-matrix and $A = M - N$ is an H-compatible splitting of A, then $\rho(M^{-1}N) < 1$. A collection of triples $(M_k, N_k, E_k), k = 1, 2, \ldots, \ell$, is called a *multisplitting* of A if $A = M_k - N_k$ is a splitting of A for $k = 1, 2, \ldots, \ell$, and E_k's are nonnegative diagonal matrices such that $\sum_{k=1}^{\ell} E_k = I$.

This paper is organized as follows. In Section 2, we study convergence of two-stage multisplitting methods with preweighting for solving the linear system (1). In Section 3, we study convergence of ILU-multisplitting method with preweighting for solving the linear system (1).

2. Two-stage multisplitting method with preweighting

In this section, we study convergence of two-stage multisplitting method with preweighting. Let $(M_k, N_k, E_k),\ k = 1, 2, \cdots, \ell$, be a multisplitting of A. Given a parameter $\lambda \in [0, 1]$ and an initial vector x_0, the corresponding multisplitting iteration method (depending on λ) for solving the linear system (1) is [8]

$$\begin{aligned} x_{i+1} &= H_\lambda x_i + G_\lambda b \\ &= x_i + G_\lambda(b - Ax_i),\ i = 0, 1, 2, \cdots, \end{aligned} \tag{2}$$

where

$$G_\lambda = \sum_{k=1}^{\ell} {E_k}^{\lambda} {M_k}^{-1} {E_k}^{1-\lambda} \ \text{ and } \ H_\lambda = I - G_\lambda A. \tag{3}$$

Here, ${E_k}^{\lambda}$ denotes the diagonal matrix obtained from E_k by replacing all diagonal entries by their λ-th power when $\lambda \neq 0$, and ${E_k}^0 := I$.

The case $\lambda = 1$ is called the *multisplitting method with postweighting* which is usually called the multisplitting method and has been extensively studied in the literature, see [2, 3, 4, 7, 9, 11, 13, 14]. The case $\lambda = 0$ is called the *multisplitting method with preweighting*. In certain situations, it was shown that $\rho(H_\lambda)$ is an increasing function of λ, which means that *preweighting technique yields the fastest method* [12].

If $\lambda = 0$ in (3), then $H_0 = I - \sum_{k=1}^{\ell} {M_k}^{-1} E_k A$ is an iteration matrix for the multisplitting method with preweighting. If $\lambda = 1$ in (3), then $H_1 =$

$I - \sum_{k=1}^{\ell} E_k {M_k}^{-1} A$ is an iteration matrix for the multisplitting method with postweighting. By simple calculation, one obtains

$$H_0{}^T = A^T \left(I - \sum_{k=1}^{\ell} E_k ({M_k}^T)^{-1} A^T \right) (A^T)^{-1},$$

$$I - \sum_{k=1}^{\ell} E_k ({M_k}^T)^{-1} A^T = \sum_{k=1}^{\ell} E_k ({M_k}^T)^{-1} {N_k}^T.$$

Let $\hat{H}_1 = \sum_{k=1}^{\ell} E_k \left({M_k}^T\right)^{-1} {N_k}^T$. Then it can be seen that $H_0{}^T$ is similar to $\hat{H}_1$. It follows that

$$\rho(H_0) = \rho(H_0{}^T) = \rho(\hat{H}_1).$$

Notice that $\hat{H}_1$ is an iteration matrix for the multisplitting method corresponding to a multisplitting $({M_k}^T, {N_k}^T, E_k)$, $k = 1, 2, \cdots, \ell$, of A^T. Hence, convergence results for multisplitting method with postweighting carry over to those for multisplitting method with preweighitng. In other words,

$$\rho(H_0) < 1 \text{ if and only if } \rho(\hat{H}_1) < 1.$$

The multisplitting method with preweighting associated with a multisplitting (M_k, N_k, E_k), $k = 1, 2, \cdots, \ell$, of A for solving the linear system (1) is as follows:

Algorithm 1: Multisplitting method with preweighting
Given an initial vector x_0
For $i = 0, 1, \ldots,$ until convergence
  For $k = 1$ to ℓ {parallel execution}
    $M_k y_k = E_k(b - Ax_i)$
  $x_{i+1} = x_i + \sum_{k=1}^{\ell} y_k$

The big advantage of Algorithm 1 is that the loop k can be executed completely in parallel by different processors. When the linear systems in Algorithm 1 are also solved iteratively in each processor using the splittings $M_k = F_k - G_k$, one obtains the following *two-stage multisplitting method with preweighting*.

Algorithm 2: Two-stage multisplitting method with preweighting
Given an initial vector x_0
For $i = 0, 1, \ldots,$ until convergence
  For $k = 1$ to ℓ {parallel execution}
    $y_{k,0} = x_{i-1}$
    For $j = 1$ to s
      $F_k y_{k,j} = G_k y_{k,j-1} + E_k(b - Ax_i)$
  $x_{i+1} = x_i + \sum_{k=1}^{\ell} y_{k,s}$

Now we further consider two-stage multisplitting method with preweighting (Algorothm 2). For $k = 1, 2, \cdots, \ell$, let

$$T_k = \left(F_k{}^{-1}G_k\right)^p + \sum_{j=0}^{p-1}\left(F_k{}^{-1}G_k\right)^j F_k{}^{-1}N_k,$$

$$P_k = \sum_{j=0}^{p-1}\left(F_k{}^{-1}G_k\right)^j F_k{}^{-1} = \left(I - (F_k{}^{-1}G_k)^p\right) M_k{}^{-1}.$$

If $\rho(F_k{}^{-1}G_k) < 1$ or $\rho(T_k) < 1$ for $k = 1, 2, \cdots, \ell$, then

$$A = B_k - B_k T_k,$$

where $B_k = P_k{}^{-1} = M_k\left(I - (F_k{}^{-1}G_k)^p\right)^{-1}$. Hence, two-stage multisplitting method with preweighting (Algorithm 2) can be written as

$$x_{i+1} = H_0 x_i + G_0 b,\ i = 0, 1, 2\ldots,$$

where

$$G_0 = \sum_{k=1}^{\ell} B_k{}^{-1}E_k \text{ and } H_0 = I - G_0 A = I - \sum_{k=1}^{\ell} B_k{}^{-1}E_k A. \tag{4}$$

Form equation (4), one obtains

$$H_0{}^T = I - A^T \sum_{k=1}^{\ell} E_k (B_k{}^{-1})^T = A^T\left(I - \sum_{k=1}^{\ell} E_k (B_k{}^{-1})^T A^T\right) A^{-T}.$$

Let $\tilde{H}_1 = I - \sum_{k=1}^{\ell} E_k (B_k{}^{-1})^T A^T$. Then $H_0{}^T$ is similar to $\tilde{H}_1$. Hence,

$$\rho(H_0) = \rho(H_0{}^T) = \rho(\tilde{H}_1).$$

By simple calculation, one obtains

$$(B_k{}^{-1})^T = \left(\sum_{j=0}^{p-1}(F_k{}^{-1}G_k)^j F_k{}^{-1}\right)^T = \sum_{j=0}^{p-1}(F_k{}^{-T}G_k{}^T)^j F_k{}^{-T},$$

$$\tilde{H}_1 = \sum_{k=1}^{\ell} E_k\left((F_k{}^{-T}G_k{}^T)^p + \sum_{j=0}^{p-1}(F_k{}^{-T}G_k{}^T)^j F_k{}^{-T}N_k{}^T\right).$$

From these equalities, it can be seen that the matrix $\tilde{H}_1$ is an iteration matrix of two-stage multisplitting method corresponding to outer splittings $A^T = M_k{}^T - N_k{}^T$ and inner splittings $M_k{}^T = F_k{}^T - G_k{}^T$ for $k = 1, 2, \ldots, \ell$.

The following theorem show the well known result for convergence of two-stage multisplitting method with postweighting when A is a monotone matrix.

Theorem 2.1 ([9])**.** *Let $A^{-1} \geq 0$, $A = M_k - N_k$ be a regular splitting of A and $M_k = F_k - G_k$ be the first type weak regular splitting of M_k for $k = 1, 2 \ldots, \ell$. Then $\rho(H_1) < 1$, where*

$$H_1 = \sum_{k=1}^{\ell} E_k T_k \ \ \textit{and} \ \ T_k = ({F_k}^{-1} G_k)^p + \sum_{j=0}^{p-1} ({F_k}^{-1} G_k)^j {F_k}^{-1} N_k.$$

The following lemma provides a convergence result of two-stage multisplitting method corresponding to outer splittings $A^T = {M_k}^T - {N_k}^T$ and inner splittings ${M_k}^T = {F_k}^T - {G_k}^T$ for $k = 1, 2, \ldots, \ell$.

Lemma 2.2. *Let $A^{-1} \geq 0$, $A = M_k - N_k$ be a regular splitting of A and $M_k = F_k - G_k$ be the second type weak regular splitting of M_k for $k = 1, 2 \ldots, \ell$. Then $\rho(\tilde{H}_1) < 1$, where*

$$\tilde{H}_1 = \sum_{k=1}^{\ell} E_k \left(({F_k}^{-T} {G_k}^T)^p + \sum_{j=0}^{p-1} ({F_k}^{-T} {G_k}^T)^j {F_k}^{-T} {N_k}^T \right).$$

Proof. Since $A = M_k - N_k$ is a regular splitting, we easily obtain that for $k = 1, 2, \ldots, \ell$,

$$A^T = {M_k}^T - {N_k}^T, \ \ ({M_k}^T)^{-1} = ({M_k}^{-1})^T \geq 0 \ \text{ and } \ {N_k}^T \geq 0.$$

Hence, $A^T = {M_k}^T - {N_k}^T$ is a regular splitting. Since $M_k = F_k - G_k$ is the first type weak regular splitting for $k = 1, 2, \ldots, \ell$,

$$\begin{gathered} {M_k}^T = {F_k}^T - {G_k}^T, \ \ ({F_k}^T)^{-1} = ({F_k}^{-1})^T \geq 0, \\ ({F_k}^T)^{-1} {G_k}^T = ({F_k}^{-1})^T {G_k}^T = (G_k {F_k}^{-1})^T \geq 0. \end{gathered}$$

Hence, ${M_k}^T = {F_k}^T - {G_k}^T$ is the first type weak regular splitting. Notice that $\tilde{H}_1$ is an iteration matrix of two-stage multisplitting method corresponding to outer splittings $A^T = {M_k}^T - {N_k}^T$ and inner splittings ${M_k}^T = {F_k}^T - {G_k}^T$ for $k = 1, 2, \ldots, \ell$. From Theorem 2.1, $\rho(\tilde{H}_1) < 1$. □

The following theorem provides a convergence result of two-stage multisplitting method with preweighting when A is a monotone matrix.

Theorem 2.3. *Let $A^{-1} \geq 0$, $A = M_k - N_k$ be a regular splitting of A and $M_k = F_k - G_k$ be the second type weak regular splitting of M_k for $k = 1, 2 \ldots, \ell$. Then $\rho(H_0) < 1$, where $H_0 = I - \sum_{k=1}^{\ell} {B_k}^{-1} E_k A$ and $B_k = M_k \left(I - ({F_k}^{-1} G_k)^p\right)^{-1}$.*

Proof. Let $\tilde{H}_1 = I - \sum_{k=1}^{\ell} E_k ({B_k}^{-1})^T A^T$. Since $\tilde{H}_1$ is similar to ${H_0}^T$, $\rho(\tilde{H}_1) = \rho(H_0)$. From Lemma 2.2, $\rho(\tilde{H}_1) < 1$. Therefore, $\rho(H_0) < 1$. □

Note that if $A = M - N$ is a regular splitting, then $A = M - N$ is the second type weak regular splitting. Hence, the following corollary is obtained.

Corollary 2.4. *Let $A^{-1} \geq 0$. If $A = M_k - N_k$ and $M_k = F_k - G_k$ are regular splittings for $k = 1, 2 \ldots, \ell$, then $\rho(H_0) < 1$, where*

$$H_0 = I - \sum_{k=1}^{\ell} {B_k}^{-1} E_k A \text{ and } B_k = M_k \left(I - ({F_k}^{-1} G_k)^p \right)^{-1}.$$

The following theorem show the well known result for convergence of two-stage multisplitting method with postweighting when A is an H-matrix.

Theorem 2.5 ([1])**.** *Let A be an H-matrix. If $A = M_k - N_k$ and $M_k = F_k - G_k$ are H-compatible splittings for $k = 1, 2 \ldots, \ell$, then $\rho(H_1) < 1$, where*

$$H_1 = \sum_{k=1}^{\ell} E_k T_k \text{ and } T_k = ({F_k}^{-1} G_k)^p + \sum_{j=0}^{p-1} ({F_k}^{-1} G_k)^j {F_k}^{-1} N_k.$$

Lemma 2.6. *Let A be an H-matrix. If $A = M_k - N_k$ and $M_k = F_k - G_k$ are H-compatible splittings for $k = 1, 2 \ldots, \ell$, then $\rho(\tilde{H}_1) < 1$, where*

$$\tilde{H}_1 = \sum_{k=1}^{\ell} E_k \left(({F_k}^{-T} {G_k}^T)^p + \sum_{j=0}^{p-1} ({F_k}^{-T} {G_k}^T)^j {F_k}^{-T} {N_k}^T \right).$$

Proof. Since $A = M_k - N_k$ is an H-compatible splitting, we easily obtain that for $k = 1, 2, \ldots, \ell$,

$$\langle A^T \rangle = \langle A \rangle^T = (\langle M_k \rangle - |N_k|)^T = \langle M_k \rangle^T - |N_k|^T = \langle {M_k}^T \rangle - |{N_k}^T|.$$

Hence, $A^T = {M_k}^T - {N_k}^T$ is an H-compatible splitting. Since $M_k = F_k - G_k$ is H-compatible splitting for $k = 1, 2, \ldots, \ell$,

$$\langle {M_k}^T \rangle = \langle M_k \rangle^T = (\langle F_k \rangle - |G_k|)^T = \langle F_k \rangle^T - |G_k|^T = \langle {F_k}^T \rangle - |{G_k}^T|.$$

Hence, ${M_k}^T = {F_k}^T - {G_k}^T$ is H-compatible splitting. Notice that $\tilde{H}_1$ is an iteration matrix of two-stage multisplitting method corresponding to outer splittings $A^T = {M_k}^T - {N_k}^T$ and inner splittings ${M_k}^T = {F_k}^T - {G_k}^T$ for $k = 1, 2, \ldots, \ell$. From Theorem 2.5, $\rho(\tilde{H}_1) < 1$. □

The following theorem provides a convergence result of two-stage multisplitting method with preweighting when A is an H-matrix.

Theorem 2.7. *Let A be an H-matrix. If $A = M_k - N_k$ and $M_k = F_k - G_k$ are H-compatible splittings for $k = 1, 2 \ldots, \ell$, then $\rho(H_0) < 1$, where $H_0 = I - \sum_{k=1}^{\ell} {B_k}^{-1} E_k A$ and $B_k = M_k \left(I - ({F_k}^{-1} G_k)^p \right)^{-1}$.*

Proof. Let $\tilde{H}_1 = I - \sum_{k=1}^{\ell} E_k ({B_k}^{-1})^T A^T$. Since $\tilde{H}_1$ is similar to ${H_0}^T$, $\rho(\tilde{H}_1) = \rho(H_0)$. From Lemma 2.6, $\rho(\tilde{H}_1) < 1$. Therefore, $\rho(H_0) < 1$. □

3. Convergence of ILU-multisplitting method with preweighting

In this section, we study convergence of ILU-multisplitting method with preweighting. Let S_n denote the set of all pairs of indices of off-diagonal matrix entries, that is,

$$S_n = \{(i,j) \mid i \neq j,\ 1 \leq i \leq n,\ 1 \leq j \leq n\}.$$

The following theorem shows the existence of the ILU factorization for an H-matrix A.

Theorem 3.1 ([6]). *Let A be an $n \times n$ H-matrix. Then, for every zero pattern set $Q \subset S_n$, there exist a unit lower triangular matrix $L = (l_{ij})$, an upper triangular matrix $U = (u_{ij})$, and a matrix $N = (n_{ij})$, with $l_{ij} = u_{ij} = 0$ if $(i,j) \in Q$ and $n_{ij} = 0$ if $(i,j) \notin Q$, such that $A = LU - N$. Moreover, the factors L and U are also H-matrices.*

In Theorem 3.1, $A = LU - N$ is called an *ILU factorization* of A corresponding to a zero pattern set $Q \subset S_n$. The following theorem shows the relations between the ILU factorization of an H-matrix A and its comparison matrix $\langle A \rangle$.

Theorem 3.2 ([15]). *Assume that A is an $n \times n$ H-matrix. Let $A = LU - N$ and $\langle A \rangle = \tilde{L}\tilde{U} - \tilde{N}$ be the ILU factorizations of A and $\langle A \rangle$ corresponding to a zero pattern set $Q \subset S_n$, respectively. Then each of the following holds:*

(a) $|L^{-1}| \leq \tilde{L}^{-1}$, (b) $|U^{-1}| \leq \tilde{U}^{-1}$,

(c) $|N| \leq \tilde{N}$, (d) $|(LU)^{-1}N| \leq (\tilde{L}\tilde{U})^{-1}\tilde{N}$.

Let A be an $n \times n$ H-matrix and $A = L_kU_k - N_k$ be the ILU-factorizations of A corresponding to a zero pattern set $Q_k \subset S_n$, $k = 1, 2, \cdots, \ell$. Then (L_kU_k, N_k, E_k), $k = 1, 2, \cdots, \ell$, is a multisplitting of A. Given an initial vector x_0, the corresponding multisplitting method with preweighting for solving $Ax = b$ is

$$x_{i+1} = H_0 x_i + G_0 b, \quad i = 0, 1, 2, \cdots,$$

where

$$G_0 = \sum_{k=1}^{\ell} (L_kU_k)^{-1}E_k \ \text{ and } \ H_0 = I - G_0 A.$$

The following theorem provides a convergence result of ILU-multisplitting method with preweighting when A is an H-matrix.

Theorem 3.3. *Let A be an $n \times n$ H-matrix and $A = L_kU_k - N_k$ be the ILU factorizations of A corresponding to a zero pattern set $Q_k \subset S_n$, $k = 1, 2, \cdots, \ell$. Then $\rho(H_0) < 1$, where $H_0 = I - \sum_{k=1}^{\ell} (L_kU_k)^{-1}E_kA$.*

Proof. Let $\langle A \rangle = \tilde{L}_k\tilde{U}_k - \tilde{N}_k$ be the ILU factorizations of $\langle A \rangle$ corresponding to a zero pattern set $Q_k \subset S_n$ for $k = 1, 2, \cdots, \ell$. Then for $k = 1, 2, \cdots, \ell$,

$$\langle A^T \rangle = \tilde{U}_k^T \tilde{L}_k^T - \tilde{N}_k^T.$$

Since $\tilde{L}_k^{-1} \geq 0$, $\tilde{U}_k^{-1} \geq 0$ and $\tilde{N}_k \geq 0$, $\langle A \rangle = \tilde{L}_k \tilde{U}_k - \tilde{N}_k$ is a regular splitting of $\langle A \rangle$ and $\langle A^T \rangle = \tilde{U}_k^T \tilde{L}_k^T - \tilde{N}_k^T$ is also a regular splitting of $\langle A^T \rangle$ for each $k = 1, 2, \cdots, \ell$. Let $\tilde{H}_1 = I - \sum_{k=1}^{\ell} E_k \left(\tilde{U}_k^T \tilde{L}_k^T \right)^{-1} \langle A^T \rangle$. Notice that $\tilde{H}_1$ is an iteration matrix for the multisplitting method corresponding to a multisplitting $(\tilde{U}_k^T \tilde{L}_k^T, \tilde{N}_k^T, E_k)$, $k = 1, 2, \cdots, \ell$, of $\langle A^T \rangle$. Since $\langle A^T \rangle^{-1} \geq 0$,

$$\rho(\tilde{H}_1) < 1. \tag{5}$$

Let $\hat{H}_1 = I - \sum_{k=1}^{\ell} E_k ({U_k}^T {L_k}^T)^{-1} A^T$. Then $\hat{H}_1$ is similar to ${H_0}^T$. Hence, $\rho(H_0) = \rho(\hat{H}_1)$. From Theorem 3.2, one obtains

$$|{L_k}^{-1}| \leq \tilde{L_k}^{-1}, \ |{U_k}^{-1}| \leq \tilde{U_k}^{-1}, \ |N_k| \leq \tilde{N}_k$$

for $k = 1, 2, \cdots, \ell$. Using these inequalities,

$$|\hat{H}_1| = |\sum_{k=1}^{\ell} E_k \left({U_k}^T {L_k}^T \right)^{-1} {N_k}^T| \leq \sum_{k=1}^{\ell} E_k \left(\tilde{U}_k^T \tilde{L}_k^T \right)^{-1} \tilde{N_k}^T = \tilde{H}_1. \tag{6}$$

From (5) and (6), $\rho(\hat{H}_1) < 1$. Hence, $\rho(H_0) < 1$ is obtained. □

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