J. Appl. Math. & Informatics Vol. **28***(2010), No. 5 - 6, pp. 1467-1476*
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AN IMPROVED EXPONENTIAL REGULA FALSI METHODS WITH CUBIC CONVERGENCE FOR SOLVING NONLINEAR EQUATIONS

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Abstract. The aim of this paper is to propose a cubic convergent regula falsi iterative method for solving the nonlinear equation f(x) = 0, where $f : [a, b] \subset \Re \to \Re$ is a continuously differentiable. In [3,6] a quadratically convergent regula falsi iterative methods for solving this nonlinear equations is proposed. It is shown there that both the sequences of diameters and iterative points sequence converge to zero simultaneously. So The aim of this paper is to accelerate further the convergence of these methods from quadratic to cubic. This is done by replacing the parameter p in the iteration of [3,5,6] by a function $p(x)$ defined suitably. The convergence analysis is carried out for the method. The method is tested on number of numerical examples and results obtained shows that our methods are better and more effective and comparable to well-known methods.

AMS Mathematics Subject Classification : 65D05, 49M15, 41A25, 35G20.
Key words and phrases : Nonlinear equations, Regula-falsi methods, Newtons methods, Cubic convergence.

1. Introduction

In numerical analysis and applied mathematics, one of the most important problems is to compute approximate solutions of the nonlinear of equation

$$f(x) = 0 \tag{1}$$

where $f : [a, b] \subset \Re \to \Re$. There exists a large number of applications that give rise to thousands of such equations depending on one or more parameters.

Many optimization problems lead to the nonlinear equations. Iterative methods requiring one or more initial guesses for the desired root are generally used to solve these equations.

All these methods satisfy a number of criteria such as they should enjoy good convergence properties, be efficient and numerically stable. Bisection and regula

Received March 23, 2010. Revised April 2, 2010. Accepted May 14, 2010.

falsi methods are globally convergent iterative methods used to find a simple root of the nonlinear equation (1) by repeated linear interpolation between the current bracketing estimates [5] . But their asymptotic convergence rate of iterative sequence $\{(x_n - \alpha)\}$, where α be a simple root, is linear. Another distinct shortcoming of these methods is that one endpoint is retained step after step, whenever a concave or convex region of f(x) has been reached. Some modifications overcoming these difficulties have been discussed in [3,5,6,7].

The well known quadratically convergent Newtons method and its variants are iterative formulae generally used for finding a root of (1). But, these methods may fail to converge in case the initial point is far from root or the derivative vanishes in the vicinity of the root. A number of third order methods are also used for solving nonlinear equations in R. Though, these methods require more computational cost, they are advantageous in applications, such as stiff system of equations, where quick convergence is required. A family of third order methods is given by

$$x_{n+1} = x_n - \left\{ (1 + \frac{1}{2} \frac{f^{''}(x_n) f(x_n)}{f^{'}(x_n)^2}) \right\} \tag{2}$$

The derivative free methods to solve nonlinear equations in R are also developed by many researchers [5 - 8]. The convergence of the sequences of diameters $\{(a_n - b_n)\}$ is important from the analysis point of view to enclose the root α. Some work in this direction is also being carried out. Recently, Chen and Li [3] had developed a class of quadratically convergent exponential iterative methods for finding a simple root α of a nonlinear equation $f(x) = 0$ in the interval [a,b].

These methods are then combined with classical regula falsi method to establish that both the sequence of diameters $\{(a_n - b_n)\}$ and sequence of iterates $\{(x_n - \alpha)\}$ produced by these methods asymptotically converges to the root.

The aim of this paper is to further accelerate the convergence of the methods of Chen and Li [3] from quadratic to cubic along with both the sequence of diameters and the sequence of iterates converging to zero. This is done by replacing the parameter p in their iteration by a function $p(x)$ defined suitably.The theoretical analysis and numerical experiments is given to show that our higher order of exponential iterative methods is effective and comparable to those given in [3] as well as Newtons method, and regula falsi method.

2. The Improved Method

Let α be a root of $f(x) = 0 \in [a, b]$. And let $f(a)f(b) < 0$ to guarantee that α be a simple root of $f(x) = 0 \in [a, b]$. The iteration formulae with a parameter under consideration of Chen and Li [4] is given by

$$x_{n+1} = x_n \exp \left\{ \frac{-f^2(x_n)}{x_n(pf^2(x_n) + f(x_n) - f(x_n - f(x_n)))} \right\}, n = 0, 1, ... \tag{3}$$

Where $p \in R$, $|p| < +\infty$. Let $e_n = x_n - \alpha$ and $f^{'}(\alpha) \neq 0$, so for $\alpha \neq 0$

$$\lim_{n\to\infty} \frac{e_{n+1}}{e_n^2} = p + \frac{f^{''}(\alpha)}{2f^{'}(\alpha)} - \frac{f^{''}(\alpha)}{2} + \frac{1}{2\alpha} \tag{4}$$

Equation (4) means that the convergence of iterative equation (3) is of order two. If we have $p = -\frac{f^{''}(\alpha)}{2f^{'}(\alpha)} + \frac{f^{''}(\alpha)}{2} - \frac{1}{2\alpha}$, then the iteration (3) is at least cubically convergent. As in [8], the parameter p at every step in (3) is chosen $p_n = sign(f(x_n) - f(x_n - f(x_n)))$ to be benefit to computing. Now we define p(x) given by

$$p(x) = -\frac{f(x - f(x))[f(x - f(x)) + f(x + f(x)) - 2f(x)]}{2[f(x) - f(x - f(x))]f^2(x)} - \frac{1}{2x} \tag{5}$$

from (4), we have

$$\lim_{n\to\infty} p(x) = -\frac{f^{''}(\alpha)}{2f^{'}(\alpha)} + \frac{f^{''}(\alpha)}{2} - \frac{1}{2\alpha} \tag{6}$$

This leads to the following modification of iteration (3)

$$x_{n+1} = x_n \exp\left\{\frac{-f^2(x_n)}{x_n(p(x_n)f^2(x_n) + f(x_n) - f(x_n - f(x_n)))}\right\}, n = 0, 1.. \tag{7}$$

The following theorem shows that this method is cubically convergent.

Theorem 1. *Let$f : [a, b] \subset R \to R$ be a continuously differentiable and $f(\alpha) = 0$ be its root. Let $U(\alpha)$ be a sufficiently small neighborhood of α such that $f^{'}(\alpha) \neq 0$,$f^{''}(\alpha)$ and $f^{'''}(\alpha)$ exist in $U(\alpha)$. Then the sequence of iteration generated by iteration formula* (7)*with* (5) *is cubically convergent.*

Proof. Let $e_n = x_n - \alpha$. Expanding left hand side by Taylor's series, we get

$$f(x_n) = f^{'}(\alpha)e_n + \frac{f^{''}(\alpha)}{2}e_n^2 + \frac{f^{'''}(\alpha)}{6}e_n^3 + O(e_n^3)$$

$$f(x_n - f(x_n)) = (1 - f^{'}(\alpha))f^{'}(\alpha)e_n + (1 - 3f^{'}(\alpha) + f^{'2}(\alpha))\frac{f^{'}(\alpha)f^{''}(\alpha)}{2}e_n^2 + O(e_n^2)$$

$$f(x_n + f(x_n)) = (1 + f^{'}(\alpha))f^{'}(\alpha)e_n + (1 + 3f^{'}(\alpha) + f^{'2}(\alpha))\frac{f^{'}(\alpha)f^{''}(\alpha)}{2}e_n^2 + O(e_n^2)$$

$$f^2(x_n) = f^{'2}(\alpha)e_n^2 + f^{'}(\alpha)f^{''}(\alpha)e_n^3 + O(e_n^3)$$

$$f(x_n) - f(x_n - f(x_n)) = f^{'2}(\alpha)e_n + (3f^{'}(\alpha) - f^{'2}(\alpha))\frac{f^{''}(\alpha)}{2}e_n^2 + O(e_n^2)$$

$$\frac{f(x_n) - f(x_n - f(x_n))}{f(x_n)} = f^{'}(\alpha) - (f^{'}(\alpha) - 2)\frac{f^{''}(\alpha)}{2}e_n + (2f^{'2}(\alpha)f^{'''}(\alpha) + 6f^{'''}(\alpha) - 3f^{''2}(\alpha) - 6f^{'}(\alpha)f^{'''}(\alpha))\frac{e_n^2}{12} + O(e_n^2)$$

So, we get

$$p(x_n) = -\frac{1}{2}\left((1-f^{'}(\alpha))\frac{f^{''}(\alpha)}{f^{'}(\alpha)} + \frac{1}{x_n}\right) + \left(\frac{f^{''2}(\alpha)}{2f^{'2}(\alpha)} + \frac{f^{'''}(\alpha)}{2} - \frac{f^{'''}(\alpha)}{2f^{'}(\alpha)} - \frac{f^{''2}(\alpha)}{4f^{'}(\alpha)}\right)e_n + O(e_n)$$

and

$$\frac{f(x_n)}{p(x_n)f(x_n) + \frac{f(x_n)-f(x_n-f(x_n))}{f(x_n)}} = e_n + \frac{1}{2x_n}e_n^2 + \left(\frac{f^{''2}(\alpha)}{4f^{'}(\alpha)} - \frac{f^{'}(\alpha)f^{'''}(\alpha)}{6} + \frac{1}{4x_n^2} - \frac{f^{''2}(\alpha)}{4f^{'2}(\alpha)} + \frac{f^{'''}(\alpha)}{6f^{'}(\alpha)}\right)e_n^3 + O(e_n^3).$$

Expanding the exponential function in(7) by Taylor's series,we get

$$\begin{aligned} x_{n+1} &= x_n \exp\left\{\frac{-f(x_n)}{x_n(p(x_n)f(x_n) + \frac{f(x_n)-f(x_n-f(x_n))}{f(x_n)}}\right\} \\ &= x_n - \frac{f(x_n)}{p(x_n)f(x_n) + \frac{f(x_n)-f(x_n-f(x_n))}{f(x_n)}} \\ &\quad + \frac{f^2(x_n)}{2x_n(p(x_n)f(x_n) + \frac{f(x_n)-f(x_n-f(x_n))}{f(x_n)})^2} \\ &\quad + O\left\{\frac{f^2(x_n)}{2x_n(p(x_n)f(x_n) + \frac{f(x_n)-f(x_n-f(x_n))}{f(x_n)})^2}\right\} \end{aligned}$$

This gives

$$e_{n+1} = \left\{\frac{f^{'}(\alpha)f^{'''}(\alpha)}{6} - \frac{f^{''2}(\alpha)}{4f^{'}(\alpha)} + \frac{f^{''2}(\alpha)}{4f^{'2}(\alpha)} + \frac{1}{12x_n^2} - \frac{f^{'''}(\alpha)}{6f^{'}(\alpha)}\right\}e_n^3 + O(e_n^3).$$

Then

$$\lim_{n\to\infty}\frac{e_{n+1}}{e_n^3} = \frac{f^{'''}(\alpha)}{6f^{'}(\alpha)}(f^{'2}(\alpha)-1) + \frac{f^{''2}(\alpha)}{4f^{'2}(\alpha)}(1-f^{'}(\alpha)) + \frac{1}{12\alpha^2}. \tag{8}$$

This shows that the iteration (7) with (5) is cubically convergent. □

Remark. For a given $h_n > 0$, we have a general formulae given by

$$x_{n+1} = x_n \exp\left\{\frac{-h_n f^2(x_n)}{x_n(p(x_n)f^2(x_n) + f(x_n) - f(x_n - h_n f(x_n)))}\right\}, n = 0, 1.. \tag{9}$$

with

$$p(x_n) = \Big\{\frac{f(x_n - h_n f(x_n))[f(x_n - f(x_n)) + f(x_n)) + f(x_n + f(x_n)) - 2f(x_n)]}{2[f(x_n) - f(x_n - f(x_n))]f^2(x_n)} + \frac{1}{2x_n}\Big\}(-h_n). \tag{10}$$

The above results of $p(x_n)$ is slightly different from that given in (5). It's easily prove that the iteration formulae (9) with (10) is also cubically convergent.

3. Higher Order Algorithm

In this section , we shall describe the integration of the method (9)to achieve asymptotic convergence of the sequence of diameters $\{(b_n - a_n)\}$ and the sequence of points $\{(x_n - \alpha)\}$ converging to zero. Let [a,b] contains a root of $f(x)$ and $y_n \in [a_n, b_n] \subset [a, b]$ is produced at nth step of regula falsi method. Now, we attempt to get a better enclosing interval by means of this point and our method(9). Let

$$q_n = \frac{\mid f(x_n) \mid}{f(b_n) - f(a_n)}, \quad h_n = \frac{b_n - a_n}{\mid f(x_n) \mid} q_n \quad \text{and} \quad y_n = a_n - f(a_n)h_n$$

Now, we define our iteration by

$$x_{n+1} = x_n \exp\left\{ - \frac{q_n(b_n - a_n)\mid f(x_n) \mid}{x_n(p(x_n)f^2(x_n) + f(x_n) - f(y_n))} \right\}, n = 0, 1, ... \tag{11}$$

For computational purposes [5], the above iteration is described below in the algorithmic form and is called Higher Order Exponential algorithm. Let $\alpha \in [a_0, b_0]$, n be the number of iterates ϵ_1 and ϵ_2 are the accurate desired by the algorithm HOEXRF to approximate α.

Algorithm HOEXRF

Begin

1. [Initialization]

$n = 0, x_n = a_n (or b_n)$

2. [Regula-Falsi Iteration]

$y_n = a_n - f(a_n)\frac{a_n - b_n}{f(a_n) - f(b_n)}$

3. [convergence Test]

If $\mid f(y_n) \mid \leq \epsilon_1$, then print y_n as a root of (1). Stop.

4. If $f(a_n)f(y_n) < 0$

$a_{n+1}^{-} = a_n, b_n^{-} = y_n$

else $a_n^{-} = y_n, b_n^{-} = b_n$

5. [HOEXRF Iteration]

$u_n = x_n \exp\{ - \frac{q_n(b_n - a_n)\mid f(x_n)\mid}{x_n(p(x_n)f^2(x_n) + f(x_n) - f(y_n))} \}$

6. If $u_n \in [a_n^{-}, b_n^{-}], x_{n+1} = u_n$

If $f(a_n^{-})f(u_n) < 0$

$a_{n+1} = a_n^{-}, b_{n+1} = u_n$

else

$a_{n+1} = u_n, b_{n+1} = b_n^{-}$

7. If $u_n \notin [a_n^{-}, b_n^{-}]$

$a_{n+1} = a_n^{-}, b_{n+1} = b_n^{-}$

If $u_n < \bar{a}_n$

$x_{n+1} = \bar{a}_n$

else

$x_{n+1} = \bar{b}_n$

8. [Convergence Test]
If $\mid f(x_{n+1}) \mid \leq \epsilon_1$,or $b_{n+1} - a_{n+1} \leq \epsilon_2$, print x_{n+1} is a root and stop.
else increase n, go to step 2.
End.

4. Convergence Theorem

Before we give the convergence theorem , we present the following lemmas which will be used for the proof of the cubic convergence theorem.

Lemma 1. *Assume that $f(\alpha) = 0$ and $U(\alpha)$ to be sufficiently small neighborhood of α, if $f^{'''}(x)$ is continuous in $U(\alpha)$ and $f^{'}(\alpha) \neq 0$. Then the sequence $\{x_n\}$ produced by the iteration formula (9) is at least cubically convergent for $h_n > 0$.*

Lemma 2. *Assume $f(x) \in C^1_{[a,b]}$ and $f(a) < 0, f(b) > 0$.$\{q_n\}$ is a real sequence with $0 < r < q_n < q < 1$. Then either the nonzero rootα of Eq.(1) in [a,b] is found in a finite number of steps or sequence of diameters $\{(b_n - a_n)\}_{n=1}^{\infty}$ generated by the algorithm HOEXRF converges to zero and*

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = \alpha, f(\alpha) = 0$$

Proof. As the proof that given in [9], we can see that the algorithm produces a sequences of $\{(b_n - a_n)\}_{n=1}^{\infty}$ and an iterative sequences x_n such that at least we have

$$\alpha \in [a_{n+1}, b_{n+1}] \subset [a_n, b_n] \subset ... \subset [a, b],$$

$$x_n \in [a_{n+1}, b_{n+1}] \subset [a_n, b_n] \subset ... \subset [a, b],$$

$$f(a_n)f(b_n) < 0, n = 0, 1, 2, ...,$$

$$a_n \leq a_{n+1} \leq b_{n+1} \leq b_n,$$

$$b_{n+1} - a_{n+1} \leq q_n(b_n - a_n) < q(b_n - a_n).$$

Since $0 < q < 1$, we obtain that $b_n - a_n \leq q^n(b - a)$.This means that

$$\lim_{n\to\infty} (b_n - a_n) = 0$$

and

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = \alpha.$$

So

$$\lim_{n\to\infty} x_n = \alpha, f(\alpha) = 0.$$

□

Theorem 2. *Under the hypothesis of lemma 2, assume that there exists a positive integer N_0 such that $|f(v_n)| < q_n|f(x_n)|$ whenever $n > N_0$ and the algorithm HOEXRF does not terminate after a finite number of steps. Then the sequence of diameter $\{(b_n - a_n)\}_{n+1}^{\infty}$ converges Q-cubically to 0, there is a constant λ such that*

$$b_{n+1} - a_{n+1} \leq \lambda(b_n - a_n)^3, \quad n = 0, 1, 2, ...$$

Proof. From lemma 1 it follows that, see [2],[3] and [9],

$$\lim_{n\to\infty}\frac{e_{n+1}}{e_n^3}=\lambda_1$$

and

$$\lim_{n\to\infty}\frac{e_{n+1}-e_n}{(e_n-e_{n-1})^3}=-\lambda_1$$

Then from eq.(8) it follows that:

$$\lim_{n\to\infty}\frac{e_{n+1}}{e_n^3}=\frac{f^{'''}(\alpha)}{6f^{'}(\alpha)}(f^{'2}(\alpha)-1)+\frac{f^{''2}(\alpha)}{4f^{'2}(\alpha)}(1-f^{'}(\alpha))+\frac{1}{12\alpha^2}=\lambda_1$$

and

$$\lim_{n\to\infty}\frac{e_{n+1}-e_n}{(e_n-e_{n-1})^3}=\lim_{n\to\infty}\frac{e_{n+1}/e_n-1}{e_n^2-3e_ne_{n-1}+3e_{n-1}^2-e_{n-1}^3/e_n}=-\lambda_1.$$

So, there exists an integer N_1 such that

$$\mid\frac{e_{n+1}-e_n}{(e_n-e_{n-1})^3}\mid<\mid\lambda_1\mid+1$$

and

$$\mid\frac{x_{n+2}-x_{n+1}}{(x_{n+1}x_n)^3}\mid<\mid\lambda_1\mid+1,$$

whenever $n>N_1$. From the assumption , whenever $n>N_0$, we have

$$\mid f(y_n)\mid<q_n\mid f(x_n)\mid,0<q_n<1.$$

From this inequality and lemma 2 ,we can deduce $z_n\in[\bar{a}_n,\bar{b}_n]$,
whenever $n>max\{N_0,N_1\}$, so $x_{n+1}=z_n$ and the above inequality means that

$$\frac{f(x_n)-f(y_n)}{f(x_n)}>0$$

Using Taylor's expansion of eq.(11), we have

$$x_{n+1}=x_n-\frac{q_n(b_n-a_n)\mid f(x_n)\mid}{x_n(p(x_n)f^2(x_n)+f(x_n)-f(y_n))}+O\left\{\frac{q_n(b_n-a_n)\mid f(x_n)\mid}{x_n(p(x_n)f^2(x_n)+f(x_n)-f(y_n))}\right\}$$

So this gives

$$b_n-a_n+O(b_n-a_n)=\frac{2(x_n-x_{n+1})(p(x_n)f^2(x_n)+f(x_n)-f(y_n))}{q_n\mid f(x_n)\mid},$$

and then, it follows that

$$\begin{aligned}&\frac{b_{n+1}-a_{n+1}+o(b_{n+1}-a_{n+1})}{(b_n-a_n+o(b_n-a_n))^3}\\&=\frac{q_n^3(x_{n+1}-x_{n+2})(p(x_{n+1})f^2(x_{n+1})+f(x_{n+1})-f(y_{n+1}))\mid f^3(x_n)\mid}{q_{n+1}(x_{n+1}-x_n)^3(p(x_n)f^2(x_n)+f(x_n)-f(y_n))^3\mid f(x_{n+1})\mid}\end{aligned}$$

Since

$$\lim_{n\to\infty}p(x)=-\frac{f^{''}(\alpha)}{2f^{'}(\alpha)}+\frac{f^{''}(\alpha)}{2}-\frac{1}{2\alpha}=L,$$

then there exists an integer N_2 such that $\mid p(x_n) \mid < \mid L \mid +1$, when $n > N_2$ and

$$\begin{aligned} \mid p(x_n)f(x_n) \mid + 1 - q \leq & \frac{\mid p(x_n)f^2(x_n) \mid + \mid f(x_n) - f(y_n) \mid}{\mid f(x_n) \mid} \\ & = \mid \frac{p(x_n)f^2(x_n) + f(x_n) - f(y_n)}{f(x_n)} \mid \\ & \leq \mid p(x_n)f(x_n) \mid + 1 + q. \end{aligned}$$

From lemma 2, we know that

$$\lim_{n\to\infty} f(x_n) = 0,$$

namely, there exists an integer N_3 such that $\mid f(x_n) < \frac{1-q}{2}$, where $n > N_3$.
Then we have

$$1 - q < \mid \frac{p(x_n)f^2(x_n) + f(x_n) - f(y_n)}{f(x_n)} \mid < 1 + q + \frac{(1-q)(1+ \mid L \mid)}{2} = \lambda_3.$$

So

$$\begin{aligned} \mid \frac{(p(x_{n+1})f^2(x_{n+1}) + f(x_{n+1}) - f(y_{n+1}))f^3(x_n)}{(p(x_n)f^2(x_n) + f(x_n) - f(y_n))^3 f(x_n)} \mid & < \frac{2 + 2q + (1-q)(1+ \mid L + 1)}{2(1-q)^3} \\ & < \frac{\lambda_3}{2(1-q)^3}. \end{aligned}$$

and if $n > \max\{N_0, N_1, N_2, N_3\}$, then

$$\mid \frac{b_{n+1} - a_{n+1} + o(b_{n+1} - a_{n+1})}{(b_n - a_n + o(b_n - a_n))^3} \mid \leq \frac{q^3(\lambda + 1)(2 + 2q + (1-q)(1+ \mid L \mid))}{8r(1-q)^3}.$$

Let $N = \max\{N_0, N_1, N_2, N_3\}$,

$$\lambda \geq \max\left\{ \frac{q^3(\lambda + 1)(2 + 2q + (1-q)(1+ \mid L \mid))}{8r(1-q)^3}, \frac{q}{(b_N - a_N)} \right\}.$$

then we have $b_{n+1} - a_{n+1} \leq \lambda(b_n - a_n)^3, \quad n = 0, 1, 2, ...$ □

5. Numerical examples

The following examples are considered to show the effectiveness of our cubically convergent method. Let x_0 is the starting approximation to the root α.
Example(1): $f(x) = x - e^{\sin(x)} + 1, [a, b] = [1, 4]$
Example(2): $f(x) = 11x^{11} - 1, [a, b] = [0.1, 1]$
Example(3): $f(x) = xe^{-x} - 0.1, [a, b] = [0, 1]$
Example(4): $f(x) = x^2 - e^{\sin(x)} + 1, [a, b] = [1, 4]$
Example(5): $f(x) = \tan^{-1}(x) + \cos(x) + x - 3, [a, b] = [0.5, 4]$.

In the numerical experiments, the initial values of the iteration methods $x_0 = b$, the computing are taken as $\epsilon_1 = \epsilon_2 = 1 \times 10^{-15}$, the maximal iterative numbers are not more than 100. Our methods HOEXRF, the methods of [1] called EXRF, Regula Falsi called RF, and Newton's methods for computing the results of examples 1-5 are given in table 1, and table 2.

From the table 1, we can see that our methods HOEXRF requires less number of iterations compared with the exponentially regula falsi iterative methods, the classical regula falsi iterative methods and the Newton's methods. And we can see also that our

methods HOEXRF, have larger convergence fields, faster convergence speed, higher convergence precision than all other methods in approximating the root.

Table 1.The numerical results of examples

Ex.	HOEXRF			EXRF		
	n	x_n	$\mid f(x_n) \mid$	n	x_n	$\mid f(x_n) \mid$
1	4	1.697e+00	4.857e-17	7	1.697e+00	8.882e-16
2	9	8.042e-01	3.435e-16	20	8.042e-01	4.441e-16
3	4	1.118e-01	1.591e-17	4	1.118e-01	1.591e-16
4	3	1.253e+00	2.143e-17	19	1.253e+00	1.213e-17
5	4	1.558e+00	1.316e-11	7	1.558e+00	1.316e-09

Table 2.The numerical results of examples

Ex.	RF			Newton		
	n	x_n	$\mid f(x_n) \mid$	n	x_n	$\mid f(x_n) \mid$
1	32	1.697e+00	4.441e-16	Not converges to α		
2	101	8.042e-01	1.254e-13	7	8.042e-01	4.441e-01
3	15	1.118 e-01	7.495e-16	Divergent		
4	100	1.253e+00	2.143e-10	Divergent		
5	25	1.558e+00	1.342e-07	Divergent		

6. Conclusions

A class of the cubic regula falsi methods for solving nonlinear equations in R is described by combining the regula falsi method with an exponential iterative method. This method is different from that given in [3]. The asymptotic convergence rate of both the sequence of iterates $\{(x_n - \alpha)\}$ and the sequence of diameter $\{(b_n - a_n)\}$ is established to be cubically convergent to zero. This is a substantial improvement over the results of [3]. An algorithm HOEXRF is also developed for computational purposes. The algorithm is then tested on a number of numerical examples and the results obtained up to the desired accuracy $\epsilon_1 = \epsilon_2 = 1 \times 10^{-15}$ are compared with our method HEXRF, the method of [1] called EXRF, Regula falsi (RF) Newtons methods. It is observed that our method takes less number of iterations and more effective in comparison with these methods.

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