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AN EXACT LOGARITHMIC-EXPONENTIAL MULTIPLIER PENALTY FUNCTION

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ABSTRACT. In this paper, we give a solving approach based on a logarithmic-exponential multiplier penalty function for the constrained minimization problem. It is proved exact in the sense that the local optimizers of a nonlinear problem are precisely the local optimizers of the logarithmic-exponential multiplier penalty problem.

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1. Introduction

Consider the following constrained minimization problem

$$\begin{aligned} &\min f(x) \\ &s.t.\ g_i(x) \le 0,\ i = 1, \cdots, m, \\ &x \in R^n, \end{aligned} \tag{1}$$

where $f(x), g_i(x) : R^n \to R, i = 0, \cdots, m$ are continuously differentiable functions. Assume that $f(x)$ is coercive, that is

$$\lim_{\|x\| \to +\infty} f(x) = +\infty$$

then there exists a big box X such that $intX$ contains all of relative local minimizers of problem (1), where $intX$ denotes the interior of X. Thus, we can consider the following equivalent problem:

$$(P) \quad \begin{aligned} &\min f(x) \\ &s.t.\ x \in S = \{x \in X : g_i(x) \le 0,\ i \in I\}. \end{aligned} \tag{2}$$

where $I = \{1, \cdots, m\}$.

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Penalty methods are an important and useful tool in constrained optimization, see, [1]-[15]. Nondifferentiable penalty function have been the first ones for which some exactness properties have been established by Zangwill [13]. The obvious difficulty with the exact penalty function is that it is non-differentiable, which prevents the use of efficient minimization algorithms. From an algorithmic viewpoint, this noniffere ntiabliliy can induce the so-called Maratos effect which prevents rapid local convergence. In order to avoid the drawback related to the nondifferentiability, some authors have introduced some classes of differentiable exact penalty functions. However, these continuously differentiable exact penalty functions always involve the derivatives of related function(e.g, [3, 7, 10]). Then Exponential penalty methods and primal-dual exponential multiplier penalty methods have been studied for linear and convex programming problems, see, [1, 2, 4, 5, 11].

In [5], the authors proposed an exponential penalty function and the associated penalty method,

$$\min f_r(x) = f(x) + r\sum_{i=1}^{m}\exp[g_i(x)/r],$$

which does not need interior starting points, but whose ultimate behavior is just like an interior penalty method. They analyzed the behavior of the method for sequences of values for parameter r that convergence quite fast to zero, but the penalty function is not exact.

In [11], the authors proposed an exact exponential penalty function

$$f(x) + \frac{1}{c}\sum_{i=1}^{m} u_i(\exp(cg_i(x)) - 1),$$

where $c > 0$ is a penalty parameter, $u_i > 0,\ i = 1, \cdots, m$ are multipliers, but they only analyzed the exponential method of multipliers for convex constrained minimization problems on R^n.

In this paper,we propose an exact logarithmic-exponential multiplier penalty function for the differentiable minimization problem on a big box $X \subset R^n$. We get the equivalence between the local optimizer of the original problem and the local optimizer of the multiplier penalty problem in Section 2 and develop the algorithm in Section 3.

2. A Logarithmic-Exponential Multiplier Penalty Function

We propose the following logarithmic-exponential multiplier penalty function:

$$Q(x, \lambda, p) = f(x) + \frac{2}{p}\sum_{i=1}^{m}\ln(1 + \exp(p\lambda_i g_i(x))) \tag{3}$$

where $\lambda = (\lambda_1, \cdots, \lambda_m) \in R^m_+$ are parameters which depend on the multipliers, $p > 0$ is a penalty parameter.

Now we consider the following logarithmic-exponential multiplier penalty problem

$$(P_{\lambda p})\ \min_{x\in X} Q(x,\lambda,p)$$

We say that x^* is a stationary point of problem $(P_{\lambda p})$, if $\nabla Q(x^*,\lambda,p)=0$. The set of local solutions of problem $(\cdot)$, we denote by $L(\cdot)$.

Definition 1. *We say that $x^*\in intX$ is a K-K-T point for problem (P), if there exists a $\lambda^*\in R^m_+$ such that*

$$\begin{aligned}&\nabla f(x^*)+\sum_{i\in I(x^*)}\lambda_i^*\nabla g_i(x^*)=0\\&\lambda_i^*\geq 0,\ g_i(x^*)\leq 0,\ i=1,\cdots,m,\\&\lambda_i^* g_i(x^*)=0,\ i=1,\cdots,m,\end{aligned}$$

Theorem 1. *Suppose that $x^*\in intX$ is a K-K-T point for problem (P), Then for any $p>0$, x^* is a stationary point of problem $(P_{\lambda^* p})$.*

Proof. Since $\lambda_i^* g_i(x^*)=0,\ i=1,\cdots,m$, we have $\lambda_i^*=0$ for $i\in I\backslash I(x^*)$. Thus

$$\begin{aligned}\nabla Q(x^*,\lambda^*,p)&=\nabla f(x^*)+\frac{2}{p}\sum_{i=1}^m\frac{\exp(p\lambda_i^* g_i(x^*))}{1+\exp(p\lambda_i^* g_i(x^*))}p\lambda_i^*\nabla g_i(x^*)\\&=\nabla f(x^*)+\sum_{i\in I(x^*)}\lambda_i^*\nabla g_i(x^*)\\&=0.\end{aligned}$$

We complete the proof. □

Remark 1. *By Theorem 1, if $x^*\in L(P)\bigcap intX$ with Lagrangian multipliers $\lambda_i^*, i=1,\cdots,m$, then x^* is a stationary point of problem $(P_{\lambda^* p})$.*

Theorem 2. *Suppose $x^*_{\lambda p}\in S\bigcap intX$ is a stationary point of problem $(P_{\lambda p})$ with $\lambda_i\geq 0,\ \lambda_i g_i(x^*_{\lambda p})=0,\ i=1,\cdots,m$, then $x^*_{\lambda p}$ is a K-K-T point of problem (P).*

Proof. Since $\lambda_i\geq 0,\ g_i(x^*_{\lambda p})\leq 0,\ \lambda_i g_i(x^*_{\lambda p})=0,\ i=1,\cdots,m$,
we have $\lambda_i=0$,when $i\in I\backslash I(x^*_{\lambda p})$. Thus

$$\begin{aligned}\nabla Q(x^*_{\lambda p},\lambda,p)&=\nabla f(x^*_{\lambda p})+\frac{2}{p}\sum_{i=1}^m\frac{\exp(p\lambda_i g_i(x^*_{\lambda p}))}{1+\exp(p\lambda_i g_i(x^*_{\lambda p}))}p\lambda_i\nabla g_i(x^*_{\lambda p})\\&=\nabla f(x^*_{\lambda p})+\sum_{i\in I(x^*_{\lambda p})}\lambda_i\nabla g_i(x^*_{\lambda p}),\end{aligned}$$

and by $\nabla Q(x^*_{\lambda p},\lambda,p)=0$, we have

$$\nabla f(x^*_{\lambda p})+\sum_{i\in I(x^*_{\lambda p})}\lambda_i\nabla g_i(x^*_{\lambda p})=0,$$

where

$$\lambda_i \geq 0,\ g_i(x^*_{\lambda p}) \leq 0,\ i = 1, \cdots, m,$$
$$\lambda_i g_i(x^*_{\lambda p}) = 0,\ i = 1, \cdots, m.$$

We complete the proof. □

Lemma 1 ([2] Lemma 3.2.1, p.298). *Let P and Q be two symmetric matrices. Assume that Q is positive semidefinite and P is positive definite on the null space of Q, that is, $x^T Px > 0$ for all $x \neq 0$ with $x^T Qx = 0$. Then there exists a scalar $\bar{c}$ such that $P + cQ$ is positive definite for all $c > \bar{c}$.*

Definition 2. *We say that the pair (x^*, λ^*) satisfies the second-order sufficient condition, if*

$$\nabla f(x^*) + \sum_{i \in I(x^*)} \lambda_i^* \nabla g_i(x^*) = 0$$
$$\lambda_i^* \geq 0,\ g_i(x^*) \leq 0,\ i = 1, \cdots, m,$$
$$\lambda_i^* g_i(x^*) = 0,\ i = 1, \cdots, m,$$
$$d^T(\nabla^2 f(x^*) + \sum_{i \in I(x^*)} \lambda_i^* \nabla^2 g_i(x^*))d > 0$$

for any $d \in D = \{d \in B : \nabla g_i(x^*)^T d \leq 0,\ \lambda_i^* \nabla g_i(x^*)^T d = 0 \ for\ all\ i \in I(x^*)\}$, where $B = \{d : \|d\| = 1\}$.

Theorem 3. *Suppose that (x^*, λ^*) satisfies the second-order sufficient condition, then x^* is a strict local minimum point of problem $(P_{\lambda^* p})$, where $p > 0$ is sufficiently large. On the other hand, if $x^*_{\lambda p} \in S \bigcap intX$ with $\lambda_i \geq 0,\ \lambda_i g_i(x^*_{\lambda p}) = 0,\ i = 1, \cdots, m$ satisfies $\nabla Q(x^*_{\lambda p}, \lambda, p) = 0,\ \nabla^2 Q(x^*_{\lambda p}, \lambda, p)$ is positive definite, then $x^*_{\lambda p} \in L(P)$.*

Proof. The Hessian matrix $H_{\lambda^* p}(x^*)$ of $Q(x, \lambda^*, p)$ at x^* is

$$\begin{aligned}
H_{\lambda^* p}(x^*) &= \nabla^2 Q(x^*, \lambda^*, p) \\
&= \nabla^2 f(x^*) + \frac{2}{p} \sum_{i=1}^{m} \frac{\exp(p\lambda_i^* g_i(x^*))}{1 + \exp(p\lambda_i^* g_i(x^*))} p\lambda_i^* \nabla^2 g_i(x^*) \\
&\quad + \frac{2}{p} \sum_{i=1}^{m} \frac{\exp(p\lambda_i^* g_i(x^*))}{1 + \exp(p\lambda_i^* g_i(x^*))} (p\lambda_i^*)^2 \nabla g_i(x^*) \nabla^T g_i(x^*) \\
&= \nabla^2 f(x^*) + \sum_{i \in I(x^*)} \lambda_i^* \nabla^2 g_i(x^*) + p \sum_{i \in I(x^*)} (\lambda_i^*)^2 \nabla g_i(x^*) \nabla^T g_i(x^*).
\end{aligned}$$

By Lemma 1, when $p > 0$ is sufficiently large, $\nabla^2 Q(x^*, \lambda^*, p)$ is positive definite. Noting that x^* is a stationary point of $(P_{\lambda^* p})$ from Theorem 2, we obtain that x^* is a strict local minimum point of problem $(P_{\lambda^* p})$.

On the other hand, we have

$$\begin{aligned}\nabla Q(x^*_{\lambda p},\lambda,p) &= \nabla f(x^*_{\lambda p}) + \frac{2}{p}\sum_{i=1}^{m}\frac{\exp(p\lambda_i g_i(x^*_{\lambda p}))}{1+\exp(p\lambda_i g_i(x^*_{\lambda p}))}p\lambda_i\nabla g_i(x^*_{\lambda p})\\ &= \nabla f(x^*_{\lambda p}) + \sum_{i=1}^{m}\lambda_i\nabla g_i(x^*_{\lambda p})\\ &= 0.\end{aligned}$$

Therefore we have

$$\begin{aligned}&\nabla f(x^*_{\lambda p}) + \sum_{i\in I(x^*_{\lambda p})}\lambda_i\nabla g_i(x^*_{\lambda p}) = 0,\\ &\lambda_i \geq 0,\ g_i(x^*_{\lambda p}) \leq 0,\ i=1,\cdots,m,\\ &\lambda_i g_i(x^*_{\lambda p}) = 0,\ i=1,\cdots,m.\end{aligned} \tag{4}$$

i.e., $x^*_{\lambda p}$ is a K-K-T point of (P). Furthermore, we have

$$\begin{aligned}0 &< d^T H_{\lambda p}(x^*_{\lambda p})d\\ &= d^T\left(\nabla^2 f(x^*_{\lambda p}) + \sum_{i\in I(x^*_{\lambda p})}\lambda_i\nabla^2 g_i(x^*_{\lambda p})\right)d + p\sum_{i\in I(x^*_{\lambda p})}(\lambda_i)^2 d^T\nabla g_i(x^*_{\lambda p})\nabla^T g_i(x^*_{\lambda p})d\\ &= d^T\left(\nabla^2 f(x^*_{\lambda p}) + \sum_{i\in I(x^*_{\lambda p})}\lambda_i\nabla^2 g_i(x^*_{\lambda p})\right)d,\end{aligned}$$

for any $d \in D' = \{d \in B : \nabla g_i(x^*_{\lambda p})^T d \leq 0,\ \lambda_i\nabla g_i(x^*_{\lambda p})^T d = 0,\ i \in I(x^*_{\lambda p})\}$. And by (4), we have that$x^*_{\lambda p}$ is a strictly local minimum of (P). □

Remark 2. *If $f(x),\ g_i(x),\ i=1,\cdots,m$ are convex, and $f(x),\ g_i(x) \in C^2,\ i=1,\cdots,m$, then for any $p>0,\ x^* \in L(P_{\lambda^* p})$.*

Theorem 4. *Suppose that*
(1) x^* is a K-K-T point for problem (P) with Lagrangian multipliers $\lambda_i^* \geq 0, i=1,\cdots,m$, furthermore, suppose that the second-order sufficient condition holds at x^*;
(2) $\nabla g_i(x^*),\ i \in I(x^*)$ are linearly independent, and $\|I(x^*)\| = n$, where $\|I(x^*)\|$ is the number of elements in $I(x^*)$. If $p>0$ is sufficiently large, $\lambda_i > 0, i=1,\cdots,m$ are finite, and $\triangle\lambda_i, i \in I(x^*)$ are appropriately chosen, then $\nabla Q(x^*,\lambda,p) = 0$, $\nabla^2 Q(x^*,\lambda,p)$ is positive definite, where $\lambda_i = \lambda_i^* + \triangle\lambda_i, i \in I(x^*)$. It means that $x^* \in L(P_{\lambda p})$.

Proof. By $\lambda_i = \lambda_i^* + \triangle\lambda_i \; for \; i \in I(x^*)$, we have

$$\begin{aligned}\nabla Q(x^*,\lambda,p) &= \nabla f(x^*) + \frac{2}{p}\sum_{i=1}^{m}\frac{\exp(p\lambda_i g_i(x^*))}{1+\exp(p\lambda_i g_i(x^*))}p\lambda_i\nabla g_i(x^*)\\ &= \nabla f(x^*) + \sum_{i\in I(x^*)}\lambda_i^*\nabla g_i(x^*) + \sum_{i\in I(x^*)}\Delta\lambda_i\nabla g_i(x^*)\\ &\quad + \sum_{i\notin I(x^*)}\frac{2\lambda_i\exp(p\lambda_i g_i(x^*))}{1+\exp(p\lambda_i g_i(x^*))}\nabla g_i(x^*).\end{aligned} \tag{5}$$

By K-K-T conditions, we have

$$\nabla f(x^*) + \sum_{i\in I(x^*)}\lambda_i^*\nabla g_i(x^*) = 0. \tag{6}$$

Furthermore, Since $\nabla g_i(x^*),\; i \in I(x^*)$ are linearly independent, there exist $\alpha_{ij},\; j \in I(x^*)$ such that $\nabla g_i(x^*) = \sum_{j\in I(x^*)}\alpha_{ij}\nabla g_j(x^*)$ for any $i \notin I(x^*)$. Thus by (5) and (6) we have

$$\begin{aligned}\nabla Q(x^*,\lambda,p) &= \sum_{i\in I(x^*)}\Delta\lambda_i\nabla g_i(x^*) + \sum_{i\notin I(x^*)}\frac{2\lambda_i\exp(p\lambda_i g_i(x^*))}{1+\exp(p\lambda_i g_i(x^*))}\nabla g_i(x^*)\\ &= \sum_{i\in I(x^*)}\Delta\lambda_i\nabla g_i(x^*) + \sum_{i\notin I(x^*)}\frac{2\lambda_i\exp(p\lambda_i g_i(x^*))}{1+\exp(p\lambda_i g_i(x^*))}\\ &\quad\times\sum_{j\in I(x^*)}\alpha_{ij}\nabla g_j(x^*)\\ &= \sum_{i\in I(x^*)}(\Delta\lambda_i + \sum_{j\notin I(x^*)}\alpha_{ji}\frac{2\lambda_j\exp(p\lambda_j g_j(x^*))}{1+\exp(p\lambda_j g_j(x^*))})\nabla g_i(x^*).\end{aligned} \tag{7}$$

In (7), let $\Delta\lambda_i = -\sum_{j\notin I(x^*)}\alpha_{ji}\frac{2\lambda_j\exp(p\lambda_j g_j(x^*))}{1+\exp(p\lambda_j g_j(x^*))})$ for $i \in I(x^*)$, then

$$\nabla Q(x^*,\lambda,p) = 0. \tag{8}$$

Furthermore, by $\lambda_i = \lambda_i^* + \triangle\lambda_i \; for \; i \in I(x^*)$, we have

$$\begin{aligned}
\nabla^2 Q(x^*, \lambda, p) &= \nabla^2 f(x^*) + \sum_{i=1}^{m} \frac{2\lambda_i \exp(p\lambda_i g_i(x^*))}{1+\exp(p\lambda_i g_i(x^*))} \nabla^2 g_i(x^*) \\
&\quad + \sum_{i=1}^{m} \frac{2p(\lambda_i)^2 \exp(p\lambda_i g_i(x^*))}{1+\exp(p\lambda_i g_i(x^*))} \nabla g_i(x^*) \nabla^T g_i(x^*) \\
&= \nabla^2 f(x^*) + \sum_{i\in I(x^*)} \lambda_i^* \nabla^2 g_i(x^*) + \sum_{i\in I(x^*)} \Delta\lambda_i \nabla^2 g_i(x^*) \\
&\quad + \sum_{i\notin I(x^*)} \frac{2\lambda_i \exp(p\lambda_i g_i(x^*))}{1+\exp(p\lambda_i g_i(x^*))} \nabla^2 g_i(x^*) \\
&\quad + \sum_{i\in I(x^*)} p(\lambda_i)^2 \nabla g_i(x^*) \nabla^T g_i(x^*) \qquad (9) \\
&\quad + \sum_{i\notin I(x^*)} \frac{2p(\lambda_i)^2 \exp(p\lambda_i g_i(x^*))}{1+\exp(p\lambda_i g_i(x^*))} \nabla g_i(x^*) \nabla^T g_i(x^*) \\
&= P + pQ + \sum_{i\in I(x^*)} \Delta\lambda_i \nabla^2 g_i(x^*) \\
&\quad + \sum_{i\notin I(x^*)} \frac{2\lambda_i \exp(p\lambda_i g_i(x^*))}{1+\exp(p\lambda_i g_i(x^*))} \nabla^2 g_i(x^*) \\
&\quad + \sum_{i\notin I(x^*)} \frac{2p(\lambda_i)^2 \exp(p\lambda_i g_i(x^*))}{1+\exp(p\lambda_i g_i(x^*))} \nabla g_i(x^*) \nabla^T g_i(x^*)
\end{aligned}$$

where

$$P = \nabla^2 f(x^*) + \sum_{i\in I(x^*)} \lambda_i^* \nabla^2 g_i(x^*), \quad Q = \sum_{i\in I(x^*)} p(\lambda_i)^2 \nabla g_i(x^*) \nabla^T g_i(x^*).$$

By Lemma 1, when $p > 0$ is sufficiently large, $P + pQ$ is positive definite. And we have

$$\begin{aligned}
\Delta\lambda_i = -\sum_{j\notin I(x^*)} \alpha_{ji} \frac{2\lambda_j exp(p\lambda_j g_j(x^*))}{1+exp(p\lambda_j g_j(x^*))}) &\to 0, \; for \; i \in I(x^*), \\
\frac{2\lambda_i \exp(p\lambda_i g_i(x^*))}{1+\exp(p\lambda_i g_i(x^*))} &\to 0, \; for \; i \notin I(x^*), \\
\frac{2p(\lambda_i)^2 \exp(p\lambda_i g_i(x^*))}{1+\exp(p\lambda_i g_i(x^*))} &\to 0, \; for \; i \notin I(x^*),
\end{aligned}$$

when $p \to +\infty$. From above and (9), we have that $\nabla^2 Q(x^*, \lambda, p)$ is positive definite when $p > 0$ is sufficiently large. This completes our proof. □

Remark 3. *By* $\|I(x^*)\| = n$, *we have* $m \geq n$, m *denotes the number of constrained functions.*

3. The Algorithm and Numerical Examples

Algorithm 1.

Step 1. *Let $p > 0$ is sufficiently large, $\lambda_i > 0, i = 1, \cdots, m$ are finite, $\epsilon > 0$.*

Step 2. *Choose any $x^0 \in X$ as an initial point. Compute*

$$\min_{x \in R^n} \{Q(x, \lambda, p)\} = \min_{x \in R^n} \left\{ f(x) + \frac{2}{p} \sum_{i=1}^{m} \ln\left(1 + \exp(p\lambda_i g_i(x))\right) \right\}$$

$x^*_{\lambda p}$ is an approximate minimizer of $Q(x, \lambda, p)$, If $\|\nabla Q(x^*_{\lambda p}, \lambda, p)\| < \epsilon$, then stop, otherwise, goto Step 3.

Step 3. *Let*

$$\bar{\lambda_i} = \frac{2\lambda_i \exp(p\lambda_i g_i(x^*_{\lambda p}))}{1 + \exp(p\lambda_i g_i(x^*_{\lambda p}))}, \ i = 1, \cdots, m$$
$$p := p + u, \ \lambda_i := \bar{\lambda_i}, \ x^0 := x^*_{\lambda p},$$

where u is a positive constant, goto Step 2.

Example 1.

$$\min f(x) = -2x_1 + x_2 \ \ s.t. \ \ (1 - x_1)^3 - x_2 \ge 0, \ \ x_2 + 0.25x_1^2 - 1 \ge 0$$

We have

$$g_1(x) = x_2 - (1 - x_1)^3 \le 0, \quad g_2(x) = 1 - x_2 - 0.25x_1^2 \le 0$$
$$Q(x, \lambda, p) = f(x) + \frac{2}{p} \sum_{i=1}^{2} \ln(1 + \exp(p\lambda_i g_i(x)))$$

Starting point $x^0 = (-0.2500000, 1.200000)$, $p = 10.0$, $\lambda_i = 1.0$, $i = 1, 2$, $u = 0.002$, $\epsilon = 1.0E-4$, we obtain results shown in table 1.

Table 1

k	x_k	∇Q	p	λ_1
0	(-0.2500000, 1.200000)	2.122941	10.000	1.000000
1	(2.180616, -1.645655)	2.5307140E-04	10.002	0.9997565
2	(1.899213, 4.3423112E-02)	6.7522039E-04	10.004	1.998612
3	(1.725489, 0.2020586)	9.3312917E-04	10.006	3.997190
4	(1.622662, 0.2977812)	9.2928542E-04	10.008	7.994380
5	(-2.1323573E-02, 1.025487)	2.0570308E-03	10.010	0.6329106
6	(9.6172364E-03, 0.9962481)	5.3806911E-04	10.012	0.6825758
7	(-3.5840159E-03, 1.001504)	1.7140124E-05		

k	λ_2	$g_1(x)$	$g_2(x)$	$f(x)$	$Q(x_k)$
0	1.000000	-0.7531250	-0.2156250	1.700000	1.722014
1	1.999999	-4.8708862E-05	1.456883	-6.006886	-2.954539
2	2.998571	0.7705120	5.4824539E-02	-3.755002	-1.937346
3	4.996642	0.5839090	5.3612988E-02	-3.248920	-0.5568899
4	8.994537	0.5391926	4.3960590E-02	-2.947543	1.823324
5	1.632543	-3.9857671E-02	-2.5600500E-02	1.068134	1.095218
6	1.682267	2.4823191E-02	3.7288107E-03	0.9770136	1.276504
7		-9.2866849E-03	-1.5071557E-03	1.008672	1.276841

We have $x = (-3.5840159E-03, 1.001504)$, $f(x) = 1.008672$. *In fact, the optimal solution* $x^* = (0.0, 1.0), f(x^*) = 1.0$.

Example 2.

$$\min f(x) = 100(x_2 - x_1^2)^2 + (1 - x_1)^2$$

$$s.t. \quad x_1^2 + x_2^2 \geq -0.25, \quad \frac{1}{3}x_2 + x_1 \geq -0.1, \quad -\frac{1}{3}x_1 + x_2 \geq -0.1$$

We have

$$g_1(x) = -x_1^2 - x_2^2 - 0.25, \quad g_2(x) = -\frac{1}{3}x_2 - x_1 - 0.1, \quad g_3(x) = \frac{1}{3}x_1 - x_2 - 0.1$$

$$Q(x, \lambda, p) = f(x) + \frac{2}{p}\sum_{i=1}^{3} \ln(1 + \exp(p\lambda_i g_i(x)))$$

Starting point $x^0 = (1.2.00000, 1.100000)$, $p = 1.0$, $\lambda_i = 1.0$, $i = 1, 2, 3$, $u = 5$, $\epsilon = 1.0E-4$, *we obtain results shown in table 2.*

Table 2

k	x_k	∇Q	p	λ_1	λ_2
0	(-0.2500000, 1.200000)	177.0257	1.0	1.000000	1.000000
1	(1.277228, 1.631926)	2.4357214E-04	6.0	4.5155751E-04	0.2393418
2	(1.111267, 1.235253)	2.2340014E-04	11.0	4.4971582E-04	7.1507618E-02
3	(1.062761, 1.129658)	1.8836425E-04	16.0	4.4676193E-04	4.0910590E-02
4	(1.040610, 1.082999)	3.0575768E-04	21.0	4.4276091E-04	2.6310483E-02
5	(1.040588, 1.082993)	1.7617838E-02	26.0	4.3760342E-04	1.8283980E-02
6	(1.021427, 1.043382)	2.5334687E-04	31.0	4.3167398E-04	1.3522550E-02
7	(1.021416, 1.043379)	8.5470350E-03	36.0	4.2479482E-04	1.0401369E-02
8	(1.021421, 1.043361)	9.9955704E-03	41.0	4.1705897E-04	8.2490547E-03
9	(1.011237, 1.022638)	3.7909663E-04	46.0	4.0879330E-04	6.7169140E-03
10	(1.009503, 1.019129)	1.6116150E-04	51.0	3.9992481E-04	5.5767386E-03
11	(1.009491, 1.019114)	4.6334486E-03	56.0	3.9051482E-04	4.7040856E-03
12	(1.009494, 1.019110)	2.3056224E-03	61.0	3.8066308E-04	4.0214998E-03
13	(1.006291, 1.012643)	2.9541465E-05			

k	λ_3	$g_1(x)$	$g_2(x)$	$g_3(x)$	$f(x)$	$Q(x_k)$
0	1.000000	-2.900000	-1.300000	-1.200000	11.60001	11.92301
1	0.1214337	-4.544495	-1.377228	-1.731926	7.6893203E-02	0.2621773
2	6.6620775E-02	-3.010764	-1.211267	-1.335253	1.2391812E-02	0.4028926
3	3.8482636E-02	-2.655590	-1.162761	-1.229658	3.9428622E-03	0.2520121
4	2.5055755E-02	-2.505755	-1.140610	-1.182999	1.6508403E-03	0.1849063
5	1.7500091E-02	-2.505697	-1.140588	-1.182993	1.6502559E-03	0.1481308
6	1.3047919E-02	-2.381960	-1.121427	-1.143382	4.5959529E-04	0.1241425
7	1.0083369E-02	-2.381930	-1.121416	-1.143379	4.5942853E-04	0.1069896
8	8.0203898E-03	-2.381903	-1.121421	-1.143361	4.5922326E-04	9.4191276E-02
9	6.5565580E-03	-2.318389	-1.111237	-1.122638	1.2641333E-04	8.4117189E-02
10	5.4604234E-03	-2.307720	-1.109503	-1.119129	9.0414032E-05	7.6029547E-02
11	4.6163662E-03	-2.307666	-1.109491	-1.119114	9.0255897E-05	6.9387071E-02
12	3.9532087E-03	-2.307663	-1.109494	-1.119110	9.0237809E-05	6.3833192E-02
13		-2.288067	-1.106291	-1.112643	3.9621602E-05	5.9106242E-02

We have $x = (1.006291, 1.012643)$, $f(x) = 3.9621602E{-}05$. *In fact, the optimal solution* $x^* = (1.0, 1.0), f(x^*) = 0.0$.

Example 3.

$$\min f(x) = x_1^2 + 2x_2^2 - 2x_1x_2 - 2x_1 - 6x_2$$
$$s.t.\quad x_1 + x_2 \le 2, \quad -x_1 + 2x_2 \le 2, \quad x_1, x_2 \ge 0$$

We have

$$g_1(x) = x_1 + x_2 - 2, \quad g_2(x) = -x_1 + 2x_2 - 2$$
$$Q(x, \lambda, p) = f(x) + \frac{2}{p}\sum_{i=1}^{2} \ln(1 + \exp(p\lambda_i g_i(x)))$$
$$X = \{(x_1, x_2) : \ 0 \le x_i \le 2; \ i = 1, 2\}, \ x \in X$$

Let $x^0 = (0.0, 0.7)$, $p = 1.0$, $\lambda_i = 1.0$, $i = 1, 2$, $u = 40$, $\epsilon = 1.0E-3$, *we obtain results shown in table 3.*

Table 3

k	x	∇Q	p	λ_1
0	(0.0000000,0.7000000)	5.576121	1.000000	1.000000
1	(2.309561,1.852993)	2.827582	41.000000	1.793672
2	(0.8102180,1.206876)	5.0201250E-04	81.000000	2.792842
3	(0.8000078,1.200013)	8.9519215E-04		

k	λ_2	$g_1(x)$	$g_2(x)$	$f(x)$	$Q(x,\lambda,p)$
0	1.000000	-1.300000	-0.6000000	-3.220000	-1.863007
1	0.7070528	2.162554	-0.6035743	-12.09504	-6.679726
2	1.4422274E-05	1.7093956E-02	-0.3964662	-7.247805	-7.174270
3		2.0682812E-05	-0.3999819	-7.200058	-7.165776

We have $x = (0.8000078, 1.200013)$, $f(x) = -7.200058$. *In fact, the optimal solution* $x^* = (0.8, 1.2), f(x^*) = -7.2$.

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